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On The Topological Complement of an Arrangement

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Abstract: *Let V be an $\ell-$ dimension vector space over the real or complex numbers. An arrangement A is a finite collection of co-dimension one subspaces. Let $M(A)=V-\bigcup_{H\in A}H$ be the complement .In this work we use the face poset $\varsigma(A)$ of A to construct the homotopy type of $M(A)$.More specifically, we would like to know the homotopy properties of $M(A)$ which are relate to various other well-known properties of arrangement ,precisely when $M(A)$ is a $K(\pi,1)$ -space.*

Keywords: poset; hyperplane; dual space.

1. Introduction

Electricity the driving force of modern civilization, is indispensable in our day to day life. There are two basic types of electricity generation. One of which is through conventional energy resources which will get extinct in near future, hence demanding an alternative arrangement. Therefore, it is of great urgency to go for non-conventional energy resources. The non-conventional

Much of the early history of the topology of arrangement revolves around the " $K(\pi,1)$ problem". The problem of determining which arrangements have a spherical complements (such an arrangement is called a $K(\pi,1)$ arrangement). Our interest in this work is in the topology, and especially the homotopy theory of the complement, which turns out to have a rich structure and we assembled many of known results .

2. Preliminaries of arrangements and their lattices

Definition 2.1[2]

Let K be a field of real or complex numbers and let V be a vector space of dimension ℓ . A hyperplane H in V is an affine subspace of dimension $\ell-1$. A hyperplane arrangement A is a finite set of hyperplanes in V .

The subscript K will be used only when we want to call attention to the field. We call A an ℓ -arrangement when we want to emphasize the dimension of V .Let ϕ_ℓ denote the empty ℓ -arrangement . Let $S=S(V^*)$ be the symmetric algebra of the dual space V^* of V .If $\{x_1,...,x_\ell\}$ is a basis for V^*,then $S\approx K[x_1,...,x_\ell]$ the symmetric polynomial algebra in the variables $x_1,...,x_\ell$.

Definition 2.2

The product $Q(A)=\prod_{H\in A}\alpha_H$ is called a defining polynomial of an arrangement A. We agree that $Q(\phi_\ell)=1$ is the defining polynomial of the empty arrangement .

Definition 2.3

An arrangement A is called Centerless,if $\bigcap_{H\in A} H=\phi$ .If $T=\bigcap_{H\in A} H\neq\varphi$ we call A a Centered with center T .If A is centered ,then the coordinates may be chosen so that,each hyperplane contains the origion.In this case we call A a central .

Remark 2.4

If A is central ,then each α_H is a linear form and $Q(A)$ is a homogenous polynomial whose degree is the cardinality of A.

Examples 2.5

(i) If we defined A by $Q(A)=xy(x+y)$,then A is centeral and consists of three lines through the origion.
(ii) If we defined A by $Q(A)=xy(x+y-1).$,then A is centerless consists of three affine lines.

Definition 2.6 [3]

The arrangement which is defined by $Q(A)=x_1x_2...x_\ell$ called Boolean arrangement.

Definition 2.7

For $1\leq i<j\leq\ell$ and $H_{ij}=\ker(x_i-x_j)$ .The arrangement which defined by $Q(A)=\prod_{1\leq i<j\leq\ell}(x_i-x_j)$ is called Braid arrangement.

Definition 2.8

Let A be an $\ell-$arrangement defined by $Q(A)\in S$,a cone over A is a centeral $(\ell+1)$ arrangement cA ,defined by $Q(cA)=x_0\overline{Q}$ where $\overline{Q}\in K[x_0,x_1,...,x_\ell]$.

Definition 2.9

Let A be a non empty central $(\ell+1)-$arrangement defined by $Q(A)\in K[x_0,x_1,...,x_\ell]$. A deconing over A is an $\ell-$arrangement dA obtained by substituting 1 for x_0 in $Q(A)$.

<u>3.The Poset (Partially ordered set)</u>

Definition 3.1 [4]

Let A be an arrangement and let $L=L(A)$ be the set of all non-empty intersections of elements of A .Define a **partial order** on L by $X\leq Y\Leftrightarrow Y\subseteq X$.

Definition 3.2

For $X\in L(A)$,define

(i) A **subarrangement** A_X of A by $A_X=\{H\in A: X\subseteq H\}$.

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(ii) An **arrangement** A^X in X by $\{X \cap H : H \in A - A_X\}$.

where X is considered as a subspace.

Definition 3.3

(i) Define a **rank function** on $L(A)$ by $r(X) = co\dim X$.

(ii) An atom of $L(A)$ is a **hyperplane.**

(iii) Let $X, Y \in L$,define their meet by the smalles subspace containing $X \cup Y$,that is $X \wedge Y = \cap\{Z \in L : X \cup Y \subseteq Z\}$.If $X \cap Y \neq \phi$, define their join by $X \vee Y = X \cap Y$.

Definition 3.4[3]

An arrangement A is called central if each hyperplane contain the origion

Remark 3.5

In a central arrangement, A is essential if and only if $T(A) = \{0\}$.

Definition 3.6[2]

A pair $(X, Y) \in L \times L$ is called modular,if for all Z with $Z \leq Y$,we have $Z \vee (X \wedge Y) = (Z \vee X) \wedge Y$.

