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Global Optimization Algorithm for Solving Rational Logarithmic Non-Linear Functions

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Abstract*—The idea of this paper stems to modify "the fact that most used optimization methods use a local quadratic representation of the objective function". It also arises from the fact that the objective function may not be represented perfectly by local quadratic representation functions and the global minimizer may be obtained for the general objective functions. Consequently, a new non-quadratic model algorithm, in this paper, is suggested for solving unconstrained nonlinear optimization problems, which modifies the classical conjugate gradient (CG) algorithms. The new algorithm is derived and evaluated numerically for some standard nonlinear test functions. The results indicate that in general the new algorithm has an improvement percentages on the previous some selected CG-algorithms.*

Keywords— Unconstrained Optimization, Conjugate Gradient Methods, Descent direction, Global convergence.

1. Introduction

Conjugate Gradient (CG) methods are iterative methods, which generate a sequence of approximation points to minimize the nonlinear function $f(x)$, of the form,

$$x_{i+1} = x_i + \lambda_i d_i , \qquad i = 0,1,2,\ldots$$

where $\lambda_i \geq 0$ is the step length compute using exact line search by the formula

$$f(x_i + \lambda_i d_i) = \min \ f(x_i + \lambda_i d_i),$$

and d_k is the current search direction computed as follow:

$$d_i = \begin{cases} -g_i \quad , & if \quad i = 0 , \\ -g_i + \beta_i d_{i-1}, & if \quad i \geq 1 , \end{cases}$$

where β_i is the conjugacy coefficient and it is , originally, defined by one of the following standard formulae:

$$\beta_i^{FR} = \frac{\|g_i\|^2}{\|g_{i-1}\|^2}$$, (Fletcher and Reeves (FR), 1964),

$$\beta_i^{HS} = \frac{g_i^T(g_i - g_{i-1})}{d_{i-1}^T(g_i - g_{i-1})}$$, (Hestenes and Stiefel (HS), 1952),

$$\beta_i^{PR} = \frac{g_i^T(g_i - g_{i-1})}{\|g_{i-1}\|^2}$$, (Polak and Ribière (PR), 1969),

$$\beta_i^{Dx} = -\frac{\|g_i\|^2}{d_{i-1}^T g_{i-1}}$$, (Dixon (Dx), 1975),

where g_{i-1} and g_i are gradients of $f(x)$ at the point x_{i-1} and x_i respectively. Also, $\|.\|$ denotes Euclidean norm of vectors.

Hestenes and Stiefel published the first CG-methods, in 1952, for solving a system of linear algebraic equations. Fletcher and Reeves, in 1964, were the first, among other scholars, to use this technique to minimize a non-linear function of several variables. Since then the method has been used successfully to tackle many nonlinear test problems.

2. CG-METHODS FOR EXTENDED QUADRATIC MODELS

In this section, a more general model than quadratic one is suggested as a basis for the CG-algorithm. If $q(x)$ is a quadratic function, then a function $f(q(x))$ is defined as a non-linear scaling of $q(x)$ if the following condition holds:

$$f = F(q(x)), \quad \frac{df}{dq} = f' > 0 \quad and \quad q(x) > 0 \tag{1}$$

where x^* is the minimizer of $q(x)$ with respect to x, (Spedicato, 1976). The following properties are immediately derived from the above condition:

i) every contour line of the quadratic function $q(x)$ is a contour line of the general function $f(x)$,

ii) if x^* is a minimizer of the quadratic function $q(x)$, then it is a minimizer of the general function $f(x)$,

iii) if x^* is a local minimizer of the quadratic function $q(x)$, then it is a local minimizer of the general function $f(x)$.

In (Boland, 1979), it was first observed that $q(x)$ and $f(q(x))$ have determined the same search directions so that the finite termination property for their algorithm is satisfied. Various authors have published related works in the area see for example; (Al-Assady, et.al, 1993), (Al-Assady and Al-Bayati, 1994), (Al-Bayati, 1992), (Hu et.at, 1994), (Gaoyi Wu, et.al, 2018), and (Yuan, G. and Hu, W., 2018). A conjugate gradient method which minimizers the function:

$$f(x) = (q(x))^P, \; P > 0 \quad \text{and} \quad x \in R^n \tag{2}$$

in at most n step has been described by (Fried, 1971) and the special case:

