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Evaluate the Double Integrals with Continuous Integrand by Driving Numerical Method

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Abstract: *This paper interested in finding the numerical value of a double – integral with a continuous integrand . Romberg acceleration used to improve the correction terms of the derivation composite rule . these rule given values of such integral on the region of integral , these outputs be better if we take in account , the accuracy as well as the number of subintervals .*

Keywords : Double integrals , Mid-point Method , Romberg acceleration .

1-Introduction

Finding the value of double integral is difficult than that of a singular one, this is due to the former that depends on two variables as well as the requirement of continuity , singularity of the integrand and beside the singularity of the partial derivatives of the integrand . It is known that ,in single integration we deal with interval , where as in double integration we deal with region or surface , so it is easy to solve such type of integrals numerically .

In this work , combined rules are derived to evaluate the correction terms of such integrals (double integrals) where the integrands are continuous and bounded on integration interval .the Romberg acceleration convergence method .

2- Singular integral for continuous integrand

Suppose that J is defined as follows :

$$J = \int_{x_0}^{x_n} f(x)\,dx = G(h) + E_G(h) + R_G \qquad ...(1)$$

Such that $f(x)$ is a continuous integrand lies above the x-axis on the interval $[x_0, x_n]$, $G(h)$ represents largranian approximation of the value of integration, *E(h)* is a series of correction terms that can be added to *G (h)* , *J* represents the area under the curve $y = f(x)$ and above x-axis and bounded by the parallel lines $x = x_n , x = x_0$.

The general form of $G(h)$ is given by

$$G(h) = h(w_0 f_0 + w_1 f_1 + w_2 f_2 + ... + w_1 f_{n-1} + w_0 f\ ) \qquad ...(2)$$

where w_i are weighted factors , and $f_r = f(x_r)$, $h = \frac{x_n - x_0}{n}$ $x_r = x_0 + rh$, $r = 0,1,2,...,n$

to simplify formula (2) we rewrite the weights in terms of w_0 provided that $w_1 = 2(1 - w_0), w_2 = 2w_0$. Now, if we let $w_0 = 1/2$, then we will gat the trapezoidal rule and then symbolized to the function $G(h)$ by the symbol

$$T(h) = \frac{h}{2}(f_0 + 2f_1 + 2f_2 + ... + f_n)$$

And when $w_0 = 0$ we get the mid-point rule and symbolized by the symbol $M(h)$:

$$M(h) = h(f_1 + f_3 + f_5 + ... + f_{n-1})$$

where *n* is the number sub-intervals.

The general formula for the suggested method (which depend on the rules of trapezoid and the mid-point) symbolized by *Su* is

$$Su = \frac{h}{4}\left(f_0 + f_n + 2f(x_0 + (n - \frac{1}{2})h) + 2\sum_{i=1}^{n-1}(f_i + f(x_0 + (i - \frac{1}{2})h) \right)$$

Such that $h = \frac{(x_n - x_0)}{n}$, $f_i = f(x_0 + ih)$, $i = 1,2,\ldots,n$. To find the correction terms $E_G(h)$ see reference [2]

The reminder $R_G(h)$ has the form $R_G = \frac{2^n}{(2k)} B_{2k} \ h^{2k+1} \ f^{(2k)}(\lambda)$,where $x_0 \prec \lambda \prec x_n$ is Bernoulli number

3- Romberg integration

The Romberg method is an application of Ralston [9] method to find the best value for J using the trapezoid, mid-point and suggested rules.

Suppose that we applied the error formulas for two different values of h , say h_1, h_2, we find that

$$J - G(h_1) = A_G h_1^2 + B_G h_1^4 + C_G h_1^6 + \qquad ...(3)$$

$$J - G(h_2) = A_G h_2^2 + B_G h_2^4 + C_G h_2^6 + \qquad ...(4)$$

substituting $h_2 = \frac{1}{2} h_1$ in the formula (4), and solving it together with formula (3) for A_G and neglecting those terms which contain $h^4, h^6,\ldots$ from both mentioned formulas we will get

$$J \cong \left(\frac{2^2\, G(h/2) - G(h)}{(2^2 - 1)} \right). \qquad ...(5)$$

Where $h = h_1$.

Formula (5) does not represent the accurate value of integration, but it is to some extend closer to the real value of the integration than the two values $G(h/2), G(h)$, it will be symbolized by

$$G(h, h/2) = \left(\frac{2^2\, G(h/2) - G(h)}{(2^2 - 1)} \right) \qquad ....(6)$$

Thus,

$$J - G(h, h/2) = A'_G\, h_1^4 + B'_G h_1^6 + \qquad ...(7)$$

where A'_G , $B'_G,\ldots$ are constants.

