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On the Existence of Nondecreasing Solution for a Quadratic Integral Equation

Amany M. Moter

Department of Mathematic, Faculty of Basic Education, University of Kufa, Najaf, Iraq
aammmm220@gmail.com

Abstract—*There are many of quadratic integral equations type but in our paper we discuss solution for quadratic integral equation of volterra type in the space of continuous function, in another way that sufficient five conditions are given to the existence of nondecreasing solution and this is the main idea. As well, we use some of basic concepts in our considerations for example measure of noncompactness.*

Keywords— quadratic Integral equation, nondecreasing solution, measure of noncompactness, Darbo's theorem.

1. Introduction and Mathematical Preliminaries

Recently, quadratic Integral equations has been received a lot of interest and their kinds, also it applied in theories in physics as theory radiative transfer, kinetic, neutron transport, and in the traffic theory (see [2],[3],[6],[8],[9],[10],[11]).
In this paper, we will solve quadratic integral equation of Volterra type and we give an example as special case of our equation.
Now, we can refer to the symbols and their meaning used here:

1- The space $(\mathcal{B}, \|.\|)$ is Banach space on the interval $[0,1]$ and $\mathcal{B}$ is Banach space with a norm $\|.\|$
2- The set $\mathcal{X}$ is subset of $\mathcal{B}$
3- The set $\mathcal{M}_{\mathcal{B}}$ is family of all nonempty and bounded subsets of $\mathcal{B}$.
4- The set $\aleph_{\mathcal{B}}$ is subfamily consisting of all relatively compact and weakly relatively compact sets.
5- The symbol $\overline{\mathcal{X}}$ stands for the weak closure of a set $\mathcal{X}$ while $\overline{\mathcal{X}}$ denotes its closure [5].

We need to talk about collection of necessary definitions and theorems

Definition: [7] A mapping $\mu : \mathcal{M}_{\mathcal{B}} \to [0, \infty)$ is said to be a measure of noncompactness in $\mathcal{B}$ if it satisfies the following conditions:

(1) the family $\ker \mu = \{\mathcal{X} \in \mathcal{M}_{\mathcal{B}} : \mu(\mathcal{X}) = 0\}$ is nonempty and $\ker \mu \subset \aleph_{\mathcal{B}}$, where ker μ is called the kernel of the measure μ.

(2) $\mathcal{X} \subset Y \Rightarrow \mu(\mathcal{X}) \leq \mu(Y)$.

(3) $\mu\,(Conv\mathcal{X}) = \mu\,(\mathcal{X})$

(4) $\mu\,[\,\zeta\,\mathcal{X} + (1-\zeta)\,Y\,] \leq \zeta\,\mu(\mathcal{X}) + (1-\zeta)\,\mu(Y), \zeta \in [0,1]$.

(5) $If\ \mathcal{X}_n \in \mathcal{M}_{\mathcal{B}}, \mathcal{X}_n = \overline{\mathcal{X}}_n\ and\ \mathcal{X}_{n+1} \subset \mathcal{X}_n\ for$

$$n = 1,2,\ldots and\ if\ \lim_{0\to\infty} \mu(\mathcal{X}_n) = 0, \qquad then\ \mathcal{X}_\infty = \bigcap_{n=1}^{\infty} \mathcal{X}_n \neq \emptyset.$$

Also, we can define the following quantities [4]:
The modules of continuity $\omega(x,\varepsilon)$ of a function $x \in \mathcal{X}, \mathcal{X}$ is a nonempty bounded subset of the class $C(I)$, $\varepsilon > 0$ is defined as:

$$\omega(x,\varepsilon) = sup\{\,|x(r) - x(q)|: q,r \in I, |q-r| \leq \varepsilon\},$$

and the function

$$\omega_0(\mathcal{X}) = \lim_{\varepsilon\to 0} sup\{\,\omega(x,\varepsilon): x \in \mathcal{X}\},$$

Also, we can define that:

$$d(x) = sup\{\,|x(r) - x(q)| - [\,x(r) - x(q)\,]: q,r \in I, q \leq r\},$$

$$d(\mathcal{X}) = sup\{\,d(x): x \in X\,\}$$

Notice that $d(\mathcal{X}) = 0$ if and only if all functions relationship to $\mathcal{X}$ are nondecreasing on I.
Now, we can define the function μ by shape

$$\mu(\mathcal{X}) = \omega_0(\mathcal{X}) + d(\mathcal{X}).$$

The last equation can be shown that the function μ is a measure of noncompactness in the space $C(I)$.