$$F(q(x)) = \varepsilon_1 q(x) + \frac{1}{2}\varepsilon_2 q^2(x) \tag{3}$$

where ε_1 and ε_2 are scalars, has investigated by (Boland et.al, 1979). (Tassopoulos and Storey, 1984a and 1984b) have proposed two different rational models that are denoted by T/S. their non-quadratic models are defined by the following formulae:

$$F(q(x)) = \frac{(\varepsilon_1 q(x) + 1)}{\varepsilon_2 q(x)} \;, \; \varepsilon_2 < 0 \text{ and } q(x) > 0 \tag{4}$$

where ε_1 and ε_2 are scalars, and

$$F(q(x)) = \frac{\varepsilon\, q(x)}{(1+q(x))} \;,\; \varepsilon > 0 \text{ and } q(x) > 0 \qquad (5)$$

Where the following logarithmic non-quadratic model:

$$F(q(x)) = \log\left(\frac{\varepsilon_1 q(x)}{\varepsilon_2 q(x)+1}\right) \;,\; \varepsilon_2 < 0 \text{ and } q(x) > 0 \qquad (6)$$

have been suggested by (Al-Mashhadany and Al- Assady, 1997).

Here we are going to list outlines of the (Tassopoulos and Storey, 1984a) CG-algorithm:

2.1 Tassopoulos and Storey (T/S) CG-Algorithm.

The outline of the **T/S** algorithm is as follows:

Given $x_0 \in R^n$ an initial estimate of the minimizer x^* and a scalar ε.

Step1: Set $d_0 = -g_0$

Step2: For $i = 1, 2, \cdots$

compute $x_i = x_{i-1} + \lambda_{i-1} d_{i-1}$, where λ_{i-1} is the minimizer of f on d_{i-1}

Step3: Define $n = \dfrac{\lambda_{i-1}\, g_{i-1}^T\, d_{i-1}}{2}$ and $w = f_i - f_{i-1}$

where $f_i = F(q(x_i))$ and $f_{i-1} = F(q(x_{i-1}))$

compute $\eta = n + w$ and $(\eta\, f_{i-1} - n\,w)$

Step4: If $|\eta| \le \varepsilon$, then set $\rho_i = 1.0$ and go to Step(7), where ε is a small number $(i.e \;\; 0.1E-10)$

Step5: If $|\eta\, f_{i-1} - nw| \le w$, then set $\rho_i = \sqrt{f_{i-1}/f_i}$, and go to Step(7) else go to step(6)

Step6: Compute $\rho_i = [n/(2n+w)]^2$

Step7: Calculate the new direction: $d_i = -g_i + \rho_i \left\{ \|g_i\|^2 / \|g_{i-1}\|^2 \right\} d_{i-1}$

(this direction is modified for standard (FR) formula for the new model)

Step8: Check for convergence,

if $\|g_i\| \le \varepsilon$, then stop, else go to Step(9)

Step9: Check for restarting criterion,

if $i = n$, go to step(7), else set $i = i+1$, and go to Step(2).

2.1 Tassopoulos and Storey (T/S) CG-Algorithm

The outline of the **T/S** algorithm is as follows:

Given $x_0 \in R^n$ an initial estimate of the minimizer x^* and a scalar ε.

Step1: Set $d_0 = -g_0$

Step2: For $i = 1, 2, \cdots$

compute $x_i = x_{i-1} + \lambda_{i-1} d_{i-1}$, where λ_{i-1} is the minimizer of f on d_{i-1}

Step3: Define $n = \dfrac{\lambda_{i-1}\, g_{i-1}^T\, d_{i-1}}{2}$ and $w = f_i - f_{i-1}$

where $f_i = F(q(x_i))$ and $f_{i-1} = F(q(x_{i-1}))$

compute $\eta = n + w$ and $(\eta f_{i-1} - nw)$

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if $\|g_i\| \le \varepsilon$, then stop, else go to Step(9)

Step9: Check for restarting criterion,

if $i = n$, go to step(7), else set $i = i+1$, and go to Step(2).

