International Journal of Academic and Applied Research (IJAAR)
ISSN: 2643-9603
Vol. 3 Issue 6, June – 2019, Pages: 40-43

A New Rectangle With Wilson's Angle in Normed Space

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Abstract: *This paper will discuss new quadrilateral in normed space. There are three theorems that have been proven in the Euclid space. These three theories will be developed and expanded in the named space.*

Keywords : Rectangle, *Normed Space*, *Wilson's angle*, *Euclid Space.*

1. Introduction

Norm space is a vector space in which a normed function is defined.

Definition 1.

Let V be a vector space over a field $\mathbb{R}$, if defined association $\|\cdot\| := V \to \mathbb{R}$, who fulfills ;

a. $\|u\| \geq 0$ for each $u \in V$.
b. Jika $u \in V$ and $\|u\| = 0$ and only if $u = 0$
c. $\|\alpha u\| = |\alpha|\|u\|$ for each $u \in V$ and $\alpha \in \mathbb{R}$.
d. $\|u + v\| \leq \|u\| + \|v\|$ for each $u, v \in V$.

Pair of vector spaces with norm functions is called normed space , symbolized by $(V, \|\,.\,\|)$.[4,5,6,7]

Let $(V, \|\,.\,\|)$ be a vector space over a field $\mathbb{R}$, for each $u, v \in V$ defined as a nonlinear function on V :

$$2\langle u, v\rangle_w := \|u\|^2 + \|v\|^2 - \|u - v\|^2$$

Of the norms possessed :

$$|\|u\| - \|v\||^2 \leq \|u - v\|^2$$

$$\Leftrightarrow \|u\|^2 - 2\|u\|\|v\| + \|v\|^2 \leq \|u - v\|^2$$

$$\Leftrightarrow \langle u, v\rangle_w \leq \|u\|.\|v\| \qquad 1.1$$

$$\|u - v\|^2 \leq (\|u\| + \|v\|)^2$$

$$\Leftrightarrow \|u - v\|^2 - \|u\|^2 - \|v\|^2 \leq 2\|u\|\|v\|$$

$$\Leftrightarrow -\langle u, v\rangle_w \leq \|u\|\|v\| \qquad 1.2$$

From the equation (1.1) and (1.2) this means $|\langle u, v\rangle| \leq \|u\|\|v\|$, for each $u, v \in V$.

Theorem 1. Let $(V, \|\,.\,\|)$ be a normed space over a field $\mathbb{R}$, for each $u, v \in V$ defined as a nonlinear function on V [1,2,3,9] :

$$2\langle u, v\rangle_w := \|u\|^2 + \|v\|^2 - \|u - v\|^2$$

then the following statement is equivalent :

1. $|\langle u, v\rangle_w| \leq \|u\|\|v\|$
2. $\|u + v\| \leq \|u\| + \|v\|$
3. $|\|u\| - \|v\|| \leq \|u - v\|$

Proof :

$(1 \Rightarrow 2)$ $\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2|\langle u, v\rangle_w|$

$$\leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\|$$

$$\leq (\|u\| + \|v\|)^2$$

$$\Leftrightarrow \|u + v\| \leq \|u\| + \|v\|$$

$(2 \Rightarrow 3)$ Note that $\|u\| = \|(u - v) + v\|$

$$\leq \|u - v\| + \|v\|$$

$$(\|u\| - \|v\|) \leq \|u - v\| \qquad 1.3$$

$$\|v\| = \|(v - u) + u\|$$

$$\leq \|u - v\| + \|u\|$$

$$-(\|u\| - \|v\|) \leq \|u - v\| \qquad 1.4$$

From the equation (1.3) dan (1.4) can be concluded that

$$|\|u\| - \|v\|| \leq \|u - v\|$$

$(3 \Rightarrow 1)$ Note that $|\|u\| - \|v\||^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|$

$$\leq \|u - v\|^2$$

$$\Leftrightarrow \|u\|^2 + \|v\|^2 - \|u - v\|^2 \leq 2\|u\|\|v\|$$

$$\Leftrightarrow 2\langle u, v\rangle_w \leq 2\|u\|\|v\|$$

$$\langle u, v\rangle_w \leq \|u\|\|v\| \qquad 1.5$$

Meanwhile $\|u - v\|^2 \leq (\|u\| + \|v\|)^2$

$$= \|u\|^2 + \|v\|^2 + 2\|u\|\|v\|$$

$$\Leftrightarrow -\|u\|^2 - \|v\|^2 + \|u - v\|^2 \leq 2\|u\|\|v\|$$

$$\Leftrightarrow -2\langle u, v\rangle_w \leq 2\|u\|\|v\|$$

$$-\langle u, v\rangle_w \leq \|u\|\|v\| \qquad 1.6$$

From the equation (1.5) and (1.6) can be concluded that

$$|\langle u, v\rangle_w| \leq \|v\|\|v\| \qquad 1.7$$

From the equation (1.7) The defined angle in a normed space called Wilson's angle :

