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On the Aboodh Transform Connections with Some Famous Integral Transforms

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Abstract: *In this paper, we establish connections existing between the new integral transform, the Aboodh transform and some well-known integral transforms in literature. In establishing these connections or dualities, we start with the most oldest transform, the Laplace transform, followed by the Natural transform which later yield the Aboodh-Sumudu duality as a consequence, and lastly the Elzaki transform. Some test examples are presented with their graphical illustrations.*

Keywords: Aboodh Transform; Elzaki Transform; Natural Transform; Laplace Transform.

1. INTRODUCTION

Integral transforms serve as mathematical tools for solving differential and integral equations for almost three centuries after the notable works of J. Fourier and P.S. Laplace respectively [1]. However, in recent times, many researchers have turned their attention to the development of this very old area resulting to the emergence of new transforms like the Sumudu transform (1993) [2], Natural transform (2008) [3], Elzaki transform (2011) [4], Aboodh transform (2013) [5], Kashuri and Fundo transform (2013) [6], Srivastava *et al* transform (2015) [7] and ZZ Transform (2016) [8] among others. Furthermore, different studies and further analysis were carried out in this regards to determine the relationships among these recent transforms by many researchers. For instance, in Belgacem and Karaballi [9], the fundamental properties and applications of Sumudu transform were investigated. Kili'cman *et al* [10] analyzed the connections between Laplace and Sumudu transform. Elzaki and Ezaki [11] studied the connections between Laplace and Elzaki transforms. Khalaf and Belgacem [12] extracted the Laplace, Fourier, and Mellin Transforms from the Sumudu transform while the extraction of Laplace, Sumudu, Fourier and Mellin Transform from the Natural transform was by Shah *et al* [13].

However, in this paper, we establish the connections existing between the Aboodh transform and some well-known integral transforms owing to its numerous applications as demonstrated in [14-18]. In establishing these connections or dualities, we start off by the most oldest transform, the Laplace transform [1], followed by the Natural transform [3] which yields the Aboodh-Sumudu duality as a consequence, and lastly the Elzaki transform [4]. Some test examples are presented with their graphical representations.

The paper is organized as follows: In Section 2, we present the concept of the Aboodh transform and some of its properties. Section 3 presents the Aboodh transform dualities, Section 4 gives some examples, and finally Section 5 gives conclusion.

2. THE ABOODH TRANSFORM

The Aboodh transform is a new integral transform similar to the Laplace transform [1] and other integral transforms that are defined in the time domain $t \geq 0$, such as the Sumudu transform [2] Natural transform [3] and Elzaki transforms [4] among others.

The Aboodh transform is defined for functions of exponential order. We consider functions in the set B defined by:

$$B = \{ f : | f(t) | < Me^{k_j |t|}, \text{ if } t \in (-1)^j \times [0, \infty), \ j = 1,2; \ (M, k_1, k_2 > 0) \}.$$

For a given function in the set B, the constant M must be finite number; k_1, k_2 may be finite or infinite. Then, the Aboodh integral transform denoted by the operator $A(.)$ is defined (by [5]) by the integral equation:

$$A(u) = A\{f(t)\} = \frac{1}{u}\int_0^{\infty} e^{-ut} f(t)\,dt, \quad t \geq 0, \quad u \in (k_1, k_2). \tag{1}$$

2.1 Linearity Property

If a and b are any constants and $f(t)$ and $g(t)$ are functions defined over the set B above, then

$$A\{af(t) \pm bg(t)\} = aA\{f(t)\} \pm bA\{g(t)\}.$$

2.2 Aboodh Transform for Derivatives

For any given function $f(t)$ defined over the set B, the Aboodh transform of derivatives are given as follows:

1. $A\{f'(t)\} = uA(u) - \frac{f(0)}{u}$,
2. $A\{f^n(t)\} = u^n A(u) - \sum_{k=0}^{n-1} \frac{f^k(0)}{u^{2-n+k}}$,
3. $A\{tf'(t)\} = -\frac{d}{du}\left[uA(u) - \frac{f(0)}{u}\right] - \frac{1}{u}\left[uA(u) - \frac{f(0)}{u}\right]$,
4. $A\{t^2 f'(t)\} = u\frac{d^2}{d^2u}A(u) + 2\frac{d}{du}A(u) - 2\frac{f(0)}{u^3}$,
5. $A\{t^2 f''(t)\} = u^2\frac{d^2}{du^2}A(u) + 4u\frac{d}{du}A(u) + 2A(u) - 2\frac{f'(0)}{u^3}$.