3. New Proposed non-quadratic CG-Algorithm

In this section, another new logarithmic model was investigated and tested on some well-known set of standard nonlinear test functions, An extended CG-algorithm is developed which is based on this new model which scales q(x) by the natural logarithmic function for the rational q(x) functions.

$$F(q(x)) = \log\left(\frac{\varepsilon_1 q(x)}{1-\varepsilon_2 q(x)}\right), \ \varepsilon_2 < 0 \qquad (7)$$

We first observe that $q(x)$ and $F(q(x))$ given by (7) have identical contours, though with different function values, and they have the same unique minimum point denoted by x^*. For any general function *f* satisfying the condition (1) it is shown in (Boland et al. 1979) that the updating process given below generates identical conjugate search directions and the same sequence of approximation points x_i to the minimizer x^*, as does the original method of (Fletcher- Reeves 1964) when applied to $f(x)=q(x)$. In order to modify the property (iii) in the following way:

" the optimal point x^* is a global minimizer of the quadratic function $q(x)$ implies that is a global minimizer of the general function $f(x)$" . In this study, we have suggested a new logarithmic model which is defined in eq.(7) based on the on standard theorem of (Renpu Ge, 1989) which is illustrated below:

Suppose F(x) has the form

$$F(x) = \frac{f_1(x_1)}{g_2(x_2)} \qquad \text{where} \quad x^T = (x_1^T, x_2^T) \qquad (8)$$

and

$$f_1(x_1) > 0 \quad and \quad g_2(x_2) > 0 \qquad (9)$$

It follows from equation (3) and (4) that:

$$\log F(x) = \log f_1(x_1) - \log g_2(x_2)$$

is a separable function.

Thus according to the following theorem which states:

Theorem 3.1. $x^{*T} = (x_1^{*T}, x_2^{*T}, \cdots, x_n^{*T})$ is a global minimizer of a rational separable function $F(x)$ if and only if every sequence of points $x_i^*, (i=1,2,\cdots,n)$ is a global minimizer of the general function $f_i(x_i)$.

Proof: See (Renpu Ge, 1989).

we can conclude that x^* is a global minimizer of $\log F(x)$ if and only if x_1^* and x_2^* are respectively global minimizers of $\log f_1(x_1)$ and $-\log g_2(x_2)$. Furthermore, the monotonicity of $\log t$ implies that x^* is a global minimizer of $F(x)$ if and only if x_1^* and x_2^* are respectively global minimizers of the function $f_1(x_1)$ and $g_2(x_2)$.

3.1 Out line of the New Proposed Algorithm

Given $x_0 \in R^n$ an initial estimate of the minimizer x*.

Step1: Set $d_0 = -g_0$.

Step2: For i = 1, 2 ...

compute $x_i = x_{i-1} + \lambda_{i-1} d_{i-1}$

where λ_{i-1} is the optimal step size obtained by the line search procedure.

Step3: Calculate $\exp(f) = 1 + f + f^2/2! + f^3/3! + \ldots$

and define $n = \lambda_{i-1} g^T_{i-1} d_{i-1} /2$.

$w = \exp(f_i) - \exp(f_{i-1})$.

$c = w - n \exp(f_i)$

Step4: If $|w| \le 1.e{-6}$ or $|c| \le 1.e{-6}$, then set $\rho_i = 1.0$ and go to step(6) else go to Step(5).

Step5: Compute

$$\rho_i = \left(\frac{\exp(f_i)}{\exp(f_{i-1})}\right)\left(\frac{n \cdot \exp(f_i)}{w}\right)^2$$

where the derivation of scaling ρ_i will be presented below.

Step6: Calculate the new search direction

$$d_i = -g_i + \beta_i d_{i-1}$$

where β_i is defined by different formulae according to variation and it is expressed as follows:

$$\beta_i^{FR} = \rho_i \left(\frac{\|g_i\|^2}{\|g_{i-1}\|^2}\right),$$ (modified Fletcher and Reeves 1964),

$$\beta_i^{HS} = \frac{g_i^T(\rho_i\, g_i - g_{i-1})}{d_{i-1}^T(g_i - g_{i-1})},$$ (modified Hestenes and stiefle 1952),

$$\beta_i^{PR} = \frac{g_i^T(\rho_i\, g_i - g_{i-1})}{\|g_{i-1}\|^2},$$ (modified Polak and Ribière 1969),

CG-methods are usually implemented by restarts in order to avoid an accumulation of errors affecting the search directions. It is therefore generally agreed that restarting is very helpful in practice, so we have used the following restarting criterion in our practical investigations. If the new direction satisfies:

$$d_i^T g_i \ge -0.8\| g_i \|^2 \quad (10)$$

then a restart is also initiated. This new direction is sufficiently downhill.