Theorem1 [1]**:** (Darbo)
Let $\mathcal{Q}$be a nonempty, bounded, closed and convex subset of $\mathcal{B}$ and let $\mathcal{A}: \mathcal{Q} \to \mathcal{Q}$ be a continuous transformation which is a contraction with respect to the measure of noncompactness μ, i.e. there exists $\wp \in [0,1)$ such that

$$\mu(\mathcal{A}(X)) \leq \wp\mu(X),$$

for any nonempty subset X of $\mathcal{B}$. Then $\mathcal{A}$ has at least one fixed point in the set $\mathcal{Q}$.

We will use all definitions and Darbo theorem to study the following quadratic integral equation of Volterra type under next five hypotheses:

$$\mathcal{H}x(q) = \Gamma(q) + \tau.\psi(q,x(q)) \int_0^q \Omega(q,r)\mho(q,r,x(r))dr, \ q \in [0,1] \quad (*)$$

Assuming the following hypotheses:

1- Let $\Gamma: [0,1] \to [0,1]$ is continuous, nondecreasing and nonnegative on $[0,1]$.
2- Let $\psi: [0,1] \times \imath \to \mathbb{R}_+$ is continuous on $[0,1] \times \imath$ and satisfies the Lipschitz condition and $L: \mathbb{R}_+ \to \mathbb{R}_+$ is bounded and nondecreasing function such that
$$|\psi(q_2,x(q_2)) - \psi(q_2,x(q_1))| \leq L|x(q_2) - x(q_1)|$$
3- Let $\Omega: [0,1] \times [0,1] \to \mathbb{R}$ is continuous function such that $\int_0^q |\Omega(q,r)|dr \leq c$, for all $q \in [0,1]$ where $c > 0$.
4- Let $\mho: [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous function and there exist a functions $\varphi, \phi: \mathbb{R}_+ \to \mathbb{R}_+$ are continuous on $\mathbb{R}_+$ with $\phi(0) = 0$ such that
$$|\mho(q,r,x(r)) - \mho(q,r,y(r))| \leq \varphi(q)\phi(|x-y|)$$
For all $r, q \in \mathbb{R}_+$ such that $r \leq q$ and $x, y \in \mathbb{R}_+$.
Also
$If\ |q_2 - q_1| \leq \delta$ than $|\mho(q_2,r,x(r)) - \mho(q_1,r,y(r))| \leq \varepsilon$.
5- The inequality
$$\|\Gamma\| + c\alpha(L\|x\| + \psi)(\varphi(q)\phi(\|x\|) + \mho) < \gamma$$
But it has a nonnegative solution γ_0 such that
$$c\alpha(L\gamma + \psi)(\varphi(q)\phi(\gamma) + \mho) < 1$$

Remark1**:** by assumption (2) of the above hypotheses we can say that the function ψ is satisfies the following inequality:

$$|\psi(q_2,x(q_2)) - \psi(q_1,x(q_1))|$$
$$\leq |\psi(q_2,x(q_2)) - \psi(q_2,x(q_1)) + \psi(q_2,x(q_1)) - \psi(q_1,x(q_1))|$$
$$\leq L|x(q_2) - x(q_1)| + \omega(\psi,\varepsilon)$$

Where

$$\omega(\psi,\varepsilon) = su\,p\{|\psi(q_2,x) - \psi(q_1,x)|, q_1, q_2 \ \in I, |q_2 - q_1| \leq \varepsilon\}.$$

Remark2: by assumption (4) of the above hypotheses, we find the following evaluation:

$$|\mho(q,r,x(r))| = |\mho(q,r,x(r)) - \mho(q,r,0) + \mho(q,r,0)|$$
$$\leq |\mho(q,r,x(r)) - \mho(q,r,0)| + |\mho(q,r,0)|$$
$$\leq \ \varphi(q)\phi(|x|) + \mho$$

That imply

$$|\mho(q,r,x(r))| \leq \ \varphi(q)\phi(|x|) + \mho$$

As well

$$|\mho(q_1,r,x(r))| \leq \varphi(q_1)\phi(|x|) + \mho_1$$

And, let us suppose that

$$|\mho(q_2,r,x(r)) - \mho(q_1,r,x(r))| = \omega(\mho,\varepsilon)$$

Where $\omega(\mho,\varepsilon) = sup\{|\mho(q_2,x) - \mho(q_1,x)|, q_1, q_2 \ \in I, |q_2 - q_1| \leq \varepsilon\}$

