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Solving Fuzzy Initial Value Problems of the Fifth Order via Fuzzy Laplace Transforms

Amal Khalaf Haydar[1] and Hawrra Fadel Mohammad Ali
[1]Department of mathematics; Faculty of Education for Girls; University of Kufa; Najaf; Iraq.
[1] amalkh.hayder@uokufa.edu.iq

Abstract: *In this work, the formula of fuzzy derivative of the fifth order and fuzzy Laplace transform for the fuzzy derivative of the fifth order are found by using generalized H-differentiability, which can be applied in finding solutions for fuzzy initial value problems of the fifth order.*

Keywords: Fuzzy differential equations, fuzzy numbers, generalized H-differentiability**,** fuzzy Laplace transform.

1. Introduction

The definition of the fuzzy derivative was first introduced in 1972, but it occupied a great place when has used in solving differential equations and its applications. Recently, many authors interested in finding solutions of fuzzy differential equations by using analytical and numerical techniques, especially, Haydar and Mohammad Ali in 2013 (1) and in 2014 (2), Abdul Rahman and Ahmad in 2015 (3), Stefanini and Bede in 2014 (4), Ahmadian and et al. in 2018 (5) and Allahviranloo and Chehlabi in 2014 (6).

This paper is arranged as follows: Basic concepts are given in Section 2. In Sections 3, a formula and fuzzy Laplace transform for the fuzzy derivative of the fifth order are introduced. In Sections 4, formulas can be used in solving FIVPs of the fifth order are constructed**.** In section 5, an example of an FIVP of fifth order are introduced. In Sections 6, conclusions are given.

2. Basic Concepts

In this section, some necessary definitions and concepts are introduced:

Definition 2.1 (7) A fuzzy number u in parametric form is a pair $(\underline{u},\overline{u})$ of functions $\underline{u}(\alpha)$ and $\overline{u}(\alpha),\ 0\le\alpha\le 1$ which satisfy the following requirements:

1. $\underline{u}(\alpha)$ is a bounded non-decreasing left continuous function in $(0,1]$, and right continuous at 0,
2. $\overline{u}(\alpha)$ is a bounded non-increasing left continuous function in $(0,1]$, and right continuous at 0,
3. $\underline{u}(\alpha)\le\overline{u}(\alpha),\ 0\le\alpha\le 1.$

Definition 2.2 (7) Let $x,y\in E$. If there exists $z\in E$ such that $x=y+z$, then z is called the H-difference of x and y, and it is denoted by $x\ominus y$.

In this paper, the sign "$\ominus$ "stands always for H-difference, and let us remark that $x\ominus y\neq x+(-1)y$.

Definition 2.3 (7) Let $f:(a,b)\to E$ and $x_0\in(a,b)$. We say that f is strongly generalized differential at x_0 if there exists an element $f'(x_0)\in E$, such that

i. For all $h>0$ sufficiently small, $\exists f(x_0+h)\ominus f(x_0),\ \exists f(x_0)\ominus f(x_0-h)$ and the limits (in the metric d)

$$\lim_{h\to 0}[(f(x_0+h)\ominus f(x_0))/h]=\lim_{h\to 0}[(f(x_0)\ominus f(x_0-h))/h]=f'(x_0)$$

or

ii. For all $h>0$ sufficiently small, $\exists f(x_0)\ominus f(x_0+h),\ \exists f(x_0-h)\ominus f(x_0)$ and the limits (in the metric d)

$$\lim_{h\to 0}[(f(x_0)\ominus f(x_0+h))/-h]=\lim_{h\to 0}[(f(x_0-h)\ominus f(x_0))/-h]=f'(x_0)$$

or

iii. For all $h>0$ sufficiently small, $\exists f(x_0+h)\ominus f(x_0),\ \exists f(x_0-h)\ominus f(x_0)$ and the limits (in the metric d)

$$\lim_{h\to 0}[(f(x_0+h)\ominus f(x_0))/h]=\lim_{h\to 0}[(f(x_0-h)\ominus f(x_0))/-h]=f'(x_0)$$

or

iv. For all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0+h),\ \exists f(x_0) \ominus f(x_0-h)$ and the limits (in the metric d)

$$\lim_{h\to 0}[(f(x_0)\ominus f(x_0+h))/-h] = \lim_{h\to 0}[(f(x_0)\ominus f(x_0-h))/h] = f'(x_0)$$

Definition 2.4 (8) Let $f(t)$ be continuous fuzzy-valued function. Suppose that $f(t)\, e^{-st}$ is improper fuzzy Rimann-integrable on $[0,\infty)$, then $\int_0^\infty f(t)\, e^{-st}dt$ is called fuzzy Laplace transforms and is denoted as $L(f(t)) = \int_0^\infty f(t) e^{-st}dt,\ s>0$. We have

$$\int_0^\infty f(t)\, e^{-st}dt = (\int_0^\infty \underline{f}(t,r)\, e^{-st}dt,\ \int_0^\infty \overline{f}(t,r)\, e^{-st}dt).$$

Also by using the definition of classical Laplace transform:

$$l[\underline{f}(t,r)] = \int_0^\infty \underline{f}(t,r)\, e^{-st}dt \text{ and } l[\overline{f}(t,r)] = \int_0^\infty \overline{f}(t,r)\, e^{-st}dt.$$

Then, we follow:

$L(f(t)) = (l(\underline{f}(t,r)), l(\overline{f}(t,r)))$.

