International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2000-000X
Vol. 1 Issue 6, August – 2017, Pages: 119-127

New Type of Linear Partial Differential Equations

Noora Ali Habeeb
University of Kufa, Faculty of Education for Girls, Department of Mathematics
nnoora35@yahoo.com

Ali Hassan Mohammed
University of Kufa, Faculty of Education for Girls, Department of Mathematics
prof.ali1957@gmail.com

Abstract*: Our aim in this paper is to introduce a new transform is known Al-Zughair transform for a function and how to use it to solve ODEs and PDEs.*

Keywords: Al-Zughair transform ,Partial Differential Equations, Alis equtions

1. Introduction

Al-Zughair transform plays an important role to solve ODE and PDE with variable coefficients and this transformation appeared for the first time at 2017 [2].

Preliminaries

Definition (1) [1]:

Let f is defined function at period $(a,)$ then the integral transformation for f whose it's symbol $F(s)$ is defined as:

$$F(s) = \int_a^b k(s,x)\, f(x)\, dx$$

Where k is a fixed function of two variables, called the kernel of the transformation and a,b are real numbers or $\mp\infty$, such that the above integral is convergent.

Definition (2) [2]: Al-Zughair transformation $[(f(x))]$ for the function $f(x)$ where $x\in[1,e]$ is defined by the following integral:

$$Z(f(x)) = \int_1^e \frac{(\ln x)^s}{x} f(x)dx = F(s)$$

Such that this integral is convergent, s is positive constant. From the above definition we can write:

$$Z(u(x,t)) = \int_1^e \frac{(\ln t)^s}{t} u(x,t)dt = v(x,s)$$

Such that (x,t) is a function of x and t .

Property (1): (Linear property)

This transformation is characterized by the linear property, that is

$$Z[Au_1(x,t) \pm Bu_2(x,t)] = AZ[u_1(x,t)] \pm BZ[u_2(x,t)]$$

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2000-000X
Vol. 1 Issue 6, August – 2017, Pages: 119-127

Where A and B are constants while, the functions $u1(x,t)$,$u2(x,t)$ are defined when $t\in[1,e]$.

Proof:

$$Z[Au_1(x,t) \pm Bu_2(x,t)] = \int_1^e \frac{(\ln t)^s}{t}\big(Au_1(x,t) \pm Bu_2(x,t)\big)dt$$

$$= \int_1^e \frac{(\ln t)^s}{t} Au_1(x,t)dt \pm \int_1^e \frac{(\ln t)^s}{t} Bu_2(x,t)dt$$

$$= A\int_1^e \frac{(\ln t)^s}{t} u_1(x,t)dt \pm B\int_1^e \frac{(\ln t)^s}{t} u_2(x,t)dt$$

$$= AZ[u_1(x,t)] \pm BZ[u_2(x,t)]$$

Al-Zughair transformation for some fundamental functions is given in table (1) [2]:

ID	Function, $f(x)$	$F(s) = \int_1^e \frac{(\ln x)^s}{x} f(x)dx = Z\big(f(x)\big)$	Regional of convergence
1	k ; k = constant	$\frac{k}{(s+1)}$	$s > -1$
2	$(\ln x)^n, n \in R$	$\frac{1}{\big(s+(n+1)\big)}$	$s > -(n+1)$
3	$\ln(\ln x)$	$\frac{-1}{(s+1)^2}$	$s > -1$

4	$(\ln(\ln x))^n, n \in z^+$	$\frac{(-1)^n n!}{(s+1)^{n+1}}$	$s > -1$
5	$sin(a\,ln(ln\,x))$	$\frac{-a}{(s+1)^2 + a^2}$	$s > -1$ a is constant
6	$cos(a\,ln(ln\,x))$	$\frac{s+1}{(s+1)^2 + a^2}$	$s > -1$ a is constant
7	$sinh(a\,ln(ln\,x))$	$\frac{-a}{(s+1)^2 - a^2}$	$\lvert s+1\rvert > a$ a is constant
8	$cosh(a\,ln(ln\,x))$	$\frac{s+1}{(s+1)^2 - a^2}$	$\lvert s+1\rvert > a$ a is constant

From Al-Zughair transform definition and the above table, we get:

Theorem (1):

If $Z\big(u(x,t)\big) = v(x,s)$ and a is constant, then $Z\big((ln\,t)^a u(x,t)\big) = v(x, s+a)$.

