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Numerical Evaluation of Number of Roots of a Random Hyperbolic Polynomial

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Abstract— *In this paper we consider a polynomial* ***of the form*** $\sum_{k=1}^{n} y_k(w) \cosh kt$ ***which is a random polynomial with hyperbolic functions and whose coefficients*** y_k***'s are normally distributed dependent random variables with mean zero, variance one, and joint density function*** $|M|^{1/4}\ (2\pi)^{-n/2} \exp\left[-\left(\frac{1}{2}\right)\bar{y}' M\bar{Y}\right]$ ***where M is the moment matrix with*** $\rho_{ij} = \rho_1\ i \neq j$***; 0< ρ<1; i, j= 1,2,……., n,*** $\bar{Y} = (y_0(w), y_1(w), \ldots. y_n(w))$ ***is a column vector. Then an upper bound of expected number of zeros of the above polynomial is A n log n, where A is a constant.***

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1. INTRODUCTION

Let $f(t) = \sum_{k=1}^{n} y_k(w) \cosh kt,$ (1.1)

Where ($y_1, y_2, \ldots\ldots\ldots\ldots y_n$) is a sequence of dependent, normally distributed random variables with mean zero, variance one and joint density function.

$|M|^{1/4}\ (2\pi)^{-n/2} \exp\left[-\left(\frac{1}{2}\right)\bar{y}' M\bar{Y}\right]$(1.1) where M is the moment matrix with $\rho_{ij} = \rho_1\ i \neq j;$ 0< ρ<1; i, j= 1,2,……., n, $Y_k = (w)$

and $\bar{Y}$ is the column vector ($Y_1, Y_2 \ldots\ldots\ldots\ldots Y_n$). The average number of real zeros of the algebraic polynomial of the form $\sum_{k=1}^{n} y_k t^k = 0,$ where y_k's are dependent random variables, has been estimated among others, by Sambadham [6]. The corresponding result for a trigonometric polynomial of the form $\sum_{k=1}^{n} y_k \cosh kt,$ has been found by Sambadham [6]. Das [1] has considered the polynomial $\sum_{k=0}^{n} y_k \cosh kt$, where y_k's are independent, normally distributed random variables.

We evaluate here an upper estimate of expected number of real roots of (1.1) where the coefficients y_k's are dependant random variables as already indicated.

We denote by N_n (α, β) the number of real roots of (6.1) in the interval (α, β) and by EN_n (α, β), it's expected number of real roots in the interval (α, β). Since (1.1) is an even function of t we have $EN_n(-\infty, \infty) = 2EN_n(0, \infty)$.In our effort to evaluate EN_n

(0, ∞), we split the interval (0, ∞) into sub intervals $\left(0,\frac{1}{2^n}\right)$, $\left(\frac{1}{2^n},\frac{\log\log n}{n}\right)$, $\left(\frac{\log\log n}{n},1\right)$, (1, n^n) and (n^n, ∞) and find average number of roots in each of these intervals in sections (1.3), (1.4), (1.5), (1.6) and (1.7).

In section (1.2) we establish the formula for $EN_n(\alpha, \beta)$, to be used later.

2. FORMULA FOR ENN (A, B)

Following the procedure of Kac [4], we observe that if f(t) is continuously differentiable for $\alpha < t < \beta$ and has finite number of turning points (i.e. finite number of points at which derivatives of f(t) vanishes in (α, β), then the number of zeros of f(t) in (α, β) is given by the formula:

$$N_n(\alpha,\beta) = (2\pi)^{-1}\int_{-\infty}^{\infty} d\theta \int_{\alpha}^{\beta} \cos\left[\theta f(t)\right]\left|f'(t)\right| dt. \qquad (1.3).$$

Here f(t) denotes the derivatives of f(t).Multiple zeros are counted once, and if α or β is a zero, it is counted as ½ .Since M is positive definite, there exists a non singular linear transformation C= (c_{ij}) such that C'MC=I and $\bar{y} = C\bar{b}$, Where I is the identity matrix and $\bar{b}$, is the transpose of $\bar{b}$, given by $\bar{b}$, =(b_1…….b_n).

Hence, $\bar{y}'M\bar{y} = \bar{b}'c'Mc\bar{b} = \bar{b}'\bar{b} = b_1^2 + \dots\dots\dots b_n^2$. Hence (1.2) reduces to

$$\left[\frac{1}{|C|(2\pi)^{n/2}}\right] \exp\left[\left(-\tfrac{1}{2}\right)\left(b_1^2 + \dots\dots\dots \dots b_n^2\right)\right] \qquad (1.4)$$

Let T= (cosh t, …………….., cosh nt). Then

$$f(t) = \sum_{k=1}^{n}\left(c_{1k}\cosh t + \dots\dots\dots\dots + c_{nk}\cosh nt\right) b_k \qquad = \sum_{k=1} x_k b_k \qquad (1.5),$$

Where $X_k = C_{1k}\cosh t + \dots\dots\dots\dots + c_{nk}\cosh nt$.

Since C MC= I, M^{-1} = CC (1.6)

Equating corresponding elements of (1.6),

we get $C_{ij}C_{ji} + \dots\dots\dots . C_{in}C_{jn} = \{1 \text{ if } i=j \quad \& \quad 0 \text{ if } i \neq j\}$ (1.7)

The mathematical expectation of $N_n(\alpha, \beta)$ is given by

$$EN_n(\alpha,\beta) = \int_{R_n} N_n(\alpha,\beta)\, dp(\bar{y}) \qquad (1.8) = \frac{\sqrt{M}}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\dots\dots\int_{-\infty}^{\infty} N_n(\alpha,\beta)\exp\left[\left(-\tfrac{1}{2}\right)\bar{y}'M\bar{y}\right].$$

$$= \frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\dots\dots\int_{-\infty}^{\infty} N_n(\alpha,\beta)\exp\left[\left(-\tfrac{1}{2}\right)(b^2{}_1 + \dots\dots\dots\dots b_n^2\right] \text{ x } db_1 \dots\dots\dots\dots db_n . \qquad (1.9)$$

$$N_n(\alpha,\beta) = (2\pi 2^{n/2}\int_{-\infty}^{\infty}\dots\dots\int_{-\infty}^{\infty}\exp\left(-\tfrac{1}{2}\|b\|^2\right)$$