Definition 2.5 (9) A fuzzy-valued function f has exponential order p if there exist constants $M>0$ and p such that for some $t_0 \ge 0$, $|f(t)| \le Me^{pt}\cdot\tilde{1},\ t\ge t_0$.

3. Fuzzy Laplace Transforms for the Fifth Order Derivative

In this section, we shall introduce the following results for fifth order derivative under generalized H-differentiability.

Theorem 3.1 Let $G(t), G'(t), G''(t), G'''(t)$ and $G^{(4)}(t)$ be differentiable fuzzy-valued functions, and if α -cut representation of fuzzy-valued function $G(t)$ is denoted by $[G(t)]^\alpha = [f_\alpha(t), g_\alpha(t)]$, then:

(a) Let $G(t), G'(t), G''(t), G'''(t)$ and $G^{(4)}(t)$ be in (i)-formula, or,
$G''(t), G'''(t)$ and $G^{(4)}(t)$ be in (i)-formula and $G(t)$ and $G'(t)$ be in (ii)-formula, or,
$G'(t), G'''(t)$ and $G^{(4)}(t)$ be in (i)-formula and $G(t)$ and $G''(t)$ be in (ii)-formula, or,
$G'(t), G''(t)$ and $G^{(4)}(t)$ be in (i)-formula and $G(t)$ and $G'''(t)$ be in (ii)-formula, or,
$G'(t), G''(t)$ and $G'''(t)$ be in (i)-formula and $G(t)$ and $G^{(4)}(t)$ be in (ii)-formula, or,
$G(t), G'''(t)$ and $G^{(4)}(t)$ be in (i)-formula and $G'(t)$ and $G''(t)$ be in (ii)-formula, or,
$G(t), G''(t)$ and $G^{(4)}(t)$ be in (i)-formula and $G'(t)$ and $G'''(t)$ be in (ii)-formula, or,
$G(t), G''(t)$ and $G'''(t)$ be in (i)-formula and $G'(t)$ and $G^{(4)}(t)$ be in (ii)-formula, or,
$G(t), G'(t)$ and $G^{(4)}(t)$ be in (i)-formula and $G''(t)$ and $G'''(t)$ be in (ii)-formula, or,
$G(t), G'(t)$ and $G'''(t)$ be in (i)-formula and $G''(t)$ and $G^{(4)}(t)$ be in (ii)-formula, or,
$G(t), G'(t)$ and $G''(t)$ be in (i)-formula and $G'''(t)$ and $G^{(4)}(t)$ be in (ii)-formula, or,
$G^{(4)}(t)$ be in (i)-formula and $G(t), G'(t), G''(t)$ and $G'''(t)$ be in (ii)-formula, or,
$G'''(t)$ be in (i)-formula and $G(t), G'(t), G''(t)$ and $G^{(4)}(t)$ be in (ii)-formula, or,
$G''(t)$ be in (i)-formula and $G(t), G'(t), G'''(t)$ and $G^{(4)}(t)$ be in (ii)-formula, or,
$G'(t)$ be in (i)-formula and $G(t), G''(t), G'''(t)$ and $G^{(4)}(t)$ be in (ii)-formula, or,
$G(t)$ be in (i)-formula and $G'(t), G''(t), G'''(t)$ and $G^{(4)}(t)$ be in (ii)-formula; then $f_\alpha(t)$ and $g_\alpha(t)$ have first order, second order, third order, fourth order, fifth order derivatives and