Proof:

$$Z\big((ln\,t)^a u(x,t)\big) = \int_1^e \frac{(ln\,t)^s}{t}(ln\,t)^a u(x,t)dt$$

$$= \int_1^e \frac{(ln\,t)^{s+a}}{t} u(x,t)dt = v(x, s+a) \blacksquare$$

Example (1): By using the table (1) of Al-Zughair transformation we will consider that:

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2000-000X
Vol. 1 Issue 6, August – 2017, Pages: 119-127

$$f(x,t) = 4x^2 \ln t + x^3$$

$$Z\big(f(x,t)\big) = \int_1^e \frac{(\ln t)^s}{t} f(x,t)dt = \int_1^e \frac{(\ln t)^s}{t}(4x^2 \ln t + x^3)dt$$

$$= 4x^2 \int_1^e \frac{(\ln t)^{s+1}}{t} dt + x^3 \int_1^e \frac{(\ln t)^s}{t} dt$$

$$= 4x^2 \frac{(\ln t)^{s+2}}{s+2}\bigg|_1^e + x^3 \frac{(\ln t)^{s+1}}{s+1}\bigg|_1^e = \frac{4x^2}{s+2} + \frac{x^3}{s+1}$$

Example (2): To find Al-zughair transform of

$$f(x,t) = \ln x\,(\ln t)^3 + x \sin \ln(\ln t)$$

$$Z\big(f(x,t)\big) = Z(\ln x\,(\ln t)^3 + x \sin \ln(\ln t))$$

$$= Z(\ln x\,(\ln t)^3) + Z(x \sin \ln(\ln t))$$

$$= \frac{\ln x}{s+4} - \frac{x}{(s+1)^2 + 1}$$

Definition (3) [2]:

Let $u(x,t)$ be a function where $t \in [1,e]$ and $Z(u(x,t))=v(x,s), u(x,t)$ is said to be an inverse for the Al-Zughair transformation and written as

$$\mathcal{T}^{-1}\big(v(x,s)\big) = u(x,t)\,,$$

Where

$$Z^{-1}$$

returns the transformation to the original function. For example

1) $Z^{-1}\left[\frac{-sinx}{(s+1)^2}\right] = \ln(\ln t)\, sinx \quad , \; s > -1.$

2) $Z^{-1}\left[\frac{x}{s+5}\right] = x(\ln t)^4 \qquad , \; s > -5.$

3) $Z^{-1}\left[\frac{sinx\,(s+1)}{(s+1)^2-4}\right] = \sin x \;\; \cosh(2\ln(\ln t)) \;\; , \; |s+1| > 2 \,.$

Property (2): If $Z^{-1}\big(v_1(x,s)\big) = u_1(x,t),\; Z^{-1}\big(v_2(x,s)\big) = u_2(x,s),$

$\ldots, Z^{-1}\big(v_n(x,t)\big) = u_n(x,s)$ and $a_1\,, a_1\,, \ldots\; a_n$ are constants then,

$$Z^{-1}[a_1 v_1(x,s) + a_2 v_2(x,s) + \cdots + a_n v_n(x,s)]$$
$$= a_1 u_1(x,t) + a_2 u_2(x,t) + \cdots + a_n\, u_n(x,t)$$

New Type of Linear Partial Differential Equations

Definition (4):
The equation

$$a_o(\ln t)^n u_t^{(n)}(x,t) + a_1(\ln t)^{n-1} u_t^{(n-1)}(x,t) + \cdots + a_{n-1}(\ln t) u_t(x,t)$$
$$+a_n u(x,t) = f(x,t)$$

Where $a_o,\; a_1,\; \ldots,\; a_n$ are constants and $f(x,t)$ is a function of x and t,

we will call it ***Ali's Equation*** in partial Differential equation .