Using (1.3), we get

$$= \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \int_{\alpha}^{\beta}\cos\left(\xi f(t)\right)\left|f'(t)\right| dt\right\} \text{x } db_1 \dots\dots\dots\dots db_n .$$

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$$= \frac{1}{2\pi}\int_{\alpha}^{\beta} dt \int_{-\infty}^{\infty} R_n(\xi, t)\, d\xi \quad (1.10)$$

Where $\|b\| = b_1^2 + b_2^2 + \dots\dots\dots .b_n^2,$

$$R_n(\xi, t) = (2\pi)^{-n/2}\int_{-\infty}^{\infty}\dots\dots\int_{-\infty}^{\infty} \exp\ (-\tfrac{1}{2}\|b\|^2)\cos(\xi f(t))|f'(t)|\, dt x\, db_1 \dots\dots\dots\dots db_n$$

Interchange of orders of integration is justified by the fact that the integrand is dominated by a quantity which is exponentially small outside of bounded set's in R_n. We now use the equality $|y| = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1-\cos(\eta y)}{\eta^2}\, d\eta \quad (1.12)$

Thus $R_n(\xi, t) = \frac{1}{n}\int_{-\infty}^{\infty}\frac{dn}{\eta^2}.(2\pi)^{-n/2}\int_{-\infty}^{\infty}\dots\dots\dots\int_{-\infty}^{\infty}\exp(-\tfrac{1}{2}\|b\|^2) \quad X[\cos(\xi f(t)) - \cos\{\xi f(t)\}\cos\{\eta f'(t)\}] x\, db_1 \dots\dots\dots\dots db_n$
(1.13)

Since $[\cos(\xi f(t))\cos\{\eta f'(t)\}] =$

$$= \tfrac{1}{2}\, R1[\exp\{i\xi f(t) + i\eta f'(t)\} + \exp\{i\xi f(t) - i\eta f'(t)\}],$$

$$= (2\pi)^{-n/2}\int_{-\infty}^{\infty}\dots\dots\int_{-\infty}^{\infty}\exp\ (-\tfrac{1}{2}\|b\|^2)\cos(\xi f(t))\cos\{\eta f'(t)\}\, db_1 \dots\dots\dots\dots db_n$$

$$= \tfrac{1}{2}(2\pi)^{-n/2} R1 \int_{-\infty}^{\infty}\exp\ (-\tfrac{1}{2}\|b\|^2) \quad x\left[\exp\left\{i\sum_{k=1}^{n}(\xi X_k + \eta X'_k)b_k\right\} + \exp\left\{i\sum_{k=1}^{n}(\xi X_k - \eta X'_k)b_k\right\}\right] x\, db_1 .db_2 \dots db_n$$

$$= \tfrac{1}{2}\left[\exp\left\{(-\tfrac{1}{2})\sum_{k=1}^{n}U_k^2\right\} + \exp\left\{(-\tfrac{1}{2})\sum_{k=1}^{n}V_k^2\right\}\right] \quad (1.14)$$

where $U_k = \xi X_k + \eta X'_k$

and $V_k = \xi X_k - \eta X'_k$; k=1,……..n.

Putting η=0 in (6.14), we get

$$(2\pi)^{-n/2}\int_{-\infty}^{\infty}\dots\dots\int_{-\infty}^{\infty}\exp\ (-\tfrac{1}{2}\|b\|^2)\cos(\xi f(t))\, db_1 \dots\dots\dots\dots .db_n$$

$$= \exp\left[(-\tfrac{1}{2})(W_1^2 + W_2^2 + \dots\dots\dots\dots .W_n^2)\right]$$

where $W_k = \xi X_k$

Hence

$$R_n(\xi, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\eta^2 d\eta\left[2\exp\left\{(-\tfrac{1}{2})\sum_{k=1}^{n}W_k^2\right\} - \exp\left\{(-\tfrac{1}{2})\sum_{k=1}^{n}U_k^2\right\} - \exp\left\{(-\tfrac{1}{2})\sum_{k=1}^{n}V_k^2\right\}\right]$$

$$= \frac{1}{n}\int_{-\infty}^{\infty}\eta^2 d\eta\left[2\exp\left\{(-\tfrac{1}{2})\sum_{k=1}^{n}\xi^2 X_k^2\right\} - \exp\left\{(-\tfrac{1}{2})\sum_{k=1}^{n}(\xi X_k - \eta X'_k)^2\right\} - \exp\left\{(-\tfrac{1}{2})\sum_{k=1}^{n}(\xi X_k + \eta X'_k)^2\right\}\right]$$

(1.15)

From (1.10) and (1.15), we get

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$$EN_n(\alpha,\beta)=\frac{1}{2\pi^2}\int_{\alpha}^{\beta}dt\int_{-\infty}^{\infty}d\xi\int_{0}^{\infty}\eta^{-2}d\eta\left[2\exp\left\{(-½)\sum_{k=1}^{n}\xi^2X_k^{\,2}\right\}\right]$$

$$-\exp\left\{(-½)\sum_{k=1}^{n}(\xi X_k-\eta X'_k)^2\right\}\text{-}\exp\left\{(-½)\sum_{k=1}^{n}(\xi X_k+\eta X'_k)^2\right\}\ ]$$

$$=\frac{1}{\pi^2}\int_{\alpha}^{\beta}dt\int_{-\infty}^{\infty}\log\left(\sum_{k=1}^{n}\frac{(vX_k-Y_k)^2}{\sum_{k=1}^{n}(vX_k)^2}dv\right), \tag{1.16}$$

Where $Y_k = X'_k$.