$[G^{(5)}(t)]^{\alpha} = [f_{\alpha}^{(5)}(t), g_{\alpha}^{(5)}(t)]$

(b) Let $G'(t), G''(t), G'''(t)$ and $G^{(4)}(t)$ be in (i)-formula and $G(t)$ be in (ii)-formula, or,
$G(t), G''(t), G'''(t)$ and $G^{(4)}(t)$ be in (i)-formula and $G'(t)$ be in (ii)-formula, or,
$G(t), G'(t), G'''(t)$ and $G^{(4)}(t)$ be in (i)-formula and $G''(t)$ be in (ii)-formula, or,
$G(t), G'(t), G''(t)$ and $G^{(4)}(t)$ be in (i)-formula and $G'''(t)$ be in (ii)-formula, or,
$G(t), G'(t), G''(t)$ and $G'''(t)$ be in (i)-formula and $G^{(4)}(t)$ be in (ii)-formula, or,
$G'''(t)$ and $G^{(4)}(t)$ be in (i)-formula and $G(t), G'(t)$ and $G''(t)$ be in (ii)-formula, or,
$G''(t)$ and $G^{(4)}(t)$ be in (i)-formula and $G(t), G'(t)$ and $G'''(t)$ be in (ii)-formula, or,
$G''(t)$ and $G'''(t)$ be in (i)-formula and $G(t), G'(t)$ and $G^{(4)}(t)$ be in (ii)-formula, or,
$G'(t)$ and $G^{(4)}(t)$ be in (i)-formula and $G(t), G''(t)$ and $G'''(t)$ be in (ii)-formula, or,
$G'(t)$ and $G'''(t)$ be in (i)-formula and $G(t), G''(t)$ and $G^{(4)}(t)$ be in (ii)-formula, or,
$G'(t)$ and $G''(t)$ be in (i)-formula and $G(t), G'''(t)$ and $G^{(4)}(t)$ be in (ii)-formula, or,
$G(t)$ and $G^{(4)}(t)$ be in (i)-formula and $G'(t), G''(t)$ and $G'''(t)$ be in (ii)-formula, or,
$G(t)$ and $G'''(t)$ be in (i)-formula and $G'(t), G''(t)$ and $G^{(4)}(t)$ be in (ii)-formula, or,
$G(t)$ and $G''(t)$ be in (i)-formula and $G'(t), G'''(t)$ and $G^{(4)}(t)$ be in (ii)-formula, or,
$G(t)$ and $G'(t)$ be in (i)-formula and $G''(t), G'''(t)$ and $G^{(4)}(t)$ be in (ii)-formula, or,
$G(t), G'(t), G''(t), G'''(t)$ and $G^{(4)}(t)$ be in (ii)-formula, then $f_{\alpha}(t)$ and $g_{\alpha}(t)$ have first order, second order, third order, fourth order and fifth order derivatives and
$[G^{(5)}(t)]^{\alpha} = [g_{\alpha}^{(5)}(t), f_{\alpha}^{(5)}(t)]$

Proof. Let $G(t), G'(t), G''(t)$ and $G'''(t)$ in (i)-formula, and $G^{(4)}(t)$ be in (ii)-formula. Since $G(t), G'(t), G''(t), G'''(t)$ be in (i)-formula then we get from (1) that:

$$[G^{(4)}(t)]^{\alpha} = [f_{\alpha}^{(4)}(t), g_{\alpha}^{(4)}(t)].$$

Since $G^{(4)}(t)$ be in (ii)-formula then by ii of definition 2.3 we get:

$$[G^{(4)}(t) \ \Theta G^{(4)}(t+h)]^{\alpha} = [f_{\alpha}^{(4)}(t) - f_{\alpha}^{(4)}(t+h), g_{\alpha}^{(4)}(t) - g_{\alpha}^{(4)}(t+h)],$$
$$[G^{(4)}(t-h) \ \Theta G^{(4)}(t)]^{\alpha} = [f_{\alpha}^{(4)}(t-h) - f_{\alpha}^{(4)}(t), g_{\alpha}^{(4)}(t-h) - g_{\alpha}^{(4)}(t)].$$

and, multiplying by $\frac{1}{-h}, h > 0$ we get:

$$\frac{1}{-h}[G^{(4)}(t) \ \Theta G^{(4)}(t+h)]^{\alpha} = [\frac{g_{\alpha}^{(4)}(t+h) - g_{\alpha}^{(4)}(t)}{h}, \frac{f_{\alpha}^{(4)}(t+h) - f_{\alpha}^{(4)}(t)}{h}],$$

and

$$\frac{1}{-h}[G^{(4)}(t-h) \ \Theta G^{(4)}(t)]^{\alpha} = [\frac{g_{\alpha}^{(4)}(t) - g_{\alpha}^{(4)}(t-h)}{h}, \frac{f_{\alpha}^{(4)}(t) - f_{\alpha}^{(4)}(t-h)}{h}].$$

Finally, using $h \longrightarrow 0$ on both sides of aforementioned relation we get:
$[G^{(5)}(t)]^{\alpha} = [g_{\alpha}^{(5)}(t), f_{\alpha}^{(5)}(t)].$
The other proofs are similar.