Theorem (2):
If the function $(x,\ln t)$ is defined for $t \in [1,e]$ and its derivatives

$u_t(x,\ln t), u_{tt}(x,\ln t)\,, \ldots, u_t^{(n)}(x,\ln t)$ are exist then:

$$Z\left[(\ln t)^n u_t^{(n)}(x,\ln t)\right] = u_t^{(n-1)}(x,1) + (-1)^n(s+n)u_t^{(n-2)}(x,1) + (-1)^{n-1}(s+n)\big(s+(n-1)\big)u_t^{(n-3)}(x,1) + \cdots + (s+n)\big(s+(n-1)\big) \ldots (s+2)u_t(x,1) + (-1)^n(s+n)\big(s+(n-1)\big) \ldots (s+2)(s+1)v(x,s).$$

Proof:

If n=1

$$Z(\ln t\, u_t(x,\ln t)) = \int_1^e \frac{(\ln t)^s}{t}(\ln t)u_t(x,\ln t)dt$$

$$= \int_1^e \frac{(ln\, t)^{s+1}}{t} u_t(x,\ln t)dt$$

Let $y = (\ln t)^{s+1} \Rightarrow dy = (s+1)\frac{(ln\, t)^s}{t}dt$

$$dh = \frac{u_t(x,\ln t)}{t}dt \;\Rightarrow h = u(x,\ln t)$$

$$\int_1^e \frac{(ln\, t)^{s+1}}{t} u_t(x,\ln t)dt$$

$$= (ln\, t)^{s+1}u(x, ln\, t)|_1^e - (s+1)\int_1^e \frac{(ln\, t)^s}{t}u(x, lnt)dt$$

$$= u(x,1) - (s+1)Z(u(x,\ln t))$$

If $n = 2$

$$Z((\ln t)^2 u_{tt}(x,\ln t)) = \int_1^e \frac{(ln\, t)^{s+2}}{t} u_{tt}(x,\ln t)dt$$

Let $y = (\ln t)^{s+2} \Rightarrow dy = (s+2)\frac{(ln\, t)^{s+1}}{t}dt$

$$dh = \frac{u_{tt}(x,\ln t)}{t}dt \;\Rightarrow h = u_t(x,\ln t)$$

$$\int_1^e \frac{(ln\, t)^{s+2}}{t} u_{tt}(x,\ln t)dt$$

$$= (\ln t)^{s+2} u_t(x, \ln t)|_1^e - (s+2)\int_1^e \frac{(\ln t)^{s+1}}{t} u_t(x, \ln t) d$$

$$= u_t(x,1) - (s+2)Z\big(\ln t\, u_t(x, \ln t)\big)$$

$$= u_t(x,1) - (s+2)Z\big(\ln t\, u_t(x, \ln t)\big)$$

$$= u_t(x,1) - (s+2)u(x,1) + (s+2)(s+1)Z\big(u(x, \ln t)$$

If $n = 3$

$$Z\big((\ln t)^3 u_{ttt}(x, \ln t)\big) = \int_1^e \frac{(\ln t)^{s+3}}{t} u_{ttt}(x, \ln t) dt$$

$$y = (\ln t)^{s+3} \quad \Rightarrow \quad dy = (s+3)\frac{(\ln t)^{s+2}}{t} dt$$

$$dh = \frac{u_{ttt}(x, \ln t)}{t} dt \quad \Rightarrow \quad h = u_{tt}(x, \ln t)$$

$$\int_1^e \frac{(\ln t)^{s+3}}{t} u_{ttt}(x, \ln t) dt$$

$$= (\ln t)^{s+3} u_{tt}(x, \ln t)|_1^e$$

$$- (s+3)\int_1^e \frac{(\ln t)^{s+2}}{t} u_{tt}(x, \ln t) dt$$

$$= u_{tt}(x,1) - (s+3)Z\big((\ln t)^2 u_{tt}(x, \ln t)\big)$$

$$= u_{tt}(x,1) - (s+3)u_t(x,1) + (s+3)(s+2)u(x,1)$$
$$- (s+3)(s+2)(s+1)Z\big(u(x, \ln t)\big).$$

And so on,

$$Z\left[(\ln t)^n u_t^{(n)}(x, \ln t)\right]$$
$$= u_t^{(n-1)}(x,1) + (-1)^n (s+n) u_t^{(n-2)}(x,1)$$
$$+ (-1)^{n-1}(s+n)\big(s+(n-1)\big)u_t^{(n-3)}(x,1) + \cdots$$
$$+ (s+n)\big(s+\ (n-1)\big)\ \ldots (\ s+2) u_t(x,1)$$
$$+ (-1)^n (s+n)\big(s+\ (n-1)\big)\ \ldots (s+2)(s+1)v(x,s) \blacksquare.$$