Thereafter, similarly way of the equation above and by assumption (2) we get the following estimate:

$$|\psi(q,x(q))| \leq L|x| + \psi.$$

Remark3:

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Consider the operator $\mathcal{J}$ transforms the ball B_{γ_0} into itself, where

$$B_{\gamma_0} = \{ x \in B_{\gamma_0} : x(q) \geq 0, q \in [0,1] \}$$

We not that B_{γ_0} is nonempty, bounded and closed, to prove B_{γ_0} is convex for if $x, y \in B_{\gamma_0}$ then $\|x\| \leq \gamma, \|y\| \leq \gamma$.
Now for $0 \leq \xi \leq 1$, we take

$$\begin{aligned}\|\xi x + (1-\xi)y\| &= \xi\|x\| + (1-\xi)\|y\| \\ &\leq \xi\gamma + \gamma - \xi\gamma \\ &= \gamma\end{aligned}$$

Remark4:
In this paper, we will refer to that there exist a small number $\alpha > 0$, suchthat $|\tau| < \alpha$ for all $\tau \in \mathbb{R}$.

2. The main existence theorem

In this section, we discuss our equation $(*)$ to finding nondecreasing solution, it is nondecreasing on the interval $[0,1]$ that means the equation $(*)$ has at least one fixed point solution x belongs to $C(I)$. To do that, we starting by the following theorem:

Theorem2: the equation $(*)$ has at least one solution $x \in C[0,1]$, if the hypotheses $(1-5)$ be satisfied.

Proof:

We consider that:

$$\mathcal{H}x(q) = \Gamma(q) + \tau.\psi\big(q, x(q)\big)\int_0^q \Omega(q,r)\,\mho\big(q,r,x(r)\big)dr, \quad q \in [0,1]$$

The operator defined by $\mathcal{H}: C[0,1] \to C[0,1]$.

$$\begin{aligned}|\mathcal{H}x(t)| &= \left|\Gamma(q) + \tau.\psi\big(q,x(q)\big)\textstyle\int_0^q \Omega(q,r)\,\mho\big(q,r,x(r)\big)dr\right| \\ &\leq |\Gamma(q)| + |\tau|\left|\psi\big(q,x(q)\big)\right|\left|\textstyle\int_0^q \Omega(q,r)dr\right|\left|\mho\big(q,r,x(r)\big)\right|\end{aligned}$$

By hypothesis $(1-3)$ and remark2 we obtain

$$|\mathcal{H}x(q)| \leq \|\Gamma\| + c\alpha(L\|x\| + \psi)(\varphi(q)\phi(\|x\|) + \mho)$$

Via last estimate, remark (3) and hypothesis (5) we have

$$|\mathcal{H}x(q)| \leq \|\Gamma\| + c\alpha(L\|x\| + \psi)(\varphi(q)\phi(\|x\|) + \mho) < \gamma$$

But if $\|x\| < \gamma_0$ and $\|y\| < \gamma_0$, we get that $\mathcal{H}$ transforms the ball B_{γ_0} into itself.
Also, $\mathcal{H}$ transforms continuously the ball $B_{\gamma_0}^{+}$ into itself.
To prove that $\mathcal{H}$ is continuous on the set Q, we chooses $\delta > 0$, let$x, y \in Q$ such that $\|x - y\| \leq \delta \ \forall\, t \in I$ and by assumption$(2-4)$, we achieve

$$|\mathcal{H}x(q) - \mathcal{H}y(q)| =$$
$$\Gamma(q) + \tau.\psi\big(q,x(q)\big)\int_0^q \Omega(q,r)\,\mho\big(q,r,x(r)\big)dr - \Gamma(q) - \tau.\psi\big(q,y(q)\big)\int_0^q \Omega(q,r)\,\mho\big(q,r,y(r)\big)dr$$
$$=$$
$$\Big|\tau.\psi\big(q,x(q)\big)\int_0^q \Omega(q,r)\,\mho\big(q,r,x(r)\big)dr - \tau.\psi\big(q,x(q)\big)\int_0^q \Omega(q,r)\,\mho\big(q,r,y(r)\big)dr +$$
$$\tau.\psi\big(q,x(q)\big)\int_0^q \Omega(q,r)\,\mho\big(q,r,y(r)\big)dr - \tau.\psi\big(q,y(q)\big)\int_0^q \Omega(q,r)\,\mho\big(q,r,y(r)\big)dr\Big|$$