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Theorem 3.2 Suppose that $g(t), g'(t), g''(t), g'''(t)$ and $g^{(4)}(t)$ are continuous fuzzy-valued functions on $[0,\infty)$ and of exponential order and that $g^{(5)}(t)$ is piecewise continuous fuzzy-valued function on $[0,\infty)$ with $g(t)=(\underline{g}(t,\alpha),\bar{g}(t,\alpha))$, then:

(1) If g, g', g'', g''' and $g^{(4)}$ be in (i)-formula, then:

$$L(g^{(5)}(t)) = s^5 L(g(t)) \ominus s^4 g(0) \ominus s^3 g'(0) \ominus s^2 g''(0) \ominus s g'''(0) \ominus g^{(4)}(0).$$

(2) If g', g'', g''' and $g^{(4)}$ and be in (i)-formula and g be in (ii)-formula , then:

$$L(g^{(5)}(t)) = -s^4 g(0) \ominus (-s^5) L(g(t)) \ominus s^3 g'(0) \ominus s^2 g''(0) \ominus s g'''(0) \ominus g^{(4)}(0).$$

(3) If g, g'', g''' and $g^{(4)}$ be in (i)-formula and g' be in (ii)-formula , then:

$$L(g^{(5)}(t)) = -s^4 g(0) \ominus (-s^5) L(g(t)) - s^3 g'(0) \ominus s^2 g''(0) \ominus s g'''(0) \ominus g^{(4)}(0).$$

(4) If g, g', g''' and $g^{(4)}$ be in (i)-formula and g'' be in (ii)-formula , then:

$$L(g^{(5)}(t)) = -s^4 g(0) \ominus (-s^5) L(g(t)) - s^3 g'(0) - s^2 g''(0) \ominus s g'''(0) \ominus g^{(4)}(0).$$

(5) If g, g', g'' and $g^{(4)}$ be in (i)-formula and g''' be in (ii)-formula , then:

$$L(g^{(5)}(t)) = -s^4 g(0) \ominus (-s^5) L(g(t)) - s^3 g'(0) - s^2 g''(0) - s g'''(0) \ominus g^{(4)}(0).$$

(6) If g, g', g'' and g''' be in (i)-formula and $g^{(4)}$ be in (ii)-formula , then:

$$L(g^{(5)}(t)) = -s^4 g(0) \ominus (-s^5) L(g(t)) - s^3 g'(0) - s^2 g''(0) - s g'''(0) - g^{(4)}(0).$$

(7) If g'', g''' and $g^{(4)}$ be in (i)-formula and g and g' be in (ii)-formula , then:

$$L(g^{(5)}(t)) = s^5 L(g(t)) \ominus s^4 g(0) - s^3 g'(0) \ominus s^2 g''(0) \ominus s g'''(0) \ominus g^{(4)}(0).$$

(8) If g', g''' and $g^{(4)}$ be in (i)-formula and g and g'' be in (ii)-formula , then:

$$L(g^{(5)}(t)) = s^5 L(g(t)) \ominus s^4 g(0) - s^3 g'(0) - s^2 g''(0) \ominus s g'''(0) \ominus g^{(4)}(0).$$

(9) If g', g'' and $g^{(4)}$ be in (i)-formula and g and g''' be in (ii)-formula , then:

$$L(g^{(5)}(t)) = s^5 L(g(t)) \ominus s^4 g(0) - s^3 g'(0) - s^2 g''(0) - s g'''(0) \ominus g^{(4)}(0).$$

(10) If g', g'' and g''' be in (i)-formula and g and $g^{(4)}$ be in (ii)-formula , then:

$$L(g^{(5)}(t)) = s^5 L(g(t)) \ominus s^4 g(0) - s^3 g'(0) - s^2 g''(0) - s g'''(0) - g^{(4)}(0).$$

(11) If g, g''' and $g^{(4)}$ be in (i)-formula and g' and g'' be in (ii)-formula , then:

$$L(g^{(5)}(t)) = s^5 L(g(t)) \ominus s^4 g(0) \ominus s^3 g'(0) - s^2 g''(0) \ominus s g'''(0) \ominus g^{(4)}(0).$$

(12) If g, g'' and $g^{(4)}$ be in (i)-formula and g' and g''' be in (ii)-formula , then:

$$L(g^{(5)}(t)) = s^5 L(g(t)) \ominus s^4 g(0) \ominus s^3 g'(0) - s^2 g''(0) - s g'''(0) \ominus g^{(4)}(0).$$

(13) If g, g'' and g''' be in (i)-formula and g' and $g^{(4)}$ be in (ii)-formula , then:

$$L(g^{(5)}(t)) = s^5 L(g(t)) \ominus s^4 g(0) \ominus s^3 g'(0) - s^2 g''(0) - s g'''(0) - g^{(4)}(0).$$

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(14) If g, g' and $g^{(4)}$ be in (i)-formula and g'' and g''' be in (ii)-formula , then:

$$L(g^{(5)}(t)) = s^5 L(g(t)) \ominus s^4 g(0) \ominus s^3 g'(0) \ominus s^2 g''(0) - s g'''(0) \ominus g^{(4)}(0).$$