3.2 Derivation of the New proposed Algorithm

The implementation of the extended CG-method has been performed for general functions F(q(x)) of the form of eq.(7). The unknown quantities ρ_i were expressed in terms of available quantities of the algorithm (i.e. function and gradient values of the objective function). It is first assumed that neither ε_1, nor ε_2 is zero in eq.(7) . Solving eq.(7) for q(x), then:

$$q(x)=\frac{\exp(f)}{\varepsilon_2(\exp(f)+\varepsilon_1/\varepsilon_2)} \tag{11}$$

and using the expression for ρ_i (henceforth ρ_i^{NEW})

$$\rho_i^{NEW}=\frac{f'_{i-1}}{f'_i}=\left(\frac{\exp(f_i)}{\exp(f_{i-1})}\right)\left(\frac{\exp(f_{i-1})+\varepsilon_1/\varepsilon_2}{\exp(f_i)+\varepsilon_1/\varepsilon_2}\right)^2 \tag{12}$$

the quantity that has to be determined explicitly is $(\varepsilon_1/\varepsilon_2)$.

During every interaction $(\varepsilon_1/\varepsilon_2)$ must be evaluated as a function of known available quantities.

From the relation:

$$g_i \;=f'_i\, G(x_i-x^*) \tag{13}$$

$$g_{i-1}=f'_{i-1}\, G(x_{i-1}-x^*) \tag{14}$$

where G is the Hessian matrix and x* is the minimum point, we have:

$$\rho_i^{NEW}=\frac{f'_{i-1}}{f'_i}=\left(\frac{g_{i-1}^T}{g_i^T}\right)\left(\frac{x_i-x^*}{x_{i-1}-x^*}\right) \tag{15}$$

Furthermore,

$$g_{i-1}^T\,(x_i-x^*)=g_{i-1}^T\,(x_{i-1}+\lambda_{i-1}d_{i-1}-x^*)$$
$$=g_{i-1}^T\,(x_{i-1}-x^*)+\lambda_{i-1}\,g_{i-1}^T\,d_{i-1}$$

and

$$g_i^T\,(x_{i-1}-x^*)=g_i^T\,(x_i-\lambda_{i-1}d_{i-1}-x^*)$$
$$=g_i^T\,(x_i-x^*)$$

since, $g_i^T\,d_{i-1}=0$

Therefore we can express ρ_i^{NEW} as follows:

$$\rho_i^T=\frac{g_{i-1}^T\,(x_{i-1}-x^*)+\lambda_{i-1}\,g_{i-1}^T\,d_{i-1}}{g_i^T\,(x_i-x^*)} \tag{16}$$

From (13) and (14), we get:

$$\rho_i^{NEW}=\frac{f'_{i-1}\,(x_{i-1}-x^*)^T\,G\,(x_{i-1}-x^*)+\lambda_{i-1}\,g_{i-1}^T\,d_{i-1}}{f'_i\,(x_i-x^*)^T\,G\,(x_i-x^*)}$$

Therefore,

$$\rho_i^{NEW} = \frac{2\, f'_{i-1}\, q_{i-1} + \lambda_{i-1}\, g_{i-1}^T\, d_{i-1}}{2\, f'_i\, q_i}$$

$$\rho_i^{NEW} = \rho_i^{NEW}\left(\frac{q_{i-1}}{q_i}\right) + \frac{\lambda_{i-1}\, g_{i-1}^T\, d_{i-1}}{2\, f'_i\, q_i} \qquad (17)$$

The quantities $\left(\frac{q_{i-1}}{q_i}\right)$ and $f_i' q_i$ can be rewritten as:

$$\frac{q_{i-1}}{q_i} = \frac{1}{\sqrt{\rho_i^{NEW}}}\sqrt{\frac{\exp(f_{i-1})}{\exp(f_i)}} \qquad (18)$$

$$f_i' q_i = \frac{\exp(f_i) + (\varepsilon_1/\varepsilon_2)}{(\varepsilon_1/\varepsilon_2)} \qquad (19)$$