$$\leq$$
$$\Big|\tau.\psi\big(q,x(q)\big)\Big(\int_0^q \Omega(q,r)\big(\mho\big(q,r,x(r)\big) - \mho\big(q,r,y(r)\big)\big)dr\Big) + \tau.\big(\psi\big(q,x(q)\big) -$$
$$\psi\big(q,y(q)\big)\big)\int_0^q \Omega(q,r)\,\mho\big(q,r,y(r)\big)dr\Big|$$
$$\leq |\tau|L\|x-y\|\left|\int_0^q \Omega(q,r)\,dr\right|\left|\mho\big(q,r,x(r)\big) - \mho\big(q,r,y(r)\big)\right|$$
$$\leq \alpha L\|x-y\|c\varphi(q)\phi(\|x-y\|)$$

At least the operator $\mathcal{H}$ is continuous on the set Q.
Next, let $q_2, q_1 \in C[0,1], q_1 \leq q_2$ such that $|q_2 - q_1| < \epsilon$, where $\epsilon > 0$ is an arbitrary small positive number, we have:

$$|\mathcal{H}x(q_2) - \mathcal{H}x(q_1)|$$
$$= \left|\Gamma(q_2) + \tau.\psi\big(q_2,x(q_2)\big)\int_0^{q_2} \Omega(q_2,r)\,\mho\big(q_2,r,x(r)\big)dr - \Gamma(q_1) - \tau.\psi\big(q_1,x(q_1)\big)\int_0^{q_1} \Omega(q_1,r)\,\mho\big(q_1,r,x(r)\big)dr\right|$$

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$\leq |\Gamma(q_2) - \Gamma(q_1)| +$
$\left|\tau.\psi\big(q_2,x(q_2)\big)\int_0^{q_2}\Omega(q_2,r)\,\mho\big(q_2,r,x(r)\big)dr - \tau.\psi\big(q_2,x(q_2)\big)\int_0^{q_2}\Omega(q_2,r)\,\mho\big(q_1,r,x(r)\big)dr +\right.$
$\tau.\psi\big(q_2,x(q_2)\big)\int_0^{q_2}\Omega(q_2,r)\,\mho\big(q_1,r,x(r)\big)dr - \tau.\psi\big(q,x(q_2)\big)\int_0^{q_2}\Omega(q_1,r)\,\mho\big(q_1,r,x(r)\big)dr +$
$\tau.\psi\big(q_2,x(q)\big)\int_0^{q_2}\Omega(q_1,r)\,\mho\big(q_1,r,x(r)\big)dr - \tau.\psi\big(q_2,x(q_2)\big)\int_0^{q_1}\Omega(q_1,r)\,\mho\big(q_1,r,x(r)\big)dr +$
$\left.\tau.\psi\big(q_2,x(q_2)\big)\int_0^{q_1}\Omega(q_1,r)\,\mho\big(q_1,r,x(r)\big)dr - \tau.\psi\big(q_1,x(q_1)\big)\int_0^{q_1}\Omega(q_1,r)\,\mho\big(q_1,r,x(r)\big)dr\right|$
$\leq |\Gamma(q_2) - \Gamma(q_1)| + |\tau|\big|\psi\big(q_2,x(q_2)\big)\big|\int_0^{q_2}|\Omega(q_2,r)|\,\big|\mho\big(q_2,r,x(r)\big) - \mho\big(q_1,r,x(r)\big)\big|dr + |\tau|\big|\psi\big(q_2,x(q_2)\big)\big|\int_0^{q_2}|\Omega(q_2,r) - \Omega(q_1,r)|\,\big|\mho\big(q_1,r,x(r)\big)\big|dr$
$+|\tau|\big|\psi\big(q_2,x(q_2)\big)\big|\int_{q_1}^{q_2}|\Omega(q_1,r)|\,\big|\mho\big(q_1,r,x(r)\big)\big|dr + |\tau|\big|\psi\big(q_2,x(q_2)\big) - \psi\big(q_1,x(q_1)\big)\big|\int_0^{q_1}|\Omega(q_1,r)|\,\big|\mho\big(q_1,r,x(r)\big)\big|dr$