3. THE ABOODH TRANSFORM DUALITIES

In this section, we present the Aboodh transform dualities of some famous integral transforms, starting with the oldest and the most used integral transform, the Laplace transform; followed by one of the most widely used transform of the last decade, the Natural transform; and lastly we consider one of the transform of the present decade; the Elzaki transform.

3.1 The Aboodh-Laplace Duality

The Laplace transform of $f(t)$ for $t \geq 0$, denoted by $F(u)$, is defined by the equation

$$F(u) = L[f(t)] = \int_0^{\infty} e^{-ut} f(t)\,dt, \ Re(u) > 0. \tag{2}$$

3.1.1 Theorem

Let $f(t) \in B$ with Aboodh transform $A(u)$ and Laplace transform $F(u)$, then

$$A(u) = \frac{1}{u} F(u) \tag{3}$$

and

$$F(u) = uA(u). \tag{4}$$

Proof:

To prove equation (3) above, let $f(t) \in B$, then from equation (1)

$$\begin{aligned} A(u) &= \frac{1}{u}\int_0^{\infty} e^{-ut} f(t)dt, \\ &= \frac{1}{u}\{\int_0^{\infty} e^{-ut} f(t)dt\}. \end{aligned}$$

Thus, from equation (2);

$$A(u) = \frac{1}{u} F(u).$$

To prove equation (4), from the equation (2), we have

$$\begin{aligned} F(u) &= \int_0^{\infty} e^{-ut} f(t)dt, \\ &= u\{\frac{1}{u}\int_0^{\infty} e^{-ut} f(t)dt\}. \end{aligned}$$

Thus, from equation (1);

$$F(u) = uA(u).$$

3.2 The Aboodh-Natural Duality

The Natural transform of $f(t)$ for $t \geq 0$, denoted by $R(s,u)$, is defined by the equation

$$R(s,u) = N[f(t)] = \int_0^{\infty} e^{-st} f(ut)dt, \; u, s \in (-k_1, k_2). \tag{5}$$

3.2.1 Theorem

Let $f(t) \in B$ with Aboodh transform $A(u)$ and Natural transform $R(s,u)$, then

$$A(u) = \frac{s}{u} R(su, s) \tag{6}$$

and

$$R(s,u) = \frac{s}{u^2} A(\frac{s}{u}). \tag{7}$$

Proof:

To prove equation (6) above, let $f(t) \in B$, then from equation (1)

$$A(u) = \frac{1}{u}\int_0^{\infty} e^{-ut} f(t)dt .$$

Let $t = sw$, $\Rightarrow dt = sdw$. Then,

$$A(u) = \frac{1}{u}\int_0^{\infty} e^{-u(sw)} f(sw)sdw ,$$
$$= \frac{s}{u}\int_0^{\infty} e^{-(us)w} f(sw)dw .$$

Hence, on comparing with equation (5) we obtain

$$A(u) = \frac{s}{u} R(su, s).$$

To prove equation (7), from equation (5) above,

$$R(s,u) = \int_0^{\infty} e^{-st} f(ut)dt$$

Let $w = ut$, $\Rightarrow t = \frac{w}{u}$ and $dt = \frac{dw}{u}$. Then,

$$R(s,u) = \int_0^{\infty} e^{-(\frac{s}{u})t} f(w)\frac{dw}{u},$$
$$= \frac{1}{u}\int_0^{\infty} e^{-(\frac{s}{u})t} f(w)dw ,$$
$$= \frac{s}{u^2}\{\frac{1}{\frac{s}{u}}\int_0^{\infty} e^{-(\frac{s}{u})t} f(w)dw\}.$$

Hence, on comparing with equation (1) we thus obtain

$$R(s,u) = \frac{s}{u^2} A(\frac{s}{u}).$$

Remarks:

The Natural transform of $f(t)$, i.e. $R(s,u)$ converges to Sumudu transform $G(u)$ at $s = 1$ and to Laplace transform $F(s)$ at $u = 1$ [13]. That is, $R(1, u) = G(u)$ and $R(s,1) = F(s)$ respectively. In view of the first fact and the Aboodh-Natural duality given in **Theorem 3.2.1** we thus state the following corollary:

3.2.2 Corollary

Let $f(t) \in B$ with Aboodh transform $A(u)$ and Sumudu transform $G(u)$, then

$$A(u) = \frac{1}{u^2} G\left(\frac{1}{u}\right).$$

3.3 The Aboodh-Elzaki Duality

The Elzaki transform of $f(t)$ for $t \geq 0$, denoted by $T(u)$, is defined by the equation

$$T(u) = E[f(t)] = u^2 \int_0^{\infty} e^{-t} f(ut)\, dt \;=,\; k_1, k_2 > 0. \tag{8}$$