For each $u, v \in V\backslash\{0\}$, defined Wilson's angle :

$$\angle_w(u, v) = \arccos\left(\frac{\|u\| + \|v\| - \|u - v\|}{2\|u\|\|v\|}\right)$$

2. RESULT

This research will develop a quadrilateral understanding in Euclid's space into a normed space. On this occasion Wilson's angle will be used to form a rectangle. This paper will describe at least three traits and several examples.

Definition 2. Let $(V, \|\,.\,\|)$ be a normed space over a field $\mathbb{R}$. *Defined* $\square[a,b,c,d]$ *is a set of vectors* $\{a,b,c,d\}$ *who fulfills* $a+b+c=d$ *with* $a,b,c,d \in V\backslash\{0\}$ *which is equipped with an angle* $\angle_W(a,d)$, $\angle_W(-a,b)$, $\angle_W(-b,c)$, $\angle_W(c,d)$.

Example .

Let $\mathcal{L}^3[0,1] = \{f \mid f:[0,1] \longrightarrow \mathbb{R},\}$ be with f is the function integrated in $[0,1]$. Space $\mathcal{L}^3[0,1]$ forming normed spaces with norms :

$$\|f\| = \left(\int_0^1 |f(t)|^3\, dt.\right)^{1/3}$$

Choose $a(t) = t+3$, $b(t) = 2t-3$, $c(t) = t^2$, and $d(t) = t^2 + 3t$, then the four vectors form $\square[a,b,c,d]$. With each norm is.

$$\|a(t)\| = \left(\int_0^1 |t+3|^3\, dt\right)^{1/3} \approx 3{,}524$$

$$\|b(t)\| = \left(\int_0^1 |2t-3|^3\, dt\right)^{1/3} \approx 2{,}154$$

$$\|c(t)\| = \left(\int_0^1 |t^2|^3\, dt\right)^{1/3} \approx 0{,}522$$

$$\|d(t)\| = \left(\int_0^1 |t^2+3t|^3\, dt\right)^{1/3} \approx 4{,}932$$

Theorem 1. Let $(V, \|\,.\,\|)$ be a normed space over a field $\mathbb{R}$. *In rectangles* $\square[a,b,c,d]$ *rules apply* $\angle_W(a,d) + \angle_W(-b,c) = \pi$ *if and only if :*

$$\sin\angle_W(a,d) = \frac{2\sqrt{(s-\|a\|)(s-\|b\|)(s-\|c\|)(s-\|d\|)}}{\|b\|\|c\| + \|a\|\|d\|}$$

Proof.

$(\Longrightarrow)$ region $\square[\mathrm{a,b,c,d}]$ = region $\Delta[a,d,e]$ + region $\Delta[b,c,e]$

$$= \tfrac{1}{2}\|\mathrm{a}\|\|\mathrm{d}\|\sin\angle_W(a,d) + \tfrac{1}{2}\|\mathrm{b}\|\|\mathrm{c}\|\sin\angle_W(-b,c),$$

because $\angle_W(a,d) = \pi - \angle_W(-b,c)$ then :

$$\sin\angle_W(a,d) = \sin(\pi - \angle_W(-b,c))$$

$$\sin\angle_W(a,d) = \sin\angle_W(-b,c) \text{ as a result :}$$

$$\|\mathrm{a}\|\|\mathrm{d}\|\sin\angle_W(a,d) + \|\mathrm{b}\|\|\mathrm{c}\|\sin\angle_W(-b,c) = (\|\mathrm{a}\|\|\mathrm{d}\| + \|\mathrm{b}\|\|\mathrm{c}\|)\sin\angle_W(a,d)$$

Meanwhile :

$$\|a\|^2 + \|d\|^2 - 2\|a\|\|d\|\cos\angle_W(a,d) = \|e\|^2 = \|b\|^2 + \|c\|^2 - 2\|b\|\|c\|\cos\angle_W(-b,c)$$

because $\cos\angle_W(-b,c) = -\cos\angle_W(a,d)$

then :

$$\|a\|^2 + \|d\|^2 - 2\|a\|\|d\|\cos\angle_W(a,d) = \|e\|^2 = \|b\|^2 + \|c\|^2 + 2\|b\|\|c\|\cos\angle_W(a,d)$$