(Putting ξ=ηv and using Frunalli's theorem [7]).

Now

$$\int_{-\infty}^{\infty}\log\left(\sum_{k=1}^{n}\frac{(vX_k-Y_k)^2}{\sum_{k=1}^{n}(vX_k)^2}\right)$$

(applying Frunalli's theorem)

$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{\exp\left\{-\sum_{k=1}^{n}(X_k\xi)^2\right\}-\exp\left\{-\sum_{k=1}^{n}(X_k\xi-Y_k\eta)^2\right\}}{\eta^2}d\eta\right]d\xi$$

$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{\exp\{(-X\xi)^2\}-\exp\{-X\xi^2+Y\eta\xi-Z\eta^2\}}{\eta^2}d\eta\right]d\xi$$

Where

$$X=\sum_{k=1}^{n}X_k^{\,2},$$

$$Y=2\sum_{k=1}^{n}X_kX'_k,$$

$$Z=\sum_{k=1}^{n}X'^2_k, \tag{1.17}$$

Let

$$I=\int_{-\infty}^{\infty}\exp\frac{\exp\{(-X\xi)^2\}-\exp\{-X\xi^2+Y\eta\xi-Z\eta^2\}}{\eta^2}d\eta$$

Then following the procedure of Das [1], it can be shown that

$$\int_{-\infty}^{\infty} I d\xi = \pi\left(\frac{4XZ-Y^2}{X^2}\right)^{1/2},$$

Hence from (1.16), we have

$$EN_n(\alpha,\beta)=\frac{1}{\pi}\int_{\alpha}^{\beta}\left(\frac{XZ-Y^2}{X^2}\right)^{1/2} dt, \qquad (1.18)$$

From the relations

$X_k = C_{1k}\cosh t + \ldots\ldots + C_{nk}\cosh nt$,

(1.7) and (1.17), we get

$$X=(1-\rho)\sum_{k=1}^{n}\cosh^2 kt+\rho\left(\sum_{k=1}^{n}\cosh kt\right)^2,$$

$$\mathrm{Y}=2(1-\rho)\sum_{k=1}^{n}k\cosh kt\sinh \mathrm{kt}+2\rho\left(\sum_{k=1}^{n}\sinh kt\right)\left(\sum_{k=1}^{n}\cosh kt\right),$$

$$\mathrm{Z}=(1-\rho)\sum_{k=1}^{n}k^2\sinh^2 kt+\rho\left(\sum_{k=1}^{n}k\sinh kt\right)^2,$$

Applying method of finite differences, we obtain, after certain elementary reductions,

$$\sum_{k=1}^{n}\cosh^2 \mathrm{kt}=\frac{2\mathrm{n}-1}{4}+\frac{\sinh(2\mathrm{n}+1)t}{4\sinh \mathrm{t}},$$

$$\sum_{k=1}^{n}k\sinh \mathrm{kt}\cosh \mathrm{kt}=\tfrac{1}{2}\left[\frac{(2n+1)\cosh(2\mathrm{n}+1)}{\sinh \mathrm{t}}-\frac{\sinh(2\mathrm{n}+1)\,\mathrm{t}\cosh \mathrm{t}}{\sinh^2 t}\right],$$

$$\sum_{k=1}^{n}k^2\sinh^2 kt=-\frac{\mathrm{n}(\mathrm{n}+1)(2\mathrm{n}+1)}{12}+\frac{1}{16}\left[\frac{(2n+1)^2\sinh(2\mathrm{n}+1)\,\mathrm{t}}{\sinh \mathrm{t}}\right.$$

$$\left.-\frac{2(2\mathrm{n}+1)\cosh(2\mathrm{n}+1)\,\mathrm{t}\cosh \mathrm{t}}{\sinh^2 t}+\frac{\sinh(2\mathrm{n}+1)\,\mathrm{t}}{\sinh t}\left(\frac{2\cosh^2 t}{\sinh^2 t}-1\right)\right],$$

$$\sum_{k=1}^{n}k\sinh \mathrm{kt}=\frac{(\mathrm{n}+1)}{2}\,\frac{\cosh(\mathrm{n}+\frac{1}{2})\,\mathrm{t}}{\sinh\frac{\mathrm{t}}{2}}-\frac{\cosh\frac{\mathrm{t}}{2}}{4\sinh^2\frac{t}{2}}\sinh(\mathrm{n}+\tfrac{1}{2})\,\mathrm{t}$$

$$-\frac{1}{4}\,\frac{\cosh(\mathrm{n}+\frac{1}{2})\,\mathrm{t}}{\sinh\frac{\mathrm{t}}{2}},$$

and

$$\sum_{k=1}^{n}\cosh kt = \frac{\sinh(n+\frac{1}{2})t}{2\sinh\frac{t}{2}} - \frac{1}{2}.$$

3. AVERAGE NUMBER OF ZEROS IN RANGE (1, NN)

In this range, the estimations of X, Y and Z are given by

$$X = (1-\rho)\left[\frac{2n-1}{4} + \frac{\sinh(2n+1)t}{4\sinh t}\right] + \rho\left[\frac{\sinh(n+\frac{1}{2})t}{2\sinh\frac{t}{2}} - \frac{1}{2}\right]^2$$

$$= \frac{(2n-1)}{4}\,\frac{\sinh(2n+1)t\cosh(2n+1)t}{\sinh t}\left(\frac{(1-\rho)\sinh t}{\sinh(2n+1)t\cosh(2n+1)t}\right.$$

$$+ \frac{(1-\rho)}{(2n-1)t\cosh(2n+1)t} + \frac{4\rho\sinh(n+\frac{1}{2})t\cosh\frac{t}{2}}{(2n-1)\cosh(n+\frac{1}{2})\cosh(2n+1)t\sinh\frac{t}{2}}$$