Here, we can use remark $(1+2)$ and hypothesis $(2-4)$ to get that
$\leq \varepsilon_1 + \alpha L(|x| + \psi_2)(c\omega(\mho,\varepsilon) + \varepsilon_2 q_2\,(\varphi(q_1)\phi(|x|) + \mho_1) + \int_{q_1}^{q_2}|\Omega(q_1,s)|\,dr(\varphi(q_1)\phi(|x|) + \mho_1) + \alpha c((L|x(q_2) - x(q_1)| + \omega(\psi,\varepsilon))((\varphi(q_1)\phi(|x|) + \mho_1)$
So,
$|\mathcal{H}x(t_2) - \mathcal{H}x(t_1)| \leq \ \alpha L(c(\varphi(t_1)\phi(|x|) + \mho_1)\omega(x,\varepsilon)$
$\leq \ \mathcal{A}\omega(X)$
Where $\alpha L(c(\varphi(t_1)\phi(|x|) + \mho_1) = \mathcal{A}$

Next step is least step to complete proof
$|\mathcal{H}x(q_2) - \mathcal{H}x(q_1)| - [\mathcal{H}x(q_2) - \mathcal{H}x(q_1)]$
$= \left|\Gamma(q_2) + \tau.\psi\big(q_2,x(q_2)\big)\int_0^{q_2}\Omega(q_2,s)\,\mho\big(q_2,r,x(r)\big)dr - \Gamma(q_1) - \tau.\psi\big(q_1,x(q_1)\big)\int_0^{q_1}\Omega(q_1,r)\,\mho\big(q_1,r,x(r)\big)dr\right|$
$-\left[\Gamma(q_2) + \tau.\psi\big(q_2,x(q_2)\big)\int_0^{q_2}\Omega(q_2,r)\,\mho\big(q_2,r,x(r)\big)dr - \Gamma(q_1) - \tau.\psi\big(q_1,x(q)\big)\int_0^{q_1}\Omega(q_1,r)\,\mho\big(q,r,x(r)\big)dr\right]$

$\leq |\Gamma(q_2) - \Gamma(q_1)| - [\Gamma(q_2) - \Gamma(q_1)] \ + |\tau|\big|\psi\big(q_2,x(q_2)\big)\big|\int_0^{q_2}|\Omega(q_2,r)|\,\big|\mho\big(q_2,r,x(r)\big) - \mho\big(q_1,r,x(r)\big)\big|dr$
$+|\tau|\big|\psi\big(q_2,x(q_2)\big)\big|\int_0^{q_2}|\Omega(q_2,r) - \Omega(q_1,r)|\,\big|\mho\big(q_1,r,x(r)\big)\big|dr$
$+|\tau|\big|\psi\big(q_2,x(q_2)\big)\big|\int_{q_1}^{q_2}|\Omega(q_1,r)|\,\big|\mho\big(q_1,r,x(r)\big)\big|dr$
$+ |\tau|\big|\psi\big(q_2,x(q_2)\big) - \psi\big(q_1,x(q_1)\big)\big|\int_0^{q_1}|\Omega(q_1,r)|\,\big|\mho\big(q_1,r,x(r)\big)\big|dr$
$- \left[\tau.\psi\big(q_2,x(q_2)\big)\int_0^{q_2}\Omega(q_2,r)\left(\mho\big(q_2,r,x(r)\big) - \mho\big(q_1,r,x(r)\big)\right)dr\right]$
$-\left[\tau.\psi\big(q_2,x(q_2)\big)\int_0^{q_2}(\Omega(q_2,r) - \Omega(q_1,r))\mho\big(q_1,r,x(r)\big)\,dr\right]$
$- \left[\tau.\psi\big(q_2,x(q_2)\big)\int_{q_1}^{q_2}\Omega(q_1,r)\,\mho\big(q_1,r,x(r)\big)dr\right]$
$-\left[\tau.(\psi\big(q_2,x(q_2)\big) - \psi\big(q_1,x(q_1)\big))\int_0^{q_1}\Omega(q_1,r)\mho\big(q_1,r,x(r)\big)\,dr\right]$
$\leq \omega(\Gamma,\varepsilon)$
$+|\tau|\big|\psi\big(q_2,x(q_2)\big) - \psi\big(q_1,x(q_1)\big)\big|\int_0^{q_1}|\Omega(q_1,r)|\big|\mho\big(q_1,r,x(r)\big)\big|\,dr$
$+2|\tau|\big|\psi\big(q_2,x(q_2)\big)\big|\int_0^{q}|\Omega(q_2,r)|\,\big|\mho\big(q_2,r,x(r)\big) - \mho\big(q_1,r,x(r)\big)\big|dr$
$+2|\tau|\big|\psi\big(q_2,x(q_2)\big)\big|\int_0^{q_2}|\Omega(q_2,r) - \Omega(q_1,r)|\,\big|\mho\big(q_1,r,x(r)\big)\big|dr$
$+2|\tau|\big|\psi\big(q_2,x(q_2)\big)\big|\int_{q_1}^{q_2}|\Omega(q_1,r)|\,\big|\mho\big(q_1,r,x(r)\big)\big|dr$
Using assumption (2+4) and remark 2, So we have
$|\mathcal{H}x(q_2) - \mathcal{H}x(q_1)| - [\mathcal{H}x(q_2) - \mathcal{H}x(q_1)]$
$\leq \alpha cL|x(q_2) - x(q_1)|(\varphi(q_1)\phi(|x|) + \mho_1) + 2\alpha L(|x| + \psi_2)(c\varepsilon + q_2\varepsilon\,(\varphi(q_1)\phi(|x|) + \mho_1) + \int_{q_1}^{q_2}|\Omega(q_1,r)|\,dr(\varphi(q_1)\phi(|x|) + \mho_1))$
This implies