3.3.1 Theorem

Let $f(t) \in B$ with Aboodh transform $A(u)$ and Elzaki transform $T(u)$, then

$$A(u) = T\left(\frac{1}{u}\right) \tag{9}$$

and

$$T(u) = A\left(\frac{1}{u}\right). \tag{10}$$

Proof:

To prove equation (9) above, let $f(t) \in B$, then from equation (1)

$$A(u) = \frac{1}{u} \int_0^{\infty} e^{-ut} f(t)\, dt.$$

Let $w = ut, \Rightarrow t = \frac{w}{u}$ and $dt = \frac{dw}{u}$. Then,

$$A(u) = \frac{1}{u} \int_0^{\infty} e^{-w} f\left(\frac{w}{u}\right) \frac{dw}{u},$$
$$= \left\{\frac{1}{u}\right\}^2 \int_0^{\infty} e^{-w} f\left(\frac{1}{u}.w\right) dw.$$

Hence, on comparing with equation (8) above, we thus obtain

$$A(u) = T\left(\frac{1}{u}\right).$$

To prove equation (10), from equation (8) above,

$$T(u) = u^2 \int_0^\infty e^{-t} f(ut)\,dt .$$

Let $z = ut$, $\Rightarrow t = \frac{z}{u}$ and $dt = \frac{dz}{u}$. Then,

$$T(u) = u^2 \int_0^\infty e^{-\frac{z}{u}} f(z) \frac{dz}{u},$$
$$= u \int_0^\infty e^{-\frac{z}{u}} f(z)\,dz .$$

Hence, on comparing with equation (1) we thus obtain

$$T(u) = A(\frac{1}{u}).$$

4. EXAMPLES

Here, we consider two examples. We compute their various integral transfroms in Tables 1-2 and further illustrate them graphically in figures $1a$, $1b$, $2a$ and $2b$ respectively, with the help of Mathematica software.

Example 1: Consider the function $f(t) = 2t^2 + t$. Thus, we get the following transforms:

Table 1: Integral transforms for $f(t) = 2t^2 + t$.

Aboodh Transform	Elzaki Transform	Laplace Transform	Natural Transform
$A(u) = \frac{4}{u^4} + \frac{1}{u^3}$	$T(u) = 4u^4 + u^3$	$F(u) = \frac{4}{u^3} + \frac{1}{u^2}$	$R(s,u) = \frac{4u^2}{s^3} + \frac{u}{s^2}$

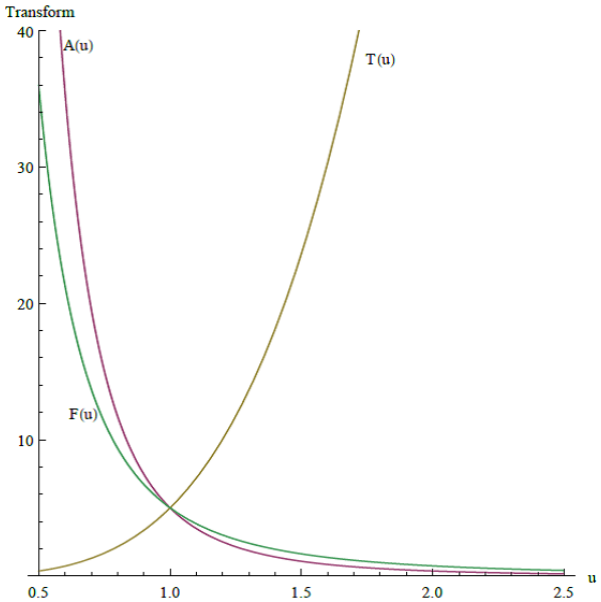

Figure 1a: Plots for Aboodh, Elzaki and Laplace Transforms of $f(t) = 2t^2 + t$.