In a similar way a closer value of the integration can be found using $G(h, h/2)$, and hence we get table of values of Romberg table and in general the values of this table can be calculated using

$$G = \frac{2^k\, G(h/2) - G(h)}{2^k - 1} \qquad ...(8)$$

where $k = 2,4,6,\ldots$, and G is the value of a new column of Romberg table, and $G(h/2), G(h)$ are present in the previous column, the first column of Romberg table represents the use of the mid-point method on the inner dimension x and the the rule of suggested method on the external dimension y, which symbolized by MSu and finely the value of Romberg table is determined according to the required accuracy which we call Eps, in which the relative error is as follows

$\left| \frac{G_2 - G_1}{G_1} \right| \le Eps$, $G_1 \neq 0$, where G_2 , G_1 are two approximate values of the integral in a single row of Romberg table with a method of numerical integration

4. Derivation of composite rule to calculate continuous double integrals and formulas for the error using the mid-point rule and the suggested method

Suppose that the integral I is defined as follows

$$I=\int_c^d\int_a^b f(x,y)\,dx\,dy=\int_a^b\int_c^d f(x,y)\,dy\,dx$$

can be written on the internal dimension (x) with the suggested method as follows

$$Su=\int_a^b f(x,y)\,dx=\frac{h}{4}\left(f(a,y)+f(b,y)+2f(a+(n-0.5)h,y)+2\sum_{i=1}^{n-1}\left(f(x_i,y)+f(x_0+(i-0.5)h,y)\right)\right)$$
$$+\frac{b-a}{24}h^2\frac{\partial^2 f(\lambda_1,y)}{\partial x^2}-\frac{b-a}{1440}h^4\frac{\partial^4 f(\lambda_2,y)}{\partial x^4}+\frac{61(b-a)}{60480}h^6\frac{\partial^6 f(\lambda_3,y)}{\partial x^6}+\ldots \qquad \ldots(10)$$

where $\lambda_j\in(a,b)$, $j=1,2,3,\ldots,$ $x_i=a+ih$ and $h=(b-a)/n$.

Integrating both sides of (10) numerically for y on the interval [c, d] using the mid-point method yiel

$$MSu=\int_c^d\int_a^b f(x,y)\,dxdy=\frac{h^2}{4}[\sum_{j=1}^{n} f(a,c+(j-0.5)h+\sum_{j=1}^{n} f(b,(j-0.5)h+$$
$$2\sum_{j=1}^{n} f(a+(n-0.5)h,c+(j-0.5)h)+2\sum_{j=1}^{n}\sum_{i=1}^{n-1} f(x_i,c+(j-0.5)h)+$$
$$2\sum_{j=1}^{n}\sum_{i=1}^{n-1} f(a+(i-0.5)h,c+(j-0.5)h)]+A_{MSu}h^2+B_{MSu}h^4+C_{MSu}h^6+\ldots.$$

Where $A_{MSu}, B_{MSu}, C_{MSu}$ constants whose value depends on the partial derivatives of the function $f(x,y)$, and that $i=1,2,3,\ldots.,n-1$, $x_i=a+ih$, $j=1,2,3,\ldots.,n$, $y_j=c+jh$

5-Examples and Results

1. $\int_1^2\int_1^2 \ln(x+y)\,dxdy$, the analytical value is (1.08913865206603) which is approximated to 14 decimal places.
2. $\int_3^4\int_0^1 xe^{-(x+y)}\,dxdy$ The analytical value is (0.06144772819733) near to 14 decimal places
3. $\int_2^3\int_2^3 (xy)^{\frac{1}{y}}\,dxdy$ has no analytical value.

N	MSu	K=2	K=4	K=6	K=6
1	1.09156956942644				
2	1.08975064597941	1.08914433816373			
4	1.08929192371573	1.08913901629450	1.08913866150322		
8	1.08917698716071	1.08913867497570	1.08913865222111	1.08913865207378	
16	1.08914823691532	1.08913865350019	1.08913865206848	1.08913865206606	1.08913865206603
$\int_1^2\int_1^2 \ln(x+y)dxdy$					

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n	MSu	K=2	K=4	K=6	K=8	K=10
1	0.054355070465128					
2	0.059616462391511	0.061370259700305				
4	0.060986172481264	0.061442742511182	0.061447574698574			
8	0.061332103831786	0.061447414281960	0.061447725733345	0.061447728130723		
16	0.061418807363908	0.061447708541281	0.061447728158570	0.061447728197065	0.061447728197325	
32	0.061440497067172	0.061447726968260	0.061447728196726	0.061447728197331	0.061447728197333	0.061447728197332

$$\int_{1}^{2}\int_{0}^{1} xe^{-(x+y)}dxdy$$

n	MSu	K=2	K=4	K=6	K=6	K=8	K=10
1	2.07631685979886						
2	2.08150417076353	2.08323327441842					
4	2.08277664179756	2.08320079880890	2.08319863376827				
8	2.08309245372090	2.08319772436201	2.08319751939889	2.08319750171049			
16	2.08317124588309	2.08319750993715	2.08319749564216	2.08319749526507	2.08319749523980		
32	2.08319093358619	2.08319749615388	2.08319749523500	2.08319749522854	2.08319749522839	2.08319749522838	
64	2.08319585486129	2.08319749528632	2.08319749522848	2.08319749522838	2.08319749522838	2.08319749522838	2.08319749522838

$$\int_{2}^{3}\int_{2}^{3} (xy)^{(1/y)}dxdy$$

Discussion and conclusion: The theorem was proved to solve double integrals over their given intervals.

From the tables corresponding to the rule *MSu*, we conclude that they give good results, but they need relatively high number of subintervals. But using Romberg acceleration after external adjustment, we reach better results which were closer to the real values integrals.

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