$$- \frac{\rho}{2(2n-1)\sinh\frac{t}{2}\cosh(n+\frac{1}{2})t\cosh(2n+1)t}$$

$$\left. + \frac{\rho\sinh t}{4(2n-1)\sinh(2n+1)t\cosh(2n+1)t}\right).$$

Thus,

$$X = \frac{\sinh(2n+1)t\cosh(2n+1)t}{4\sinh t}\,O(2n-1)$$

$$= \frac{\sinh(4n+2)t}{\sinh t}\,O(n).$$

$$Y = 2(1-\rho)\left[\frac{(2n+1)\cosh(2n+1)t}{4\sinh t} - \frac{\sinh(2n+1)t\cosh t}{4\sinh^2 t}\right]$$

$$+ 2\rho\left[\frac{(n+1)\cosh(n+\frac{1}{2})t}{2\sinh\frac{t}{2}} - \frac{\cosh\frac{t}{2}}{4\sinh^2\frac{t}{2}}\sinh(n+\frac{1}{2})t\right.$$

$$-\frac{1}{4}\frac{\cosh(n+\tfrac12)t}{\sinh\frac{t}{2}}\Bigg]\Bigg[\frac{\sinh(n+\tfrac12)t}{2\sinh\frac{t}{2}}-\tfrac12\Bigg]$$

$$=\frac{(2n+1)}{2}\frac{\cosh(2n+1)t\sinh(2n+1)t}{\sinh t}\Bigg[\frac{1-\rho}{\sinh(2n+1)t}$$

$$-\frac{(1-\rho)\cosh t}{(2n+1)\ \sinh t\cosh(2n+1)t}+\frac{\rho(n+1)}{(2n+1)}\frac{\cosh\frac{t}{2}}{\sinh\frac{t}{2}\cosh(2n+1)t}$$

$$-\frac{2\rho}{(2n+1)}\frac{\cosh t\cosh^2\frac{t}{2}\sinh^2(n+\tfrac12)t}{\sinh^2\frac{t}{2}\cosh(2n+1)t\sinh(2n+1)t}$$

$$-\frac{\rho}{2(2n+1)}\frac{\cosh\frac{t}{2}}{\sinh\frac{t}{2}\cosh(2n+1)t}-\frac{(n+1)\rho\cosh\frac{t}{2}}{4(2n+1)\ \sinh(n+\tfrac12)t\cosh(2n+1)t}$$

$$+\frac{\rho}{2(2n+1)}\frac{\cosh^2\frac{t}{2}}{\sinh\frac{t}{2}\cosh(n+\tfrac12)t\cosh(2n+1)t}$$

$$-\frac{\rho}{2(2n+1)}\frac{\cosh\frac{t}{2}}{\sinh(n+\tfrac12)t\cosh(2n+1)t}\Bigg].$$

Thus,

$$Y=\frac{\sinh(4n+2)t}{\sinh t}O(n),$$

$$Z=(1-\rho)\Bigg[\frac{-n(n+1)(2n+1)}{12}\frac{1}{16}\Bigg\{\frac{(2n+1)^2\sinh(2n+1)t}{\sinh t}$$

$$-\frac{2(2n+1)\cosh(2n+1)t\cosh t}{\sinh^2 t}+\frac{2\sinh(2n+1)t\cosh^2 t}{\sinh^3 t}$$

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$$-\frac{\sinh(2n+1)t}{\sinh t}\Big\}\Big]+\rho\left[\frac{(n+1)}{2}\frac{\cosh(n+\tfrac{1}{2})t}{\sinh\frac{t}{2}}\right.$$

$$\left.-\frac{\cosh\frac{t}{2}}{4\sinh^2\frac{t}{2}}\sinh(n+\tfrac{1}{2})t-\frac{1}{4}\frac{\cosh(n+\tfrac{1}{2})t}{\sinh\frac{t}{2}}\right]^2$$

$$=(2n+1)^3\sinh(2n+1)t\cosh(2n+1)t\left[-\frac{(1-\rho)(n+1)n}{12(2n+1)^3}\frac{\sinh^2 t}{\sinh(2n+1)t\cosh(2n+1)t}\right.$$

$$+\frac{(1-\rho)}{16(2n+1)}\frac{\sinh t}{\cosh(2n+1)t}-\frac{(1-\rho)}{8(2n+1)^2}\frac{\cosh t}{\cosh(2n+1)}$$

$$+\frac{(1-\rho)}{8(2n+1)^3}\frac{(1+\sinh^2)t}{\sinh t\cosh(2n+1)t}-\frac{(1-\rho)}{16(2n+1)^3}\frac{\sinh t}{\cosh(2n+1)t}$$

$$+\frac{\rho(n+1)^2}{4(2n+1)^3}\frac{\{\cosh(2n+1)t+1\}\{\cosh t+1\}}{\cosh(2n+1)t\sinh(2n+1)t}$$

$$+\frac{\rho}{(2n+1)^3}\frac{(1+\sinh^2 t/2)(\cosh t+1)\{\cosh(2n+1)t-1\}}{\sinh^2 t/2\cosh(2n+1)t\sinh(2n+1)t}$$

$$+\frac{\rho}{16(2n+1)^3}\frac{\{1+\cosh(2n+1)t\}(1+\cosh t)}{\sinh(2n+1)t\cosh(2n+1)t}$$

$$-\frac{\rho(n+1)}{4(2n+1)^3}\frac{(1+\sinh^2 t/2)\sinh t}{\cosh(2n+1)t\sinh^2 t/2}-\frac{\rho(n+1)}{4(2n+1)^3}\frac{\{1+\cosh(2n+1)t\}(1+\cosh t)}{\sinh(2n+1)t\cosh(2n+1)t}$$

$$+\frac{\rho}{8(2n+1)^3}\frac{(1+\sinh^2 t/2)\sinh t}{\cosh(2n+1)t\sinh^2 t/2}=\frac{\sinh(2n+1)t\cosh(2n+1)t}{\sinh^2 t}O(n)^3$$