Substituting (18) and (19) in (17), gives :

$$\rho_i^{NEW} = \rho_i^{NEW}\frac{1}{\sqrt{\rho_i^{NEW}}}\sqrt{\frac{\exp(f_{i-1})}{\exp(f_i)}} + \frac{(\varepsilon_1/\varepsilon_2)\left(\lambda_{i-1}\, g_{i-1}^T\, d_{i-1}/2\right)}{\exp(f_i) + \varepsilon_1/\varepsilon_2} \qquad (20)$$

Using the transformation:-

$$\lambda_{i-1}\, g_{i-1}^T\, d_{i-1} = 2n \qquad (21)$$

From (13) and (21), it follows that:

$$\left(\frac{\exp(f_i)}{\exp(f_{i-1})}\right)\left(\frac{\exp(f_{i-1}) + \varepsilon_1/\varepsilon_2}{\exp(f_i) + \varepsilon_1/\varepsilon_2}\right)^2 = \sqrt{\frac{\exp(f_i)}{\exp(f_{i-1})}}\left(\frac{\exp(f_{i-1}) + \varepsilon_1/\varepsilon_2}{\exp(f_i) + \varepsilon_1/\varepsilon_2}\right)\sqrt{\frac{\exp(f_{i-1})}{\exp(f_i)}} + \frac{n(\varepsilon_1/\varepsilon_2)}{(\exp(f_i) + \varepsilon_1/\varepsilon_2)}$$

$$(\exp(f_{i-1}) + \varepsilon_1/\varepsilon_2)^2 = \left(\frac{\exp(f_{i-1})}{\exp(f_i)}\right)(\exp(f_{i-1}) + \varepsilon_1/\varepsilon_2)(\exp(f_i) + \varepsilon_1/\varepsilon_2) + n(\varepsilon_1/\varepsilon_2)\frac{\exp(f_{i-1})}{\exp(f_i)}(\exp(f_i) + \varepsilon_1/\varepsilon_2)$$

$$\frac{\varepsilon_1}{\varepsilon_2} = \frac{(\exp(f_{i-1}))^2 - \exp(f_{i-1})\cdot\exp(f_i) + n\cdot(\exp(f_i))^2}{\exp(f_i) - \exp(f_{i-1}) - n\cdot\exp(f_i)}$$

$$= \frac{-\exp(f_{i-1})(\exp(f_i) - \exp(f_{i-1})) + n\cdot(\exp(f_i))^2}{\exp(f_i) - \exp(f_{i-1}) - n\cdot\exp(f_i)}$$

Using the following transformation:

$$w = \exp(f_i) - \exp(f_{i-1})$$

The above equation can be rewritten as:

$$\frac{\varepsilon_1}{\varepsilon_2} = \frac{-\left(w\cdot\exp(f_{i-1}) - n\cdot(\exp(f_i))^2\right)}{w - n\cdot\exp(f_i)} \qquad (22)$$

and substituting equation (22) in equation(13), we get:

$$\rho_i^{NEW} = \left(\frac{\exp(f_i)}{\exp(f_{i-1})}\right)\left(\frac{n\cdot\exp(f_i)}{w}\right)^2$$

4. Numerical Results and Conclusions

In order to test the effectiveness of the new proposed CG-algorithm which has been used to extend the standard CG- method, the comparative tests involve several well-known test functions (see Appendix) has been chosen and solved numerically by utilizing the new and established methods.

Tables (1), (2) and (3) utilize the comparisons between our proposed new algorithm which is corresponding to the new rational non-quadratic model represented in eq.(7), denoted by (NEW), the classical CG-method, denoted by (CG), the rational model of Tassopoulos and Storey, denoted by (T/S), the method of (Al-Mashhadany and Al- Assady, 1997), denoted by (H/N) for low; intermediate and high dimension well-known nonlinear test problems.
The identical linear search procedure was used, namely, the cubic fitting procedure described by (Bundy, 1984) and also the used in each case so that $\|g_{i-1}\| < 1\times10^{-5}$. Specifically quantity the number of function calls (NOF), the number of iterations (NOI).