(15) If g, g' and g''' be in (i)-formula and g'' and $g^{(4)}$ be in (ii)-formula , then:

$$L(g^{(5)}(t)) = s^5 L(g(t)) \ominus s^4 g(0) \ominus s^3 g'(0) \ominus s^2 g''(0) - s g'''(0) - g^{(4)}(0).$$

(16) If g, g' and g'' be in (i)-formula and g''' and $g^{(4)}$ be in (ii)-formula , then:

$$L(g^{(5)}(t)) = s^5 L(g(t)) \ominus s^4 g(0) \ominus s^3 g'(0) \ominus s^2 g''(0) \ominus s g'''(0) - g^{(4)}(0).$$

(17) If g''' and $g^{(4)}$ be in (i)-formula and g, g' and g'' be in (ii)-formula , then:

$$L(g^{(5)}(t)) = -s^4 g(0) \ominus (-s^5) L(g(t)) \ominus s^3 g'(0) - s^2 g''(0) \ominus s g'''(0) \ominus g^{(4)}(0).$$

(18) If g'' and $g^{(4)}$ be in (i)-formula and g, g' and g''' be in (ii)-formula , then:

$$L(g^{(5)}(t)) = -s^4 g(0) \ominus (-s^5) L(g(t)) \ominus s^3 g'(0) - s^2 g''(0) - s g'''(0) \ominus g^{(4)}(0).$$

(19) If g'' and g''' be in (i)-formula and g, g' and $g^{(4)}$ be in (ii)-formula , then:

$$L(g^{(5)}(t)) = -s^4 g(0) \ominus (-s^5) L(g(t)) \ominus s^3 g'(0) - s^2 g''(0) - s g'''(0) - g^{(4)}(0).$$

(20) If g' and $g^{(4)}$ be in (i)-formula and g, g'' and g''' be in (ii)-formula , then:

$$L(g^{(5)}(t)) = -s^4 g(0) \ominus (-s^5) L(g(t)) \ominus s^3 g'(0) \ominus s^2 g''(0) - s g'''(0) \ominus g^{(4)}(0).$$

(21) If g' and g''' be in (i)-formula and g, g'' and $g^{(4)}$ be in (ii)-formula , then:

$$L(g^{(5)}(t)) = -s^4 g(0) \ominus (-s^5) L(g(t)) \ominus s^3 g'(0) \ominus s^2 g''(0) - s g'''(0) - g^{(4)}(0).$$

(22) If g' and g'' be in (i)-formula and g, g''' and $g^{(4)}$ be in (ii)-formula , then:

$$L(g^{(5)}(t)) = -s^4 g(0) \ominus (-s^5) L(g(t)) \ominus s^3 g'(0) \ominus s^2 g''(0) \ominus s g'''(0) - g^{(4)}(0).$$

(23) If g and $g^{(4)}$ be in (i)-formula and g', g'' and g''' be in (ii)-formula , then:

$$L(g^{(5)}(t)) = -s^4 g(0) \ominus (-s^5) L(g(t)) - s^3 g'(0) \ominus s^2 g''(0) - s g'''(0) \ominus g^{(4)}(0).$$

(24) If g and g''' be in (i)-formula and g', g'' and $g^{(4)}$ be in (ii)-formula , then:

$$L(g^{(5)}(t)) = -s^4 g(0) \ominus (-s^5) L(g(t)) - s^3 g'(0) \ominus s^2 g''(0) - s g'''(0) - g^{(4)}(0).$$

(25) If g and g'' be in (i)-formula and g', g''' and $g^{(4)}$ be in (ii)-formula , then:

$$L(g^{(5)}(t)) = -s^4 g(0) \ominus (-s^5) L(g(t)) - s^3 g'(0) \ominus s^2 g''(0) \ominus s g'''(0) - g^{(4)}(0).$$

(26) If g and g' be in (i)-formula and g'', g''' and $g^{(4)}$ be in (ii)-formula , then:

$$L(g^{(5)}(t)) = -s^4 g(0) \ominus (-s^5) L(g(t)) - s^3 g'(0) - s^2 g''(0) \ominus s g'''(0) - g^{(4)}(0).$$

(27) If $g^{(4)}$ be in (i)-formula and g, g', g'' and g''' be in (ii)-formula , then:

$$L(g^{(5)}(t)) = s^5 L(g(t)) \ominus s^4 g(0) - s^3 g'(0) \ominus s^2 g''(0) - s g'''(0) \ominus g^{(4)}(0).$$