$$\left.=\frac{\sinh(4n+2)t}{\sinh^2 t}O(n)^2\right].$$

From the above estimations, we have

$$\frac{(4ZX-Y^2)}{X^2}$$

$$=4\frac{Z}{X}-\left(\frac{Y}{X}\right)^2$$

$$=4e^{-t}O(n^2),$$

Hence

$$\frac{(4ZX-Y^2)^{1/2}}{X} = 2e^{-t/2}\, O(n).$$

From (6.18), we find that for all t*≥1,

$$EN_n(1, t^*) = O(n) \int_1^{t^*} 2e^{-t/2}\, dt$$

$$= O(n)\,(e^{-1}\, e^{-t^*})$$

$$= O(n).$$

Taking $t^* = n^n$, we have

$$EN_n(1, n^n) = O(n). \quad (1.19).$$

4. AVERAGE NUMBER OF ZEROS IN THE RANGE $\left(\frac{\log\log n}{n}, 1\right)$

Inside this range $\frac{\sinh nt}{(\sinh t)^p}$ is an increasing function of t, because

$$\cosh t < \frac{e}{t} < \frac{en}{\log \log n},$$

and the derivative of $\frac{\sinh nt}{(\sinh t)^p}$ is given by

$$\cosh nt\, (\operatorname{cosech} t)^p\, (n - p \tanh nt \operatorname{cothh} nt)$$

$$\geq \cosh nt\, (\operatorname{cosech} t)^p \left(n - \frac{en}{\log \log n}\right)$$

$$> 0,$$

for large n.

Hence,

$$\frac{\sinh nt}{(\sinh t)^p} \geq \frac{\sinh(\log\log n)}{\left[\sinh\left(\frac{\log \log n}{n}\right)\right]^p}$$

$$\geq \sinh(\log\log n)\left(\frac{n}{4\log\log n}\right)^p$$

$$> \frac{n^p \log n}{256\,(\log \log n)^p}.$$

for p= 1,2,3 and 4.

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Hence, we have used fact that sin ht ≤ 4t in (0,1). This is due to the observation that (sign ht-4t) is a decreasing function of t and vanishes at the origin.

$$\text{Now, } X = (1-\rho)\left[\frac{2n-1}{4} + \frac{\sinh(2n+1)t}{4\sinh t}\right]$$

$$+\rho\left[\frac{\sinh\left(n+\frac{1}{2}\right)t}{2\sinh t/2} - \frac{1}{2}\right]^2$$

$$\le \frac{\sinh(2n+1)t}{\sinh^2 t/2}\left[(1-\rho)\left\{\frac{\sinh t/2}{8\cosh t/2} + \frac{(2n-1)}{4}\frac{\sinh^2 t/2}{\sinh(2n+1)t}\right\}\right.$$

$$\left. + \rho\left\{\frac{\sinh(n+½)t}{4\cosh(n+½)t} + \frac{\sinh^2 t/2}{2\sinh(2n+1)t}\right\}\right]$$

$$\le \frac{\sinh(2n+1)t}{\sinh^2 t/2}\left[\left\{(1-\rho)\frac{\sinh t/2}{8\cosh t/2} + \frac{\rho}{8}\frac{\sinh(n+½)t}{\cosh(n+½)t} + \right.\right.$$

$$\left.\left.\frac{\rho}{8}\frac{\sinh t}{\sinh(2n+1)t}\right\} + (1-\rho)\frac{(2n-1)}{8}\frac{\sinh t}{\sinh(2n+1)t}\right]$$

$$\le C_1\frac{\sinh(2n+1)t}{\sinh^2 t/2}\left[1 + \frac{(1-\rho)(2n-1)\,256\log\log(2n+1)}{8C_1(2n+1)\log(2n+1)}\right].$$

Here C_1 is a constant.

$$\text{So } X = C_1\frac{\sinh(2n+1)t}{\sinh^2\frac{t}{2}}\left[1 + O\left(\frac{\log\log n}{\log n}\right)\right].$$

$$\frac{Y}{2} < (1-\rho)(2n+1)\frac{\cosh(2n+1)t}{4\sinh t} + \rho\frac{(n+1)}{8}\frac{\sinh(2n+1)t}{\sinh^2\frac{t}{2}}$$

$$+\frac{\rho}{8}\frac{\cosh\frac{t}{2}}{\sinh^2\frac{t}{2}}\sinh(n+½)t + \frac{\rho}{8}\frac{\cosh(n+½)t}{\sinh t}$$

$$= (2n+1)\cosh(2n+1)t\sinh(2n+1)t\left[\frac{1-\rho}{4}\frac{\sinh\frac{t}{2}\sinh^2\frac{t}{2}}{\sinh t\sinh(2n+1)t}\right.$$

$$+\frac{\rho}{8}\frac{(n+1)}{(2n+1)}\frac{\sinh\frac{t}{2}}{\cosh(2n+1)\,t}+\frac{\rho}{8}\frac{\cosh\frac{t}{2}\sinh\frac{t}{2}}{\cosh(2n+1)\,t\sinh(2n+1)\,t}$$

$$\left.+\frac{\rho}{8}\frac{\sinh^{2}\frac{t}{2}}{\cosh(2n+1)\,t\sinh(2n+1)\,t}\right]$$

$$\leq\frac{4(2n+1)\cosh(2n+1)\,t\sinh(2n+1)\,t}{\sinh^{3}\frac{t}{2}}\left[\frac{\sinh t}{\sinh(2n+1)\,t}\right]$$

$$<1024\frac{\log\log(2n+1)}{\log(2n+1)}\frac{\cosh(2n+1)\,t\sinh(2n+1)\,t}{\sinh^{3}\frac{t}{2}},$$

