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Hyperbolic Functions Acceleration Methods For Improving The Values of Integrations Numerically of Second Kinds

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Abstract*: The aims of this study are to introduce acceleration methods that are called hyperbolic acceleration methods, which come within the series of several acceleration methods that generally known as Al-Tememe's acceleration methods of the second kind which are discovered by (Ali Hassan Mohammed).These methods are useful in improving the results of determining numerical integrals of continuous integrands where the main error is of the forth order with respect to accuracy, subintervals and the fasting of calculating the results specifically to accelerate results come out by Simpson's method. Also, it is possible to make use of it to improve the results of solving differential equations numerically of the main error of the forth order.*

1. Introduction

There are numerical methods for calculating single integrals that are bounded in their integration intervals.

1. Trapezoidal Rule
2. Midpoint Rule
3. Simpson's Rule

It is called Newton–Cotes formulas.
This study will introduce Simpson's method to find approximate values of single integrals of continuous integrands through using Hyperbolic acceleration methods, which come within Al-Tememe's acceleration series of the second kind. We will compare these methods with respect to accuracy and the fasting of approaching these values to the real value (analytical) of those integrals.
Let's assume the integration J:

$$J=\int_{x_0}^{x_{2n}} f(x)dx \qquad (1)$$

Such that f(x) is a continuous integrand Lies above X - axis in the interval $[x_0, x_{2n}]$,and the approximate value of J is required. Generally, Newton–Cotes formula for integration can be written in the following form:

$$\int_{x_0}^{x_{2n}} f(x)dx = G(h)+E_G(h)+R_G$$

Such that G (h) represents (Lagrangian – Approximation) of integration value J, G refers to the type of the rule, E_G (h) is the correction terms that can be added to G(h)and R_G is the remainder.
The Simpson 's rule value G(h) will referred by S(h) and it is given by:
S(h)=$\frac{h}{3}[f(a)+4f(a+h)+2f(a+2h)+4f(a+3h)+\cdots+2f(a+(2n-2)h)+4f(a+(2n-1)h)+f(b)]$;h=$\frac{b-a}{2n}$ and n=1,2,…
And the general formula for E_G(h) is given by:

$$h^5D^4+\ldots \qquad h^4D^3+\frac{1}{180}E_S(h)=\frac{1}{180}$$

Fox [1]
So, when integrals of integration is a continuous function and their derivatives are in each point of integration intervals$[x_0, x_{2n}]$, it is possible to write error formula as:
$E=J-s(h)=A_1h^4+A_2h^6+A_3h^8+\ldots$
Such that$A_1, A_2, A_3, \ldots$ are constants that their values do not depend on h but on the values of the derivatives in the end of the integration interval.

2.Al-Tememe's Hyperbolic acceleration

We will introduce six rules of Al-Tememe's hyperbolic acceleration, which come within Al-Tememe's acceleration series of the second kind.
It is mentioned above that the error in Simpson's rule can be written as the following:

$E=A_1h^4+A_2h^6+\dots$

$=h^3(A_1+A_2h^3+\dots\dots\dots..)$ (3)

$\cong h^3\sinh$ since$(\sinh h=h+\frac{h^3}{6}+\frac{h^5}{120}+\frac{h^7}{5040}+\cdots)$ [2]

And by assuming that S (h) is the approximation value of the Simpson integration rule,so:

$$E = J-S(h) \cong h^3 \sinh h$$

If we assume that we calculated two values of J numerically in the Simpson rule, $S_1(h_1)$ when $h = h_1$, $S_2(h_2)$ when $h = h_2$,so, it is:

$J-S_1(h_1) \cong h_1^3\sinh h_1$ (4)

$J-S_2(h_2) \cong h_2^3\sinh h_2$ (5)

From equations (4) and (5) we obtained:

$$A^S{}_{Sinh}\cong\frac{(h_1{}^3\sinh h_1)\,S_2(h_2)-(h_2{}^3\sinh h_2)\,S_1(h_1)}{h_1{}^3\sinh h_1-h_2{}^3\sinh h_2} \quad (6)$$

The formula (6) is called Al-Tememe's sine hyperbolic acceleration rule of the second kind .