(28) If g''' be in (i)-formula and g, g', g'' and $g^{(4)}$ be in (ii)-formula , then:

$$L(g^{(5)}(t)) = s^5 L(g(t)) \ominus s^4 g(0) - s^3 g'(0) \ominus s^2 g''(0) - sg'''(0) - g^{(4)}(0).$$

(29) If g'' be in (i)-formula and g, g', g''' and $g^{(4)}$ be in (ii)-formula , then:

$$L(g^{(5)}(t)) = s^5 L(g(t)) \ominus s^4 g(0) - s^3 g'(0) \ominus s^2 g''(0) \ominus sg'''(0) - g^{(4)}(0).$$

(30) If g' be in (i)-formula and g, g'', g''' and $g^{(4)}$ be in (ii)-formula , then:

$$L(g^{(5)}(t)) = s^5 L(g(t)) \ominus s^4 g(0) - s^3 g'(0) - s^2 g''(0) \ominus sg'''(0) - g^{(4)}(0).$$

(31) If g be in (i)-formula and g', g'', g''' and $g^{(4)}$ (ii)-formula, then:

$$L(g^{(5)}(t)) = s^5 L(g(t)) \ominus s^4 g(0) \ominus s^3 g'(0) - s^2 g''(0) \ominus sg'''(0) - g^{(4)}(0).$$

(32) If g, g', g'', g''' and $g^{(4)}$ be in (ii)-formula, then:

$$L(g^{(5)}(t)) = -s^4 g(0) \ominus (-s^5) L(g(t)) \ominus s^3 g'(0) - s^2 g''(0) \ominus sg'''(0) - g^{(4)}(0).$$

Proof First, we state the notations as follows: $\underline{g}', \underline{g}'', \underline{g}''', \underline{g}^{(4)}$ and $\underline{g}^{(5)}$ are the lower endpoints function's derivatives, $\bar{g}', \bar{g}'', \bar{g}''', \bar{g}^{(4)}$ and $\bar{g}^{(5)}$ are the upper endpoints function's derivatives. Also $\underline{g'}, \underline{g''}, \underline{g'''}, \underline{g^{(4)}}$ and $\underline{g^{(5)}}$ are the lower endpoints of the derivatives. $\overline{g'}, \overline{g''}, \overline{g'''}, \overline{g^{(4)}}$ and $\overline{g^{(5)}}$ are the upper endpoints of the derivatives. For arbitrary fixed $\alpha \in [0,1]$, we have $g(t) = (\underline{g}(t,\alpha), \bar{g}(t,\alpha))$.

Now, we prove (6) as follows: Since g, g', g'' and g''' are (i)-formula and $g^{(4)}$ is (ii)-formula then by theorem 3.1(b) we get

$$g^{(5)}(t) = (\bar{g}^{(5)}(t,\alpha), \underline{g}^{(5)}(t,\alpha)).$$

Therefore, we get:

$$\underline{g^{(5)}}(t,\alpha) = \bar{g}^{(5)}(t,\alpha), \ \overline{g^{(5)}}(t,\alpha) = \underline{g}^{(5)}(t,\alpha). \tag{3.1}$$

Then from (3.1), we get

$$L(g^{(5)}(t)) = L(\underline{g^{(5)}}(t,\alpha), \overline{g^{(5)}}(t,\alpha))$$
$$= (l(\bar{g}^{(5)}(t,\alpha)), l(\underline{g}^{(5)}(t,\alpha))). \tag{3.2}$$

We know from the ordinary differential equations that:

$$l(\underline{g}^{(5)}(t,\alpha)) = s^5 l(\underline{g}(t,\alpha)) - s^4 \underline{g}(0,\alpha) - s^3 \underline{g}'(0,\alpha) - s^2 \underline{g}''(0,\alpha) - s\underline{g}'''(0,\alpha) - \underline{g}^{(4)}(0,\alpha),$$

$$l(\bar{g}^{(5)}(t,\alpha)) = s^5 l(\bar{g}(t,\alpha)) - s^4 \bar{g}(0,\alpha) - s^3 \bar{g}'(0,\alpha) - s^2 \bar{g}''(0,\alpha) - s\bar{g}'''(0,\alpha) - \bar{g}^{(4)}(0,\alpha).$$

Since g, g', g'' and g''' are (i)-formula, we get:

$$\underline{g}'(0,\alpha) = \underline{g'}(0,\alpha), \quad \bar{g}'(0,\alpha) = \overline{g'}(0,\alpha) \ ,$$

$$\underline{g}''(0,\alpha) = \underline{g''}(0,\alpha), \ \bar{g}''(0,\alpha) = \overline{g''}(0,\alpha),$$

$$\underline{g}'''(0,\alpha) = \underline{g'''}(0,\alpha), \bar{g}'''(0,\alpha) = \overline{g'''}(0,\alpha),$$

$$\underline{g}^{(4)}(0,\alpha) = \underline{g^{(4)}}(0,\alpha), \bar{g}^{(4)}(0,\alpha) = \overline{g^{(4)}}(0,\alpha).$$