So that it is obtained :

$$\cos\angle_W(a,d) = \frac{\|a\|^2 + \|d\|^2 - \|b\|^2 - \|c\|^2}{2(\|b\|\|c\| + \|a\|\|d\|)}$$

pay attention : $\sin^2\angle_W(a,d) = 1 - \cos^2\angle_W(a,d)$

$$= 1 - \left(\frac{\|a\|^2 + \|d\|^2 - \|b\|^2 - \|c\|^2}{2(\|b\|\|c\| + \|a\|\|d\|)}\right)^2$$

$$= \frac{\{2(\|b\|\|c\| + \|a\|\|d\|)\}^2}{4(\|b\|\|c\| + \|a\|\|d\|)^2} \cdot \frac{\{\|a\|^2 + \|d\|^2 - \|b\|^2 - \|c\|^2\}^2}{4(\|b\|\|c\| + \|a\|\|d\|)^2}$$

$$= \frac{2(\|b\|\|c\| + \|a\|\|d\|)}{2(\|b\|\|c\| + \|a\|\|d\|)} + \frac{(\|a\|^2 + \|d\|^2 - \|b\|^2 - \|c\|^2)}{2(\|b\|\|c\| + \|a\|\|d\|)}$$

$$= \frac{2(\|b\|\|c\| + \|a\|\|d\|)}{2(\|b\|\|c\| + \|a\|\|d\|)} - \frac{(\|b\|^2 - 2\|b\|\|c\| + \|c\|^2)}{2(\|b\|\|c\| + \|a\|\|d\|)}$$

$$= \frac{(\|b\|^2 + 2\|b\|\|c\| + \|c\|^2) - (\|a\|^2 - 2\|a\|\|d\| + \|d\|^2)}{2(\|b\|\|c\| + \|a\|\|d\|)}$$

$$= \frac{\{(\|a\| + \|d\|)^2 - (\|b\| - \|c\|)^2\}}{2(\|b\|\|c\| + \|a\|\|d\|)} \cdot \frac{\{(\|b\| + \|c\|)^2 - (\|a\| - \|d\|)^2\}}{2(\|b\|\|c\| + \|a\|\|d\|)}$$

$$= \frac{(\|a\| + \|b\| + \|d\| - \|c\|)(\|a\| + \|c\| + \|d\| - \|b\|)}{2(\|b\|\|c\| + \|a\|\|d\|)} \cdot \frac{(\|a\| + \|b\| + \|c\| - \|d\|)(\|b\| + \|c\| + \|d\| - \|a\|)}{2(\|b\|\|c\| + \|a\|\|d\|)}$$

Let $2s = \|a\| + \|b\| + \|c\| + \|d\|$, be then :

$$\sin^2\angle_W(a,d) = \frac{2(s-\|d\|)2(s-\|a\|)2(s-\|c\|)2(s-\|b\|)}{4(\|b\|\|c\| + \|a\|\|d\|)^2}$$

$$\sin\angle_W(a,d) = \frac{2\sqrt{(s-\|a\|)(s-\|b\|)(s-\|c\|)(s-\|d\|)}}{\|b\|\|c\| + \|a\|\|d\|}$$

International Journal of Academic and Applied Research (IJAAR)
ISSN: 2643-9603
Vol. 3 Issue 6, June – 2019, Pages: 40-43

$(\Leftarrow)$

$\sin^2 \angle_W(a,d)$

$$= \frac{2(s-\|a\|)2(s-\|b\|)2(s-\|c\|)2(s-\|d\|)}{4(\|b\|\|c\|+\|a\|\|d\|)^2}$$

$$= \frac{4(s-\|a\|)(s-\|b\|)(s-\|c\|)(s-\|d\|)}{(\|b\|\|c\|+\|a\|\|d\|)^2}$$

$$= \frac{(\|a\|+\|b\|+\|c\|-\|d\|)(\|a\|+\|b\|+\|c\|-\|a\|)}{\|b\|\|c\|+\|a\|\|d\|} \cdot \frac{(\|a\|+\|b\|+\|c\|-\|c\|)(\|a\|+\|b\|+\|c\|-\|b\|)}{\|b\|\|c\|+\|a\|\|d\|}$$

$$= 1 - \left(\frac{\|b\|^2+\|c\|^2-\|a\|^2-\|d\|^2}{2(\|b\|\|c\|+\|a\|\|d\|)}\right)^2$$

$$= 1 - \cos^2 \angle_W(-b,c)$$

$$= \sin^2 \angle_W(-b,c)$$

Thus because $\sin \angle_W(a,d) = \sin \angle_W(-b,c)$ and $\angle_W(a,d), \angle_W(-b,c) \in [0,\pi]$ then $\angle_W(a,d) + \angle_W(-b,c) = \pi$ ■

Theorem 2. Let $(V, \|.\|)$ be a normed space over a field $\mathbb{R}$. *In* $\square[a,b,c,d]$ *then the rules apply:*

$$\angle_W(-a,b) + \angle_W(c,d) = \pi \iff \|a+b\|\|b+c\| = \|a\|\|c\| + \|b\|\|d\|$$

Proof.