$$d(\mathcal{H}x) \leq \ \alpha cLd(\psi x)(\varphi(q_1)\phi(|x|) + \mho_1)$$

So,

$$d(X) \leq \alpha c(\varphi(q_1)\phi(|x|) + \mho_1)Ld(X)$$

Therefor,

$$\mu(\mathcal{H}x) \leq \alpha cL(\varphi(q_1)\phi(|x|) + \mho_1)\mu(X)$$

■

3. Example

Example: let the following quadratic integral equation

$$x(q) = 3q + \mu(5q^2 + 10x(q)) \int_0^q \frac{r}{q+r}(qx(q) + \frac{\sqrt{q-r}}{t^2})$$

The above equation satisfy hypotheses (1-4), we can be easily seen that
$\Gamma(q) = 3q$ is continuous , nondecreasing and nonnegative on $[0,1]$.
To prove that the equation ψ satisfies the hypotheses (2) we assume $q_1, q_2 \in [0,1]$ such that $q_1 \leq q_2$ and $x \in C[0,1]$, also it we obtain

$$\begin{aligned}
&\left|\psi\big(q_2, x(q_2)\big) - \psi\big(q_1, x(q_1)\big)\right| - \left[\psi\big(q_2, x(q)\big) - \psi\big(q_1, x(q)\big)\right] \\
&\qquad = |5q_2^2 + 10x(q_2) - 5q_2^2 - 10x(q_1) + 5q_2^2 + 10x(q_1) - 5q_1^2 + 10x(q_1)| \\
&\qquad - [5q_2^2 + 10x(q_2) - 5q_2^2 - 10x(q_1) + 5q_2^2 + 10x(q_1) - 5q_1^2 + 10x(q_1)] \\
&\leq 10|x(q_2) - x(q_1)| + 5|q_2^2 - q_1^2| - 10[x(q_2) - x(q_1)] - 5|q_2^2 - q_1^2| \\
&= 10\{|x(q_2) - x(q_1)| - [x(q_2) - x(q_1)]\} \\
&= 10d(X)
\end{aligned}$$

And we not that the equation ψ satisfies the Lipschitz condition on $\mathbb{R}_+$, also $\Omega(q,r) = \frac{r}{q+r}$ is continuous and nondecreasing function on $[0,1] \times [0,1]$, for each variable q and r.
The following function satisfies hypotheses (1)

$$\mho\big(q, r, x(r)\big) = rx(q) + \frac{\sqrt{q-r}}{q^2}$$

For all $r, q \in \mathbb{R}_+$ such that $r \leq q$ and $x, y \in \mathbb{R}_+$.

$$\begin{aligned}
\left|\mho\big(q, r, x(r)\big) - \mho\big(q, r, y(r)\big)\right| &= \left|qx(q) + \frac{\sqrt{q-r}}{q^2} - qy(q) - \frac{\sqrt{q-r}}{q^2}\right| \\
&\leq q|x(q) - y(q)|
\end{aligned}$$

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Authors

Amny M. Moter, lecturer