Then, equation (3.2) becomes:

$$L(g^{(5)}(t)) = (s^5 l(\bar{g}(t,\alpha)) - s^4 \bar{g}(0,\alpha) - s^3 \overline{g'}(0,\alpha) - s^2 \overline{g''}(0,\alpha) - s\overline{g'''}(0,\alpha) - \overline{g^{(4)}}(0,\alpha),$$
$$s^5 l(\underline{g}(t,\alpha)) - s^4 \underline{g}(0,\alpha) - s^3 \underline{g'}(0,\alpha) - s^2 \underline{g''}(0,\alpha) - s\underline{g'''}(0,\alpha) - \underline{g^{(4)}}(0,\alpha))$$
$$= -s^4 g(0) \ominus (-s^5) L(g(t)) - s^3 g'(0) - s^2 g''(0) - g'''(0) - g^{(4)}(0).$$

The other proofs are similar.

4. Constructing Formulas for Solving FIVPs of the Fifth Order

We consider the FIVP of the fifth order in the form:

$$y^{(5)}(t)=f\,(t,y(t),y'(t),y''(t),y'''(t),y^{(4)}(t)), \tag{4.1}$$

$$y(0)=(\underline{y}(0,\alpha),\overline{y}(0,\alpha)),\ y'(0)=(\underline{y}'(0,\alpha),\overline{y}'(0,\alpha)),\ y''(0)=(\underline{y}''(0,\alpha),\overline{y}''(0,\alpha)),$$

$$y'''(0)=(\underline{y}'''(0,\alpha),\overline{y}'''(0,\alpha)),\ y^{(4)}(0)=(\underline{y}^{(4)}(0,\alpha),\overline{y}^{(4)}(0,\alpha)),$$

where $f:R^{+}\times E\times E\times E\times E\times E\rightarrow E$ is a continuous fuzzy mapping. By using Laplace transform method we have:

$$L(y^{(5)}(t))=L(f\,(t,y(t),y'(t),y''(t),y'''(t),y^{(4)}(t))). \tag{4.2}$$

Then, according to theorem 3.2, we have $2^5=32$ cases for solving (4.1). We shall consider the same cases given in theorem 3.2 respectively, as follows:

Case 1 Using (1) of theorem 3.2, equation (4.2) can be written as follows:

$$\begin{aligned} &s^5 l(\underline{y}(t,\alpha))-s^4\underline{y}(0,\alpha)-s^3\underline{y}'(0,\alpha)-s^2\underline{y}''(0,\alpha)-s\,\underline{y}'''(0,\alpha)-\underline{y}^{(4)}(0,\alpha)=l(\underline{f}),\\ &s^5 l(\overline{y}(t,\alpha))-s^4\overline{y}(0,\alpha)-s^3\overline{y}'(0,\alpha)-s^2\overline{y}''(0,\alpha)-s\,\overline{y}'''(0,\alpha)-\overline{y}^{(4)}(0,\alpha)=l(\overline{f}).\end{aligned} \tag{4.3}$$

Case 2 Using (2) of theorem 3.2, equation (4.2) can be written as follows:

$$\begin{aligned} &s^5 l(\underline{y}(t,\alpha))-s^4\underline{y}(0,\alpha)-s^3\overline{y}'(0,\alpha)-s^2\overline{y}''(0,\alpha)-s\,\overline{y}'''(0,\alpha)-\overline{y}^{(4)}(0,\alpha)=l(\overline{f}),\\ &s^5 l(\overline{y}(t,\alpha))-s^4\overline{y}(0,\alpha)-s^3\underline{y}'(0,\alpha)-s^2\underline{y}''(0,\alpha)-s\,\underline{y}'''(0,\alpha)-\underline{y}^{(4)}(0,\alpha)=l(\underline{f}).\end{aligned} \tag{4.4}$$

Case 3 Using (3) of theorem 3.2, equation (4.2) can be written as follows:

$$\begin{aligned} &s^5 l(\underline{y}(t,\alpha))-s^4\underline{y}(0,\alpha)-s^3\underline{y}'(0,\alpha)-s^2\overline{y}''(0,\alpha)-s\,\overline{y}'''(0,\alpha)-\overline{y}^{(4)}(0,\alpha)=l(\overline{f}),\\ &s^5 l(\overline{y}(t,\alpha))-s^4\overline{y}(0,\alpha)-s^3\overline{y}'(0,\alpha)-s^2\underline{y}''(0,\alpha)-s\,\underline{y}'''(0,\alpha)-\underline{y}^{(4)}(0,\alpha)=l(\underline{f}).\end{aligned} \tag{4.5}$$