Similarly, the second cosine hyperbolic acceleration rule can be written. Because the error mentioned abovecan be written as :

$$E=h^4(A_1+A_2h^2+A_3h^4+\dots)\cong h^4\cosh h,\qquad \text{since}(\cosh h=1+\frac{1}{2}h^2+\frac{1}{24}h^4+\frac{1}{720}h^6+\dots) \quad [2]$$

Based on the same sine hyperbolic acceleration method mentioned above, we get the following:

$$A^S{}_{cosh}\cong\frac{(h_1{}^4cosh\ h_1)S_2(h_2)-(h^{2^4}\cosh h_2)\,S_1(h_1)}{h_1{}^4\cosh h_1-h_2{}^4\cosh h_2} \quad (7)$$

The formula (7) is called Al-Tememe's cosine hyperbolic acceleration of the second kind.

Similarly, we find the third hyperbolic acceleration rule that we will call Al-Tememe's tangent hyperbolic acceleration of the second kind, which is referred to by ($A^S{}_{Tanh}$) and Al-Tememe's forth hyperbolic acceleration rule of the second kind that we will call it Al-Tememe's secant hyperbolic acceleration of the second kind, which is referred to by($A^S{}_{Sech}$). These laws are:

$$A^S{}_{Tanh}\cong\frac{(h_1{}^3\tanh h_1)\,S_2(h_2)-(h_2{}^3\tanh h_2)\,S_1(h_1)}{h_1{}^3\tanh h_1-h_2{}^3\tanh h_2} \quad (8)$$

$$A^S{}_{Sech}\cong\frac{(h_1{}^4\ \mathrm{sech}\, h_1)\,S_2(h_2)-(h_2{}^4\ \mathrm{sech}\, h_2)\,S_1(h_1)}{h_1{}^4 sech\ h_1-h_2{}^4\,\mathrm{sech}\, h_2} \quad (9)$$

since $(\tanh h=1-\frac{1}{3}h^3+\frac{2}{15}h^5-\frac{17}{315}h^7+\cdots)$ [2]

since$(\mathrm{sech}\ h= 1-\frac{1}{2}h^2+\frac{5}{24}h^4-\frac{61}{720}h^6+\dots..)$ [2]

Now we will derive the fifth hyperbolic acceleration rule: since the error is:

$E(h)=A_1h^4 + A_2h^6 + \dots = h^2(A_1h^2 + A_2h^4 + \dots\dots)$ (10)

$\cong h^2(\cosh h-1)$ (11)

$\cong 2h^2\sinh^2\left(\frac{h}{2}\right)$ (12)

The fifth hyperbolic acceleration law is as follows:

$$A^S{}_{Sinh^2}\cong\frac{\left(h^2{}_2 sinh^2\left(\frac{h_2}{2}\right)\right)S_2(h_2)-\left(h_1{}^2\sinh^2\left(\frac{h_1}{2}\right)\right)S_1(h_1)}{h_2{}^2\sinh^2\left(\frac{h_2}{2}\right)-h_1{}^2\sinh^2\left(\frac{h_1}{2}\right)} \quad (13)$$

The formula (13) is called Al-Tememe's quadratic sine hyperbolic acceleration of the second kind. Also, can be written $E=2\cosh^2\frac{h}{2}$

.

Similarly, the sixth hyperbolic acceleration rule of the second kind is given by

$$\mathbf{A^S{}_{cosh^2}}\cong\frac{\left(h_2{}^2cosh^2\left(\frac{h_2}{2}\right)\right)S_2(h_2)-\left(h^2{}_1\cosh^2\left(\frac{h}{2}\right)\right)S_1(h_1)}{h_2{}^2\cos h^2\left(\frac{h_2}{2}\right)-h_1{}^2\cosh^2\left(\frac{h_1}{2}\right)} \quad \mathbf{(14)}$$

The formula (14) is called Al-Tememe's quadratic cosine hyperbolic cosine acceleration of the second kind.