Definition 3.7

An element $X \in L$ is called a modular if (X, Y) is a modular pair for all $\boldsymbol{Y} \in \boldsymbol{L}$.

Definition 3.8

An essential $\ell-$arrangement A is called a supersolvable if $L(A)$ has a maximal chain of modular elements $V = X_0 < X_1 < ... < X_\ell = T$

Examples 3.9

(i) In the Boolean arrangement, every element is modular, so we may take any maximal chain.
(ii) The braid arrangement is supersolvable with the maximal chain of modular elements given by $V < \{x_1 = x_2\} < \{x_1 = x_2 = x_3\} < ... < \{x_1 = ... = x_\ell\} = T$

4.Homotopy type of an arrangement

Definition 4.1

The set $V - \bigcup_{H \in A} H$ is a disjoint union of open sets called chambers.

Definition 4.2

Let Let $C(A)$ be the set of chambers of A ,the face poset of A is the collection $\varsigma(A) = \bigcup_{X \in L(A)} C(A^X)$, partially ordered by reseve inclusion.

Definition 4.3

Let $\varsigma(\mathrm{A})$ be the face poset of A. An element $P \in \varsigma(\mathrm{A})$ is called a face of A.

Definition 4.4[1]

Let P be a partially ordered set. Let $K = K(P)$ be the simplicial complex associated to P as follows

(i) The vertices of K are the elements of P.

(ii) A set of vertices $\{X_0, \ldots, X_q\}$ span a $q-$simplex if and only if it is a linearly ordered subset of P; after relabeling $X_0 < \ldots < X_q$

Then $K(P)$ be the corresponding geometric simplicial complex called the order complex.

Definition 4.5[2]

Given two points $u_0, u_1 \in V$, their join is the line segment between u_0 and u_1: $u_0 * u_1 = \{(1-\chi)u_0 + \chi u_1, \chi \in [0,1]\}$.

This may be iterated for the affine independent points $u_0, \ldots, u_k$, $k \leq \ell$,

to obtain their convex hull a k-simplex denoted by $u_0 * \ldots * u_k$.

Definition 4.6

Let A be an essential arrangement.

The mapping $\varphi : K(\varsigma) \to V$ which defined by

$\varphi((Q_0, \ldots, Q_k)) = v(Q_0) * \ldots * v(Q_k)$ is an embedding.

Theorem 4.7

Let $P_k \in \varsigma$ be a face of codimension k. Let S be the set of all saturated chains $Q_0 < \ldots < Q_{k-1} < P_k$.

Then $D_p^k = \bigcup_S v(Q_0) * \ldots * v(Q_{k-1}) * v(P_k)$ is a triangulated K-cell in V whose boundary is $S_p^{k-1} = \bigcup_S v(Q_0) * \ldots * v(Q_{k-1})$.

Proposition 4.8

For $P \in \varsigma$, let $\varsigma_p = \{Q \in \varsigma : Q \leq P\}$ be the segment below P. Let $E(P) = K(\varsigma_p)$.

Then $\mathrm{M}(A)$ has the same homotopy type of $\bigcup_{P \in G} E(P)$ where G is the set of faces whose union is $\mathrm{M}(A)$.

Example 4.9

Let A defined by $Q(A) = xy$. Then

$\mathrm{M}(A)$ has the same homotopy type of $S^1 \times S^1$.

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Then $\pi_1 \mathrm{M}(A) \cong \pi_1(S^1 \times S^1)$

$$\cong \pi_1(S^1) \times \pi_1(S^1)$$
$$\cong Z \times Z$$

Definition 4.10

An arrangement A is called a $K(\Pi,1)$ if $\mathrm{M}(A)$ is a $K(\Pi,1)$.

Proposition 4.11

Every central 2-arrangement is $K(\Pi,1)$ and $\mathrm{M}(A)$ has the homotopy type of $(\vee_{n-1} S^1) \times S^1$ where $n = |A|$.

Proposition 4.12

For any arrangement A,if $L(A)$ is supersolvable ,then $\mathrm{M}(A)$ is a $K(\Pi,1)$ space.

Example 4.13

The boolean arrangement is $K(\Pi,1)$.

Proposition 4.14

Let A be an affine arrangement and cA be the cone over A.Then $\mathrm{M}(cA) \cong \mathrm{M}(A) \times S^1$.

Similarly if A is a central arrangement,then $\mathrm{M}(A) \cong \mathrm{M}(dA) \times S^1$.

Example 4.15

Let A be the Boolean arrangement .

$$\mathrm{M}(A) \cong \underbrace{S^1 \vee S^1 \vee \ldots \vee S^1}_{\ell\text{ - times}} \qquad \text{see } 3.4$$

$$\cong \vee_{\ell - times} \mathrm{S}^1$$

Since $\mathrm{M}(A)$ is a $K(\Pi,1)$,thus $\Pi_1(\vee_\ell S^1) \cong$ free abelian group of rank ℓ.

$$\cong \underbrace{Z \oplus \ldots \oplus Z}_{\ell\text{ - times}}$$

$$\cong \mathrm{Z} \times \ldots \times \mathrm{Z}.$$

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