Table1:- Comparisons of different methods for non-quadratic models

$$2 \le n \le 10$$

Test Function	n	CG		T/S		H/N		NEW	
		NOI	NOF	NOI	NOF	NOI	NOF	NOI	NOF
Rosen	2	31	73	31	73	31	73	29	68
Powell	4	50	114	38	96	51	109	34	80
Wood	4	28	61	36	78	36	75	30	64
Miele	4	57	178	46	139	57	178	44	124
Dixon	10	22	46	18	44	21	44	18	44
Total		188	472	169	430	196	479	155	389

It is obvious that the new proposed CG-algorithm, for this set of low dimensionality test functions, improve the classical CG-algorithm in about (18%) NOI and (20%) NOF. Also the new proposed CG-algorithm improve the T/S algorithm in (9%) NOI and (12%) NOF. Finally the new proposed CG- improve H/N algorithm in about (21%) NOI and (21%) NOF.

Table2:- Comparison of different methods for non-quadratic models

$$20 \ge n \le 80$$

Test Function	n	CG		T/S		H/N		NEW	
		NOI	NOF	NOI	NOF	NOI	NOF	NOI	NOF
Dixon	20	15	37	15	37	15	37	15	37
Non-dig.	20	24	61	24	61	24	61	24	60
Wood	40	51	105	43	90	35	74	34	71
OSP	40	25	62	18	51	23	65	23	65
Powell	60	67	136	43	90	36	74	28	59
Rosen	60	23	56	23	57	18	41	27	63
Wood	80	69	140	43	90	50	105	31	67
Powell	80	112	239	75	167	95	194	47	106
Total		386	836	284	643	296	651	229	528

Clearly the new proposed CG-algorithms beats CG-method in (41%)NOI and (37%) NOF; T/S method in (20%) NOI and (18%) NOF; H/N method in (23%) NOI and (19%) NOF.

Table3:- Comparisons of different method for non-quadratic model $100 \le n \le 400$

Test Function	n	CG		T/S		H/N		NEW	
		NOI	NOF	NOI	NOF	NOI	NOF	NOI	NOF
OSP	100	25	59	28	68	23	64	35	43
Non-dig.	100	25	62	25	62	25	59	23	59
Rosen	100	23	56	23	57	20	49	25	57
Wood	200	69	140	47	98	48	98	37	80
Miele	200	209	472	154	351	59	174	47	128
Powell	100	129	263	78	183	111	249	69	151
Wood	400	69	140	47	98	34	170	29	61
Rosen	400	23	56	23	57	22	51	23	65
Total		572	1248	425	974	342	914	288	644

Clearly the new proposed CG-algorithms beats CG-method in (50%)NOI and (49%) NOF; T/S method in (33%) NOI and (34%) NOF; H/N method in (16%) NOI and (30%) NOF.

APPENDIX

1.Rosen-Brock Function:

$$F(x) = \sum_{i=1}^{n/2} [100(x_{2i} - x_{2i-1}^2)^2 + (1 - x_{2i-1})^2],$$

$$x_0 = (-1.2; 1.0; \dots)^T$$

2. Generalized Powell Quadratics Functions:

$$F(x) = \sum_{i=1}^{n/4} (x_{4i-3} + 10x_{4i-2})^2 + 5(x_{4i-1} - x_{4i})^2 + (x_{4i-2} - 2x_{4i-1})^4 + 10(x_{4i-3} - x_{4i})^4 ,$$

$$x_0 = (\ 3.0;\ -1.0;\ 0.0;\ 1.0)^T$$

3. Wood Function:

$$F(x) = \sum_{i=1}^{n/4} 100 \begin{bmatrix} (x_{4i-2} + x^2_{4i-3})^2 + (1 - x_{4i-3})^2 + 90(x_{4i} - x^2_{4i-1})^2 + (1 - x_{4i-1})^2 + 10.1(x_{4i-2} - 1)^2 \\ +(x_{4i} - 1)^2 + 19.8(x_{4i-2} - 1)(x_{4i} - 1) \end{bmatrix},$$

$$x_0 = (\ -3.0;\ -1.0; -3.0; -1.0; \dots\dots)^T$$

4. Miele Function:

$$F(x) = \sum_{i=1}^{n/4} [\exp(x_{4i-3}) - x_{4i-2}]^2 + 100(x_{4i-2} - x_{4i-1})^6 +$$
$$[\tan(x_{4i-1} - x_{4i})]^4 + x_{4i-3}^8 + (x_{4i} - 1)^2 ,$$