3.Examples:

Below are some of the integrations whose integrands are continuous in the integration interval and we use the hyperbolic acceleration methods to improve their results.

3.1:$I=\int_1^2\frac{1}{\sqrt{x}}dx$and its analytical value is 0.82842712474619 .which is rounded to 14 decimal order.

3.2:$I=\int_0^1 tan^{-1}(x)\,dx$and its analytical value is 0.43882457311747.which is rounded to 14 decimal order.

3.3: $I = \int_0^{0.5} \sin^{-1}(x)\, dx$ and its analytical value is0. 12782479158358.which is rounded to 14 decimal.

4.the Results

The integrand of integration $I = \int_1^2 \frac{1}{\sqrt{x}}\, dx$ is continuous in the integration interval [1,2], and the formula of correction terms of Simpson's rule as above mentioned (equation 3).
We put EPS = 10^{-12}(which represents of the subsequent value absolute error - the previous value) and obtained the results listed in Table (1).

We obtained a correct value by accelerating $A^S_{COSh(h)}$and other accelerations to eleven decimal order where n = 32,34,36,38,40,42.While Simpson's method without acceleration was correct to eight decimal places when n = 42, while accelerating $A^S_{seh(h)}$,$A^S_{sinh^2(h)}$,we obtained the same accuracy , when n=30,32,34,36,38,40. the integrand of integration $I = \int_0^1 \tan(x)^{-1}\, dx$ is continuous in the integration interval [0,1], and the formula of correction terms of Simpson's rule as above mentioned (equation 3).
We put EPS = 10^{-12}(which represents the of subsequent value absolute error - the previous value) and obtained the results listed in Table (2).

We obtained a correct value by accelerating $A^S_{COSh(h)}$to eleven decimal order when n = 32,34,36,38,40,42 While by Simpson's method without acceleration was correct to eight decimal order when n = 42, while by accelerating $A^S_{sinh(h)}$we obtained the same accuracy, when n=36,38,40 We obtained the same accuracy by accelerating $A^S_{tanh(h)}$ when n=34,36,38,40.Also we obtained the same accuracy by accelerating $A^S_{cosh^2(h)}$when n=36,38,40.
integrand of integration $I = \int_0^{0.5} \sin(x)^{-1}\, dx$ is continuous in the integration interval [0,0.5], and the formula of correction terms of Simpson's rule as above mentioned (equation 3).
We put EPS=10^{-12} (which represents the of subsequent value absolute error - the previous value) and obtained the results listed in Table (3).

We get correct value by accelerating $A^S_{COSh(h)}$ and for all other accelerations to 11 decimal order when n=22,24,26,28, While the value by using Simpson's method without acceleration was correct to 8 decimal order when n=28.

5-Conclusion:

We conclude from the mentioned tables that these acceleration methods have the same efficiency and give high accuracy of results during limited number of sub partial intervals by simple variation.