$(\Rightarrow)$ Let $\angle_W(-a,b) + \angle_W(c,d) = \pi$, be

Will be shown $\|e\|\|f\| = \|a\|\|c\| + \|b\|\|d\|$

Because :

$$\cos \angle_W(-a,b) = \frac{\|a\|^2+\|b\|^2-\|c\|^2-\|d\|^2}{2(\|a\|\|b\|+\|c\|\|d\|)}$$

Then :

$$\|e\|^2 = \|a\|^2 + \|b\|^2 - 2\|a\|\|b\| \cos \angle_W(-a,b)$$

$$= \|a\|^2 + \|b\|^2 - 2\|a\|\|b\| \left\{\frac{\|a\|^2+\|b\|^2-\|c\|^2-\|d\|^2}{2(\|a\|\|b\|+\|c\|\|d\|)}\right\}$$

$$= \frac{\{\|a\|^2+\|b\|^2\}\|c\|\|d\| + \|a\|\|b\|\{\|c\|^2+\|d\|^2\}}{\|a\|\|b\|+\|c\|\|d\|}$$

$$= \frac{\{\|a\|\|c\|+\|b\|\|d\|\}\{\|a\|\|d\|+\|b\|\|c\|\}}{\|a\|\|b\|+\|c\|\|d\|}$$

Because :

$$\cos \angle_W(a,d) = \frac{\|a\|^2+\|d\|^2-\|b\|^2-\|c\|^2}{2(\|a\|\|d\|+\|b\|\|c\|)}$$

Then :

$$\|f\|^2 = \|a\|^2 + \|d\|^2 - 2\|a\|\|d\| \cos \angle_W(a,d)$$

$$= \|a\|^2 + \|d\|^2 - 2\|a\|\|d\| \left\{\frac{\|a\|^2+\|d\|^2-\|b\|^2-\|c\|^2}{2(\|a\|\|d\|+\|b\|\|c\|)}\right\}$$

$$= \frac{\{\|a\|^2+\|d\|^2\}\|b\|\|c\| + \|a\|\|d\|\{\|b\|^2+\|c\|^2\}}{\|a\|\|d\|+\|b\|\|c\|}$$

$$= \frac{\{\|a\|\|b\|+\|c\|\|d\|\}\{\|a\|\|c\|+\|b\|\|d\|\}}{\|a\|\|d\|+\|b\|\|c\|}$$

So that :

$$\|e\|^2\,\|f\|^2 = \frac{\{\|a\|\|c\|+\|b\|\|d\|\}\{\|a\|\|d\|+\|b\|\|c\|\}}{\|a\|\|b\|+\|c\|\|d\|} \cdot \frac{\{\|a\|\|b\|+\|c\|\|d\|\}\{\|a\|\|c\|+\|b\|\|d\|\}}{\|a\|\|d\|+\|b\|\|c\|}$$

$$= \{\|a\|\|c\|+\|b\|\|d\|\}^2$$

Then :

$$\|a+b\|\|b+c\| = \|a\|\|c\| + \|b\|\|d\|$$

$(\Leftarrow)$ Known $\|a+b\|\|b+c\| = \|a\|\|c\| + \|b\|\|d\|$ will be proven $\angle_W(-a,b) + \angle_W(c,d) = \pi$

$$\cos \angle_W(-a,b) = \frac{\|a\|^2 + \|b\|^2 - \|e\|^2}{2\|a\|\|b\|}$$

Because $\|a\| = \|c\|, \|b\| = \|d\|$, and $\|e\| = \|f\|$, then :

$$\cos \angle_W(-a,b) = \frac{\|e\|^2 - \|e\|^2}{2\|a\|\|b\|} = 0$$

or $\angle_W(-a,b) = \pi/2$

Meanwhile :

$$\cos \angle_W(c,d) = \frac{\|c\|^2 + \|d\|^2 - \|e\|^2}{2\|c\|\|d\|}$$

$$\cos \angle_W(c,d) = \frac{\|f\|^2 - \|f\|^2}{2\|c\|\|d\|} = 0$$

or $\angle_W(c,d) = \pi/2$

So that it is obtained $\angle_W(-a,b) + \angle_W(c,d) = \pi$ □

Theorem 3. Let $(V, \|.\|)$ be a normed space over a field $\mathbb{R}$. *In* $\square[a,b,c,d]$ *with* $\angle_W(-a,b) + \angle_W(c,d) = 2\alpha$ *then the rectangular area is :*

$$L = \sqrt{(s-\|a\|)(s-\|b\|)(s-\|c\|)(s-\|d\|) -} \cdot \sqrt{\|a\|\|b\|\|c\|\|d\| \cos^2 \alpha}$$

Proof.