Hence,

$$Y=\frac{\cosh(2n+1)\,t\sinh(2n+1)\,t}{\sinh^{3}\frac{t}{2}}O\left(\frac{\log\log n}{\log n}\right),$$

$$Z<\frac{(1-\rho)}{16}(2n+1)^{2}\frac{\sinh(2n+1)\,t}{\sinh t}+2(1-\rho)\sinh(2n-1)\,t\frac{\cosh^{2}t}{\sinh^{3}t}$$

$$+\frac{\rho(n+1)^{2}}{4}\frac{\cosh^{2}(n+\tfrac{1}{2})\,t}{\sinh^{2}\frac{t}{2}}+\frac{\rho}{16}\frac{\cosh^{2}\frac{t}{2}}{\sinh^{4}\frac{t}{2}}\sinh^{2}\left(n+\tfrac{1}{2}\right)t$$

$$+\frac{\rho}{16}\frac{\cosh^{2}(n+\tfrac{1}{2})\,t}{\sinh^{2}\frac{t}{2}}+\frac{\rho}{32}\frac{\sinh(2n+1)\,t\cosh\frac{t}{2}}{\sinh^{3}\frac{t}{2}}$$

$$=(2n+1)^{2}\frac{\sinh(2n+1)\,t}{\sinh^{2}\frac{t}{2}}\left[\frac{(1-\rho)}{16}\frac{\sinh^{2}\frac{t}{2}}{\sinh t}+\frac{2(1-\rho)}{(2n+1)^{2}}\frac{\sinh^{2}\frac{t}{2}\cosh^{2}t}{\sinh^{3}t}\right.$$

$$+\frac{\rho}{8}\frac{(n+1)^{2}}{(2n+1)^{2}}\frac{\cosh(n+\tfrac{1}{2})t}{\sinh(n+\tfrac{1}{2})\,t}+\frac{\rho}{16(2n+1)^{2}}\frac{\cosh^{2}\frac{t}{2}\sinh^{2}(n+\tfrac{1}{2})t}{\sinh^{2}\frac{t}{2}\sinh(2n+1)\,t}$$

$$+\frac{\rho}{16(2n+1)^2}\frac{\cosh(n+\frac{1}{2})t}{\sinh(n+\frac{1}{2})\,t}+\frac{\rho}{32(2n+1)^2}\frac{\cosh\frac{t}{2}}{\sinh\frac{t}{2}}\Bigg]$$

$$\le(2n+1)^2\frac{\sinh(2n+1)\,t}{\sinh^2\frac{t}{2}}\Bigg[\frac{(1-\rho)}{32}\frac{\sinh\frac{t}{2}}{\cosh\frac{t}{2}}+\frac{(1-\rho)}{(2n+1)^2}\frac{\sinh\frac{t}{2}}{\cosh\frac{t}{2}}\left(\frac{en}{\log\log n}\right)^2$$

$$+\frac{\rho}{8}\frac{(n+1)^2}{(2n+1)^2(n+\frac{1}{2})}\left(\frac{en}{\log\log n}\right)+\frac{\rho}{32(2n+1)^2}\left(\frac{en}{\log\log n}\right)^2\frac{\sinh(n+\frac{1}{2})}{\cosh(n+\frac{1}{2})}$$

$$+\frac{\rho}{32(2n+1)^2(n+\frac{1}{2})}\left(\frac{en}{\log\log n}\right)+\frac{\rho}{32(2n+1)^2}\left(\frac{en}{\log\log n}\right)\Bigg].$$

Thus,

$$Z=\frac{\sinh(2n+1)\,t}{\sinh^2\frac{t}{2}}O(2n+1)^2.$$

Hence, $4XZ-Y^2$

$$=\left[4C_1\frac{\sinh(2n+1)\,t}{\sinh^2\frac{t}{2}}\left\{1+O\left(\frac{\log\log n}{\log n}\right)\right\}\frac{\sinh(2n+1)t}{\sinh^2\frac{t}{2}}O(2n+1)^2\right]$$

$$-\left(\frac{\sinh^2(2n+1)t\cosh^2(2n+1)t}{\sinh^6\frac{t}{2}}\right)O\left(\frac{\log\log n}{n}\right)^2$$

$$\frac{4C_1\sinh^2(2n+1)t}{\sinh^4\frac{t}{2}}\left[1+O\left(\frac{\log\log n}{\log n}\right)\right]O(2n+1)^2$$

$$-\left(\frac{\sinh^2(2n+1)t\cosh^2(2n+1)t}{\sinh^6\frac{t}{2}}\right)O\left(\frac{\log\log n}{\log n}\right)^2$$

$$= \frac{4C_1 \sinh^2(2n+1)t}{\sinh^6 \frac{t}{2}} \left[1+O\left(\frac{\log\log n}{\log n}\right)\right] O(2n+1)^2 \left[\sinh^2 \frac{t}{2}\right.$$

$$\left. - \frac{(\cosh^2(2n+1)t)\, O\left(\frac{\log\log n}{\log n}\right)^2}{4C_1 \left\{1+O\left(\frac{\log\log n}{\log n}\right)\right\} O(2n+1)^2}\right]$$

$$\leq \frac{4C_1 \sinh^2(2n+1)t}{\sinh^6 \frac{t}{2}} \left[1+O\left(\frac{\log\log n}{\log n}\right) O(2n+1)^2\right].$$

Also, we have

$$X^2 = \frac{C^2{}_1 \sinh^2(2n+1)t}{\sinh^4 \frac{t}{2}} \left[1+0\left(\frac{\log\log n}{\log n}\right)\right]^2$$

Hence, $$\frac{4XZ-Y^2}{X^2} = \frac{0(n)^2}{\left[1+0\left(\frac{\log\log n}{\log n}\right)\right] \sinh^2 \frac{t}{2}},$$

or $$\frac{(4XZ-Y^2)^{1/2}}{X} = \frac{0(n)}{\left[1+0\left(\frac{\log\log n}{\log n}\right)\right]^{1/2} \sinh \frac{t}{2}},$$