n	Values of simpson's rule	$A^S_{\cosh(h)}$	$A^S_{\sinh(h)}$	$A^S_{\tanh(h)}$	$A^S_{\mathrm{sech}(h)}$	$A^S_{sinh^2(x)}$	$A^S_{cosh^2(x)}$
2	0.82884885081624						
4	0.82846037425962	0.82843681954937	0.82843531547490	0.82843281207941	0.82843188276574	0.82843490256643	0.82843572129128
6	0.82843407964131	0.82842774334070	0.82842765345726	0.82842751491532	0.82842746737707	0.82842763039053	0.82842767642536
8	0.82842937419453	0.82842721535456	0.82842720267621	0.82842718341849	0.82842717691189	0.82842719946716	0.82842720587904
10	0.82842805592764	0.82842714568144	0.82842714280705	0.82842713846399	0.82842713700486	0.82842714208322	0.82842714353007
12	0.82842757645745	0.82842713116833	0.82842713029577	0.82842712898069	0.82842712854006	0.82842713007659	0.82842713051479
14	0.82842736944488	0.82842712713432	0.82842712681190	0.82842712632664	0.82842712616429	0.82842712673102	0.82842712689274
16	0.82842726852109	0.82842712576661	0.82842712562940	0.82842712542308	0.82842712535412	0.82842712559502	0.82842712566378
18	0.82842721464946	0.82842712523043	0.82842712516550	0.82842712506792	0.82842712503532	0.82842712514924	0.82842712518176
20	0.82842718380028	0.82842712499562	0.82842712496224	0.82842712491210	0.82842712489535	0.82842712495388	0.82842712497060
22	0.82842716511570	0.82842712488340	0.82842712486507	0.82842712483753	0.82842712482834	0.82842712486048	0.82842712486966
24	0.82842715326849	0.82842712482586	0.82842712481523	0.82842712479926	0.82842712479393	0.82842712481257	0.82842712481789
26	0.82842714546469	0.82842712479458	0.82842712478813	0.82842712477844	0.82842712477521	0.82842712478651	0.82842712478974
28	0.82842714015597	0.82842712477672	0.82842712477266	0.82842712476655	0.82842712476451	0.82842712477164	0.82842712477367
30	0.82842713644350	0.82842712476610	0.82842712476345	0.82842712475947	0.82842712475814	0.82842712476278	0.82842712476411
32	0.82842713378452	0.82842712475954	0.82842712475776	0.82842712475510	0.82842712475421	0.82842712475732	0.82842712475821
34	0.82842713183983	0.82842712475538	0.82842712475415	0.82842712475232	0.82842712475171	0.82842712475385	0.82842712475446
36	0.82842713039109	0.82842712475265	0.82842712475179	0.82842712475050	0.82842712475007	0.82842712475157	0.82842712475200
38	0.82842712929398	0.82842712475082	0.82842712475020	0.82842712474928	0.82842712474897	0.82842712475005	0.82842712475036
40	0.82842712845090	0.82842712474957	0.82842712474912	0.82842712474845	0.82842712474822	0.82842712474901	0.82842712474923
42	0.82842712779442	0.82842712474869	0.82842712474836			0.82842712474827	0.82842712474844

Table no.(1) to calculate integration $I = \int_1^2 \frac{1}{\sqrt{x}} dx = 0.82842712474619$ by simpson's rule with the triangnlation methods of Al-tememe

n	Values of simpson's rule	$A^S{}_{\cosh(h)}$	$A^s{}_{\sinh(h)}$	$A^S{}_{\tanh(h)}$	$A^S{}_{\mathrm{sech}(h)}$	$A^S{}_{sinh^2(x)}$	$A^s{}_{cosh^2(x)}$
2	0.43999809990011						
4	0.43888437242331	0.43881684317971	0.43881253113299	0.43880535412246	0.43880268986344	0.43881134736142	0.43881369457220
6	0.43883570720747	0.43882398019045	0.43882381383714	0.43882355742826	0.43882346944602	0.43882377114599	0.43882385634573
8	0.43882803518206	0.43882451528781	0.43882449461632	0.43882446321744	0.43882445260871	0.43882448938409	0.43882449983839
10	0.43882598036308	0.43882456153722	0.43882455705682	0.43882455028717	0.43882454801279	0.43882455592857	0.43882455818381
12	0.43882524900757	0.43882456978983	0.43882456845889	0.43882456645294	0.43882456578083	0.43882456812456	0.43882456879297
14	0.43882493706103	0.43882457192406	0.43882457143821	0.43882457070697	0.43882457046233	0.43882457131633	0.43882457156002
16	0.43882478612065	0.43882457261880	0.43882457241360	0.43882457210502	0.43882457200189	0.43882457236217	0.43882457246501
18	0.43882470595220	0.43882457288429	0.43882457278767	0.43882457264245	0.43882457259395	0.43882457276347	0.43882457281187
20	0.43882466020391	0.43882457299859	0.43882457294909	0.43882457287473	0.43882457284990	0.43882457293670	0.43882457296149
22	0.43882463256520	0.43882457305256	0.43882457302544	0.43882457298471	0.43882457297111	0.43882457301865	0.43882457303223
24	0.43882461507360	0.43882457307999	0.43882457306429	0.43882457304072	0.43882457303285	0.43882457306037	0.43882457306822
26	0.43882460356855	0.43882457309481	0.43882457308529	0.43882457307101	0.43882457306625	0.43882457308291	0.43882457308767
28	0.43882459575086	0.43882457310322	0.43882457309723	0.43882457308823	0.43882457308523	0.43882457309573	0.43882457309873
30	0.43882459028882	0.43882457310821	0.43882457310431	0.43882457309845	0.43882457309650	0.43882457310333	0.43882457310528
32	0.43882458637963	0.43882457311127	0.43882457310866	0.43882457310473	0.43882457310343	0.43882457310800	0.43882457310931
34	0.43882458352232	0.43882457311322	0.43882457311142	0.43882457310873	0.43882457310783	0.43882457311097	0.43882457311187
36	0.43882458139478	0.43882457311449	0.43882457311322	0.43882457311133	0.43882457311070	0.43882457311291	0.43882457311354
38	0.43882457978432	0.43882457311534	0.43882457311443	0.43882457311308	0.43882457311262	0.43882457311421	0.43882457311466
40	0.43882457854719		0.43882457311526	0.43882457311427	0.43882457311394	0.43882457311509	0.43882457311542
42	0.43882457758418			0.43882457311510	0.43882457311486		