Suppose the area is wide $\square[a,b,c,d]$ symbolized L then :

L = area $\Delta[a,d,e]$ + area $\Delta[b,e,c]$

$$= \frac{1}{2}\|a\|\|d\| \sin \angle_W(a,d) + \frac{1}{2}\|b\|\|c\| \sin \angle_W(-b,c)$$

$$4L = 2\|a\|\|d\| \sin \angle_W(a,d) + 2\|b\|\|c\| \sin \angle_W(-b,c) \qquad (3.1)$$

Meanwhile the rules of cosine apply :

$$\|a\|^2 + \|d\|^2 - 2\|a\|\|d\| \cos \angle_W(a,d) = \|b\|^2 + \|c\|^2 - 2\|b\|\|c\| \cos \angle_W(-b,c)$$

$$\|a\|^2 + \|d\|^2 - \|b\|^2 - \|c\|^2 = 2\|a\|\|d\| \cos \angle(a,d)$$

$$-2\|b\|\|c\| \cos \angle_W(-b,c) \qquad (3.2)$$

From equations (3.1) and (3.2) are obtained :

$$16L^2 + (\|a\|^2 + \|d\|^2 - \|b\|^2 - \|c\|^2)^2$$
$$= 4\|a\|^2\|d\|^2 \cos^2 \angle(a,d) + 8\|a\|\|b\|\|c\|\|d\| \sin \angle_W(a,d) \sin \angle_W(-b,c)$$
$$+ 4\|b\|^2\|c\|^2 \sin^2 \angle_W(-b,c) + 4\|a\|^2\|d\|^2 \cos^2 \angle_W(a,d) +$$
$$4\|b\|^2\|c\|^2 \cos^2 \angle_W(-b,c) - 8\|a\|\|b\|\|c\|\|d\| \cos \angle_W(a,d) \cos \angle_W(-b,c)$$
$$= (4\|a\|^2\|d\|^2 + 4\|b\|^2\|c\|^2)(\cos^2 \angle_W(a,d) + \sin^2 \angle_W(-b,c))$$
$$- 8\|a\|\|b\|\|c\|\|d\| (\cos \angle_W(a,d) \cos \angle_W(-b,c) - \sin \angle_W(a,d) \sin \angle_W(-b,c))$$
$$= (4\|a\|^2\|d\|^2 + 4\|b\|^2\|c\|^2) - 8\|a\|\|b\|\|c\|\|d\|(\cos(\angle_W(a,d) + \angle_W(-b,c))$$
$$= (4\|a\|^2\|d\|^2 + 4\|b\|^2\|c\|^2) - 8\|a\|\|b\|\|c\|\|d\|(\cos(2\alpha))$$
$$= (4\|a\|^2\|d\|^2 + 4\|b\|^2\|c\|^2) - 8\|a\|\|b\|\|c\|\|d\|(2\cos^2 \alpha - 1)$$
$$= 4(\|a\|\|d\| + \|b\|\|c\|)^2 - 16\|a\|\|b\|\|c\|\|d\| \cos^2 \alpha$$
$$16L^2 = (2(\|a\|\|d\| + \|b\|\|c\|))^2 - (\|a\|^2 + \|d\|^2 - \|b\|^2 - \|c\|^2)^2$$
$$- 16\|a\|\|b\|\|c\|\|d\| \cos^2 \alpha$$
$$16L^2 = 16(s - \|a\|)(s - \|b\|)(s - \|c\|)(s - \|d\|) - 16\|a\|\|b\|\|c\|\|d\| \cos^2 \alpha$$
$$L^2 = (s - \|a\|)(s - \|b\|)(s - \|c\|)(s - \|d\|) - \|a\|\|b\|\|c\|\|d\| \cos^2 \alpha \ ,$$

With $2s = \|a\| + \|b\| + \|c\| + \|d\|$.

□

3. ACKNOWLEDGMENT

The authors would like to express their gratitude to Hasanuddin University and RISTEKDIKTI Indonesia, and supported by LP3M UNHAS with Research Grant No. 005 / ADD / SP2H / LT / DPRM / VIII / 2018.

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