Therefore,

$$EN_n\left(\frac{\log\log n}{n}, 1\right) = \frac{1}{2\pi} \int_{\frac{\log\log n}{n}}^{1} \frac{O(n)}{\left[1+0\left(\frac{\log\log n}{\log n}\right)\right]^{1/2} \sinh \frac{t}{2}} dt$$

$$= \frac{O(n)}{2\pi\left[1+0\left(\frac{\log\log n}{\log n}\right)\right]^{1/2}} \left[\log\tanh\frac{t}{2}\right]_{\frac{\log\log n}{2n}}^{1/2} \qquad (1.19)$$

$$\leq C_2 \frac{n\log n}{\left[1+0\left(\frac{\log\log n}{\log n}\right)\right]^{1/2}},$$

5. AVERAGE NUMBER OF ZEROS IN THE RANGE $\left(\frac{1}{2^n}, \frac{\log\log n}{n}\right)$

Let

$$t \in \left(\frac{1}{2^n}, \frac{\log n}{n}\right),$$

We have always

$$\frac{\left(4XZ - Y^2\right)^{1/4}}{X} \le 2\left(\frac{Z}{X}\right)^{1/2}.$$

Here

$$Z = (1-\rho)\sum_{k=1}^{n} k^2 \sinh^2 kt + \rho\left(\sum_{k=1}^{n} k \sinh kt\right)^2$$

$$\le (1-\rho)\left(\sum_{k=1}^{n} k \sinh kt\right)^2 + \rho\left(\sum_{k=1}^{n} k \sinh kt\right)^2$$

$$= \left(\sum_{k=1}^{n} k \sinh kt\right)^2$$

$$\le \left(\sum_{k=1}^{n} k\right)^2 \qquad \text{(Since } \sinh^2 kt<1 \text{ here)}$$

$$= \frac{n^2 (n+1)^2}{4} \qquad (1.20).$$

Similarly,

$$X = (1-\rho)\sum_{k=1}^{n} k^2 \cosh^2 kt + \rho\left(\sum_{k=1}^{n} k \cosh kt\right)^2$$

$$\le (1-\rho)\left(\sum_{k=1}^{n} k \cosh^2 kt\right) + \rho\left(\sum_{k=1}^{n} k \cosh^2 kt\right)$$

$$= \left(\sum_{k=1}^{n} \cosh^2 kt \ge n.\right)$$

Hence,

$$\frac{(4XZ - Y^2)}{X} \le \left[\frac{n^2 (n+1)^2}{2n}\right]^{1/2} < \left[\frac{n^3}{2}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)\right]^{1/2} < \left(\frac{3}{2}\right)^{1/2} n^{\frac{3}{2}}.$$

Therefore

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$$EN_n\left(\frac{1}{2^n},\frac{\log\log n}{n}\right)$$
$$\leq \frac{1}{2\pi}\int_{\frac{1}{2^n}}^{\frac{\log\log n}{n}}\left(\frac{3}{2}\right)^{1/2} n^{3/2}dt$$
$$=\frac{(3/2)^{1/2}n^{3/2}}{2\pi}\left(\frac{\log\log n}{n}-\frac{1}{2^n}\right)$$
$$< n^{1/2}\log\log n. \qquad (1.21)$$

6. AVERAGE NUMBER OF ZEROS IN THE RANGE $\left(0,\frac{1}{2^N}\right)$

Let F(s,t)= f(y,t)= $\sum_{k=1}^{n} Y_k(s)\cosh kt,$

Where Y_k's are considered as function of sε (0,1). The following is the generalization of the work of Dunnage [2].

When s does not belong to an experimental set of small measure, we can apply Jensen's theorem to the function, namely

$$f(s,z)=\sum Y_k(s)\cosh kz.$$

Let n (s,r) denote the number of real zeros of f(s,z) in |z| ≤r.

Let $\varepsilon=\frac{1}{2^n}$.

Then

n (s,z) ≤ n (s,z) , for 0 ≤ |z| ≤ ε.

Now, P(f (s,0) ≠0)=1.

Let E_1 ={s: f(s,0)=0}.

Then, $P(E_1)$=0.Now if $s\notin E_1$, then applying Jensen's theorem we have

$$n(s,\in)\leq\frac{1}{2\pi\log 2}\int_0^{2\pi}\log\left|\frac{f(s,2\in e^{i\theta})}{f(s,0)}\right|d\theta.$$

$$f(s,2\varepsilon e^{i\theta})=\sum_{k=1}^{n} y_k(s)\cosh k(2\varepsilon e^{i\theta})$$

Now $=\sum_{k=1}^{n} y_k(s)\,\delta_k + i\sum_{k=1}^{n} y_k(s)\,\mu_k$ =S+iQ,

Where $S=\sum_{k=1}^{n} y_k(S)\,\delta_k,$

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$$V = \sum_{k=1}^{n} y_k(S)\,\mu_k,$$

δ_k= cosh (2kε cos θ) cos (2kε sin θ),

μ= sinh (2kε cos θ) sin (2kε sin θ).

In the range $0 \le t \le 1/2^n$, we have

$\delta_k \le 1 + 4k^2\varepsilon^2$ and $\mu_k \le 8k^2\varepsilon^2$.