Case 4 Using (4) of theorem 3.2, equation (4.2) can be written as follows:

$$\begin{aligned} &s^5 l(\underline{y}(t,\alpha))-s^4\underline{y}(0,\alpha)-s^3\underline{y}'(0,\alpha)-s^2\underline{y}''(0,\alpha)-s\,\overline{y}'''(0,\alpha)-\overline{y}^{(4)}(0,\alpha)=l(\overline{f}),\\ &s^5 l(\overline{y}(t,\alpha))-s^4\overline{y}(0,\alpha)-s^3\overline{y}'(0,\alpha)-s^2\overline{y}''(0,\alpha)-s\,\underline{y}'''(0,\alpha)-\underline{y}^{(4)}(0,\alpha)=l(\underline{f}).\end{aligned} \tag{4.6}$$

Case 5 Using (5) of theorem 3.2, equation (4.2) can be written as follows:

$$\begin{aligned} &s^5 l(\underline{y}(t,\alpha))-s^4\underline{y}(0,\alpha)-s^3\underline{y}'(0,\alpha)-s^2\underline{y}''(0,\alpha)-s\,\underline{y}'''(0,\alpha)-\overline{y}^{(4)}(0,\alpha)=l(\overline{f}),\\ &s^5 l(\overline{y}(t,\alpha))-s^4\overline{y}(0,\alpha)-s^3\overline{y}'(0,\alpha)-s^2\overline{y}''(0,\alpha)-s\,\overline{y}'''(0,\alpha)-\underline{y}^{(4)}(0,\alpha)=l(\underline{f}).\end{aligned} \tag{4.7}$$

The other cases can be found in a similar way.

5. An Illustrative Example

Consider the following fifth order FIVP:

$$y^{(5)}(t)=\sigma_0,\ \ \sigma_0=(e^r,e^{2-r})$$

$$y(0)=y'(0)=y''(0)=y'''(0)=y^{(4)}(0)=(3r-3,3-3r).$$

We have $2^5=32$ cases for solving this FIVP. We shall consider the same cases given in theorem 3.2 respectively, as follows:

Case 1 By using system (4.3), we get the r-cut representation of solution as following:

$$\underline{y}(t,r)=\frac{e^r}{120}t^5+(3r-3)(1+t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}),$$

$$\overline{y}(t,r)=\frac{e^{2-r}}{120}t^5-(3r-3)(1+t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}).$$

Case 2 By using system (4.4), we get the r-cut representation of solution as following :

$$\underline{y}(t,r)=\frac{e^{2-r}}{120}t^5+(3r-3)(1-t-\frac{t^2}{2}-\frac{t^3}{6}-\frac{t^4}{24}),$$

$$\bar{y}(t,r)=\frac{e^{r}}{120}t^{5}-(3r-3)(1-t-\frac{t^{2}}{2}-\frac{t^{3}}{6}-\frac{t^{4}}{24}).$$

Case 3 By using system (4.5), we get the r -cut representation of solution as following :

$$\underline{y}(t,r)=\frac{e^{2-r}}{120}t^{5}+(3r-3)(1+t-\frac{t^{2}}{2}-\frac{t^{3}}{6}-\frac{t^{4}}{24}),$$

$$\bar{y}(t,r)=\frac{e^{r}}{120}t^{5}-(3r-3)(1+t-\frac{t^{2}}{2}-\frac{t^{3}}{6}-\frac{t^{4}}{24}).$$

Case 4 By using system (4.6), we get the r -cut representation of solution as following :

$$\underline{y}(t,r)=\frac{e^{2-r}}{120}t^{5}+(3r-3)(1+t+\frac{t^{2}}{2}-\frac{t^{3}}{6}-\frac{t^{4}}{24}),$$

$$\bar{y}(t,r)=\frac{e^{r}}{120}t^{5}-(3r-3)(1+t+\frac{t^{2}}{2}-\frac{t^{3}}{6}-\frac{t^{4}}{24}).$$

Case 5 By using system (4.7), we get the r -cut representation of solution as following :

$$\underline{y}(t,r)=\frac{e^{2-r}}{120}t^{5}+(3r-3)(1+t+\frac{t^{2}}{2}+\frac{t^{3}}{6}-\frac{t^{4}}{24}),$$

$$\bar{y}(t,r)=\frac{e^{r}}{120}t^{5}-(3r-3)(1+t+\frac{t^{2}}{2}+\frac{t^{3}}{6}-\frac{t^{4}}{24}).$$

The other solutions can be found in a similar way.

6. Conclusions

The formula of fuzzy derivative of the fifth order is found. In addition, fuzzy Laplace transforms for the fuzzy derivatives of the fifth order are found. We used fuzzy Laplace transforms in solving FIVP of the fifth order, and multiple solutions are provided for this FIVP.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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