Table no.(2) to calculate integration $I = \int_0^1 tan^{-1}(x)\, dx$ =0.43882457311747 by simpson's rule with the triangnlation methods of Al-tememe

n	Values of simpson's rule	$A^S{}_{\cosh(h)}$	$A^S{}_{\sinh(h)}$	$A^S{}_{\tanh(h)}$	$A^S{}_{\mathrm{sech}(h)}$	$A^S{}_{sinh^2(h)}$	$A^S{}_{cosh^2(h)}$
2	0.12785998301388						
4	0.12782740452239	0.12782528552955	0.12782525059573	0.12782519676141	0.12782517831065	0.12782524164668	0.12782525950546
6	0.12782532903263	0.12782482088687	0.12782481906224	0.12782481630641	0.12782481538106	0.12782481860305	0.12782481952093
8	0.12782496424560	0.12782479577530	0.12782479552628	0.12782479515156	0.12782479502623	0.12782479546383	0.12782479558870
10	0.12782486282229	0.12782479254179	0.12782479248607	0.12782479240233	0.12782479237437	0.12782479247211	0.12782479250002
12	0.12782482607694	0.12782479187586	0.12782479185905	0.12782479183382	0.12782479182540	0.12782479185485	0.12782479186326
14	0.12782481024782	0.12782479169190	0.12782479168572	0.12782479167643	0.12782479167333	0.12782479168417	0.12782479168727
16	0.12782480254170	0.12782479162977	0.12782479162714	0.12782479162320	0.12782479162189	0.12782479162649	0.12782479162780
18	0.12782479843220	0.12782479160547	0.12782479160423	0.12782479160237	0.12782479160175	0.12782479160392	0.12782479160454
20	0.12782479608051	0.12782479159485	0.12782479159421	0.12782479159326	0.12782479159294	0.12782479159405	0.12782479159437
22	0.12782479465684	0.12782479158978	0.12782479158943	0.12782479158890	0.12782479158873	0.12782479158934	0.12782479158952
24	0.12782479375448	0.12782479158718	0.12782479158698	0.12782479158667	0.12782479158657	0.12782479158693	0.12782479158703
26	0.12782479316025	0.12782479158577	0.12782479158565	0.12782479158546	0.12782479158540	0.12782479158562	0.12782479158568
28	0.12782479275611	0.12782479158496	0.12782479158489	0.12782479158477	0.12782479158473	0.12782479158487	0.12782479158491

Table no.(1) to calculate integration $I = \int_0^{0.5} \sin^{-1}(x)\, dx$ =0. 12782479158358 by simpson's rule with the triangnlation methods of Al-tememe

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