Example (1): To solve the differential equation

$$\ln t\, u_t(x,\ln t) - 3u(x,\ln t) = x sin(2\ln(\ln t)) \;\; ; \;\; u(x,1) = -5$$

we take Z-transform to both sides of above equation we get :

$$Z[\ln t\, u_t(x,\ln t)] - 3Z[u(x,\ln t)] = xZ[sin(2\ln(\ln t))]$$

$$u(x,1) - (s+1)Z[u(x,ln\, t)] - 3Z[u(x,ln\, t)] = \frac{-2x}{(s+1)^2+4}$$

$$-5 - (s+4)Z[u(x,ln\, t)] = \frac{-2x}{(s+1)^2+4}$$

$$Z[u(x,ln\, t)] = \frac{2x}{(s+4)((s+1)^2+4)} - \frac{5}{(s+4)}$$

By take Z^{-1}-transform to both side of above equation we get:

$$u(x,t) = Z^{-1}\left[\frac{A(x)(s+1)+B(x)}{((s+1)^2+4)} + \frac{C(x)}{(s+4)}\right] - 5(ln\, t)^3$$

$$A(x) = \frac{-2x}{13} \quad , \quad B(x) = \frac{6x}{13} \quad , \quad C(x) = \frac{2x}{13}$$

$$u(x,\ln t) = Z^{-1}\left[\frac{-2x}{13}\frac{(s+1)}{((s+1)^2+4)}\right] + Z^{-1}\left[\frac{6x}{13}\frac{1}{((s+1)^2+4)}\right] + Z^{-1}\left[\frac{2x}{13}\frac{1}{(s+4)}\right] - 5(ln\, t)^3$$

$$= \frac{-2x}{13}\cos(2\, ln(ln\, t)) - \frac{6x}{26}\sin(2\ln(\ln t)) + \frac{2x}{13}(\ln t)^3 - 5(ln\, t)^3$$

Example (2): To find the solution of the differential equation

$$(\ln t)^2 u_{tt}(x,\ln t) - \ln t\, u_t(x,\ln t) + u(x,\ln t) = \ln(\ln t)\sin x \;\; ;$$

$$u(x,1) = 1 \;, \qquad u_t(x,1) = 3$$

we take Z-transform to both sides of above equation we get :

$$Z[(\ln t)^2 u_{tt}(x,\ln t)] - Z[\ln t\, u_t(x,\ln t)] + Z[u(x,\ln t)] = \sin x\, Z[\ln(\ln t)]$$

$$u_t(x,1) - (s+2)u(x,1) + (s+2)(s+1)Z[u(x,\ln t)] - u(x,1) + (s+1)Z[u(x,\ln t)] + Z[u(x,\ln t)] = \frac{-\sin x}{(s+1)^2}$$

$$-s + (s+2)^2 Z[u(x,\ln t)] = \frac{-\sin x}{(s+1)^2}$$

$$Z[u(x,\ln t)] = \frac{-\sin x}{(s+1)^2(s+2)^2} + \frac{s}{(s+2)^2}$$

$$= \frac{-\sin x}{(s+1)^2(s+2)^2} + \frac{s+2}{(s+2)^2} - \frac{2}{(s+2)^2}$$

By take Z^{-1}-transform to both side of above equation we get:

$$u(x,\ln t) = Z^{-1}\left[\frac{-\sin x}{(s+1)^2(s+2)^2}\right] + \ln t + 2(\ln t)(\ln(\ln t))$$

$$= \frac{A(x)(s+1) + B(x)}{(s+1)^2} + \frac{C(x)(s+2) + D(x)}{(s+2)^2} - \ln t + 2\ln(\ln t)$$

$$A(x) = 2\sin x\,, B(x) = -\sin x\,, C(x) = -2\sin x\,, D(x) = -\sin x$$

$$u(x,\ln t) = 2\sin x + (\ln(\ln t))\sin x - 2(\ln t)\sin x + (\ln t)(\ln(\ln t))\sin x + \ln t + 2(\ln t)(\ln(\ln t))$$

REFERENCES:

[1] Gabriel Nagy, “Ordinary Differential Equations” Mathematics Department, Michigan State University, East Lansing, MI, 48824.October 14, 2014.

[2] Mohammed, A.H., Sadiq B. A., Hassan, A.M. "Solving New Type of Linear Equations by Using New Transformation" EUROPEA ACADEMIC RESEARCH Vol. IV, Issue 8/ November 2016.