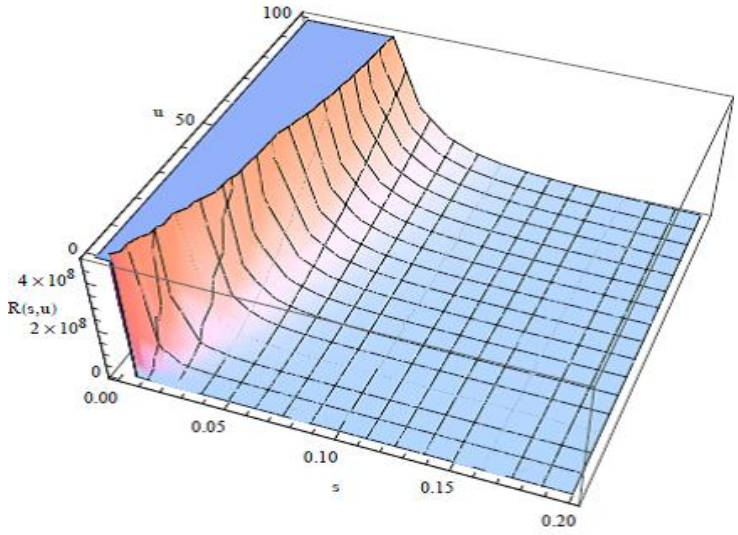

Figure 1b: Plot for Natural Transform of $f(t) = 2t^2 + t$ **.**

Example 2 Consider the function $f(t) = e^{-4t}$. Therefore we get the following transforms:

Table 2: Integral transforms for $f(t) = e^{-4t}$.

Aboodh Transform	Elzaki Transform	Laplace Transform	Natural Transform
$A(u) = \frac{1}{u^2 - 4u}$	$T(u) = \frac{u^2}{1 + 4u}$	$F(u) = \frac{1}{u + 4}$	$R(s,u) = \frac{1}{s + 4u}$

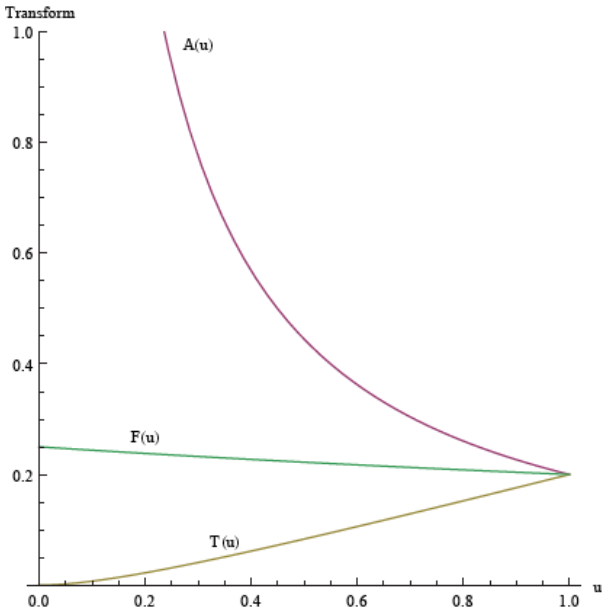

Figure 2a: Plots for Aboodh, Elzaki and Laplace Transforms of $f(t) = e^{-4t}$ **.**

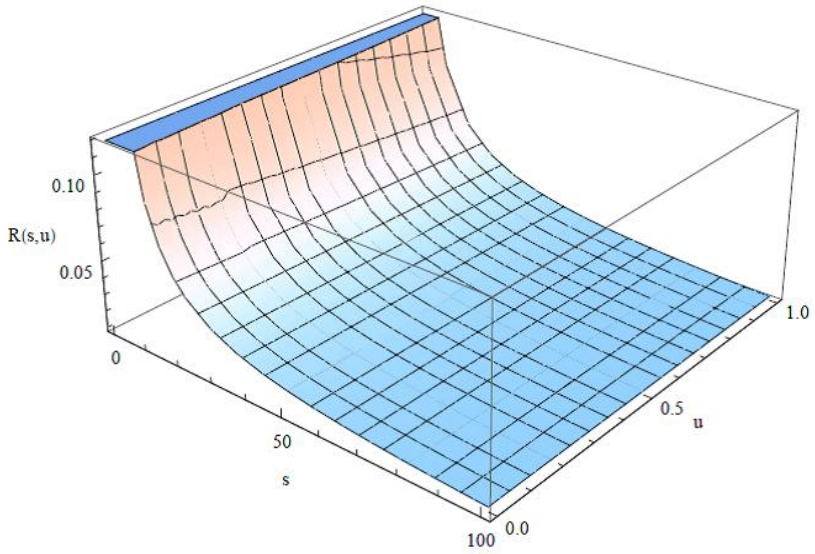

Figure 2b: Plot for Natural Transform of $f(t) = e^{-4t}$ **.**

5 CONCLUSION

In conclusion, the duality relation existing between the Aboodh integral transform and the Laplace, Sumudu, Natural and Elzaki integral transforms is derived along side with their converse dualities. The established theorems were all proved in an easy way and supported with some simple examples which were later illustrated graphically.

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