Let $R = f(s,0) = \sum_{k=1}^{n} y_k(s).$

Then $|S - R| = \left| \sum_{k=1}^{n} y_k(s)(\delta_k - 1) \right| = 4n^3\varepsilon^2 \max|y_k(s)|$

and $|Q| = \left| \sum_{k=1}^{n} y_k(s)\delta_k \right| = 8n^3\varepsilon^2 \max|y_k(s)|$

Let $h_n = \max|y_k(s)|$

It follows from lemma-1 of Samal and Mishra [5] that

$$P(h_n > n) < \frac{2^{1+\alpha} C_3}{(2+\alpha)\,n^2} < \frac{8C_3}{n^2}; \;(\; C_3 \text{ is a constant})$$

i.e. $P\,(h_n \le n) > 1 - \dfrac{8C_3}{n^2}.$

Now $|S - R| + |Q| \le 12n^3\varepsilon^2 \max_{1\le k\le n} |y_k(s)| \le \dfrac{12n^3 h_n}{4^n}.$

Let $E_2 = \left\{ S : \max_{1\le k\le n} |y_k(s)| > n \right\}$

We can prove that $P(E_2) < \dfrac{8C_4}{n^2},$

when $t \notin E_2,\ h_n \le n$

and $|S - R| + |Q| \le \dfrac{12n^3 h_n}{4^n} < \dfrac{12n^4}{4^n}.$

It follows from Gnedenko and Kolmogorov [3] that

$$P(|y_k(s)| < 1) \le P(|ny_k(s)| < 1) = \left(|y_k(s)| < \frac{1}{n} \right) < \frac{1}{\pi n}\Gamma(\tfrac{1}{2}),$$

i.e. $P\left(\left| \sum_{k=1}^{n} y_k(s) \ge 1 \right| \right) > 1 - \dfrac{1}{\pi n}\Gamma(\tfrac{1}{2}).$

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Let $E_3=\left\{s:\left|\sum_{k=1}^{n} y_k(s)\right| \le\right\}$. Then $P(E_3) < \frac{\Gamma(1/2)}{n}$.

Let $E=E_1 \cup E_2 \cup E_3$.

Then, $$P(E) < \frac{8C_4}{n^2} + \frac{\Gamma(\frac{1}{2})}{n} < \frac{C_5}{n},$$ (C_5 is a constant).

If $s \notin E$, then $$n(s,\in) \frac{1}{2\pi \log 2} \int_0^{2\pi} \log \left| \frac{f(s, 2 \in \in^{i\theta}}{} \right| d\theta$$

$$\le \frac{1}{2\pi \log 2} \int_0^{2\pi} \log\left(1 + \frac{|S-R| + |Q|}{|R|}\right) d\theta$$

$$\le \frac{1}{2\pi \log 2} \int_0^{2\pi} \log\left(1 + \frac{12n^4}{|R|}\right) d\theta$$

$$\le \frac{12n^4}{4^n} \to 0, \text{as } n \to \infty,$$

As $n(s,\varepsilon)$ can have only non-negative integral values, $n(s,\varepsilon)=0$ for all $s \notin E$.

Thus $f(s,t)$ does not possess a zero in $0 \le t \le 1/2^n$ if s lies outside the set E. For

$s \varepsilon E$, $f(s,t)$ can have at most 2n zeros in $0 \le t \le 1/2^n$ and we obtain

$$EN_n(0, \frac{1}{2^n}) \le 2n\, P(E) \le C_6, \text{ where } C_6 \text{ is a constant} \qquad (1.22).$$

7. AVERAGE NUMBER OF ZEROS IN THE RANGE (n^n, ∞)

We show that for n sufficiently large, $EN_n(n^n, \infty)$ is negligible.

Let $F_1= \{s : |y_n(s)| \le 1/n\}$.

By Gnedenko and Kolmogorov [4], we have

$$P(F_1) \le \frac{1}{n\pi} \Gamma(\tfrac{1}{2}) < \frac{C_7}{n};$$

i.e. when $S \notin F_1$ we have $|Y_n(S)| > 1/n$ and $P(F_1) < C_8/n$.

Let $F_2= \{s: |Y_k(s)| > n, 1 \le k \le n-1\}$.

Now $P(s : |y_k(s)| \le n, 1 \le k \le n-1)$

$$\prod_{k=1}^{n} \{P(|y_k(s)| \le n)\} = \left\{1 - \sqrt{\frac{2}{\pi}} \int_n^{\infty} e^{-\frac{1}{2}u^2} du\right\}^n$$

$$\geq 1-n\sqrt{\frac{2}{\pi}\int_{n}^{\infty}e^{-\frac{1}{2}u^{2}}du} > 1-e^{-\frac{1}{3}n^{2}} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Since $$\left(\int_{n}^{\infty}e^{-\frac{1}{2}u^{2}}du = \int_{0}^{\frac{1}{n}}e^{-\frac{1}{2}\left(\frac{1}{v^{2}}\right)}\frac{dv}{v^{2}} < \frac{1}{n}e^{-\frac{1}{2}n^{2}}\right)$$

Hence $P(F_2)<C_9/n$.

Let $F=F_1 \cup F_2$.

Then for S€F, we have

$$|f(s,t)| = \left|y_n(s)\cosh nt - \sum_{k=1}^{n-1} y_k(s)\cosh kt\right| \geq \left(e^{nt}/2n\right) - \log n(e^{t}+e^{2t}+\dots\dots\dots+e^{(n-1)t}+\delta$$

$$>0, \ \forall\, t \geq n^{n} \text{ and } \delta \text{ small.}$$

Hence $f(s,t) \neq 0$ if s€F and $t \geq n^n$.

For s€F, f (s,t) can have at most 2n zeros in $n^n \leq t \leq \infty$.

Thus, we have $$EN_n(n^n,\infty) \leq \int_{S\varepsilon F} N(n^n,\infty)ds \leq 2n\,P(F) < C_{10} \quad (1.23)$$

8. PROOF OF THEOREM:- Since f (t) is an even function of t, we obtain from (1.19), (1.20), (1.21), (1.22) and (1.23) that

$$EN_n(-\infty,\infty) = 2EN_n(0,\infty)$$
$$= 2EN_n\left(0,\frac{1}{2^n}\right) + 2EN_n\left(\frac{1}{2^n},\frac{\log\log n}{n}\right)$$
$$+ 2EN_n\left(\frac{\log\log n}{n},1\right) + 2EN_n(1,n^n) + 2EN_n(n^n,\infty)$$
$$< A\,n\log n, \qquad \text{where A is a constant}$$

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