$$x_0 = (1.0; 2.0; 2.0; 2.0; ...)^T$$

5. Non-Diagonal Variant of Rosen-Brock Function:

$$F(x) = \sum_{i=2}^{n} [100\left(x_i - x_i^2\right)^2 + \left(1 - x_i\right)^2] \quad ; \quad n > 1,$$

$$x_0 = \left(-1.0; ...\right)^T$$

6. OSP Oren and Spedicato Powell Function:

$$F(x) = \left[\sum_{i=1}^{n} i\, x_i^2\right]^2, \qquad x_0 = (1.0; \ ...)^T.$$

7. Dixon Function:

$$F(x) = (1 - x_1)^2 + (1 - x_{10})^2 + \sum_{i=2}^{9} (x_i - x_{i+1}),$$

$$x_0 = (-1.0; ...)^T$$

REFERENCES

[1] Al-Assady, N.H.; Dawood, A. I. and Aziz, M.M., 1993. "Generalized CG. Based on Quasi-Quadratic Model", J. Ed. and Sc., No. 13, pp.210-218.
[2] Al-Assady, N.H., and Al-Bayati, A.Y., 1994 . "Minimization of Extended Quadratic Functions with Inexact Line searches", JOTA, Vol.82, No.1, pp.139-147.
[3] Al-Bayati, A.Y., 1993. "A New Non-Quadratic Model for Unconstrained Non-Linear Optimization", Mu'tah Journal for Research and Studies, Natural and Applied Science Series, Jordan, Vol. 8, No. 1, pp. 131-155.
[4] Al-Bayati, A.Y. and Al-Naemi, G.M., 1995."New Extended CG-Methods for Non-Linear Optimization", Mu'tah Journal for Research and Studies, Natural and Applied Science Series, Jordan, Vol. 10, No. 6, pp. 69-87.
[5] Boland, W.R.; Kamgnia, E.R. and Kowalik, J.S., 1979. "A conjugate Gradient Method Invariant to Non-Linear Scaling", Journal of Optimization Theory and Applications 27, pp.221-230.
[6] Boland, W.R. and Kowalik, J.S., 1979. "Extended Conjugate Gradient Methods with Restarts", Journal of Optimization Theory and Applications 28, pp.1-9,
[7] Bunday, B., 1984. "Basic Optimization Methods", Edward Arnold, Bedford Square, London,
[8] Dixon, L. G. W., 1975. "Conjugate Gradient Algorithm: Quadratic Termination without Linear Searches", Journal of Institute of Mathematics and its Applications, 15. pp. 9-18.
Fletcher, R. and Reeves, C.M., 1964. "Function Minimization by Conjugate Gradients", Computer Journal, 7.
[9] Fried, I., 1971. "N-step Conjugate Gradient Minimization Scheme for Non-Quadratic Functions", AIAA, J., 9.
[10] Gaoyi Wu, Yong Li, and Gonglin Yuan, 2018. "A Three-Term Conjugate Gradient Algorithm with Quadratic Convergence for Unconstrained Optimization Problems". Mathematical Problems in Engineering, Volume 2018, Article ID 4813030, 15 pages.
[11] Hestenes, M.R. and Stiefel, E., 1952. "Methods of Conjugate Gradients for Solving Linear Systems, J. Res. N.B.S., 49.
[12] Huda K. Mohammed and Nidhal H. Al-Assady, 1997. A Rational Logarithmic Model for Unconstrained Non-linear Optimization. Qatar Univ. Sc. J., 17 (2), pp. 217 – 223.
[13] Renpu Ge., 1989. A Parallel Global Optimization Algorithm for Rational Separable-Factorable Function, Journal of Applied Mathematics and Computation, Vol. 32, No.1, pp.61-72.3.
[14] Spedicato, E., 1976. A Variable Metric Method for Function Minimization Derived From Invariancy to Non-linear Scaling, JOTA, 20.
[15] Tassopoulos, A. and Storey, C., 1984a. A Conjugate Direction Method Based on a Non-Quadratic Model, JOTA, 43.
[16] Tassopoulos, A. and Storey, C., 1984b. Use of a Non-Quadratic Model in a Conjugate Gradient Searches, JOTA, 43.

[17] Yuan, G. and Hu, W., 2018. "A conjugate gradient algorithm for large-scale unconstrained optimization problems and nonlinear equations". Journal of Inequalities and Applications.2018: 113.

Authors

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