International Journal of Academic Multidisciplinary Research (IJAMR)
ISSN: 2643-9670
Vol. 3 Issue 9, September – 2019, Pages: 35-63

Internal Resonances in 3-D Nonlinear Dynamical System

Mousa A. ALshawish[1], Ahmed N. Alshembari[2]

[1]Al Azhar University, Gaza, Palestine
mousa.alshawish@gmail.com
[2]Technische Universität Kaiserslautern, Germany

Abstract: *This paper is concerned with the three dimensional motion of a nonlinear dynamical system. The motion is described by nonlinear partial differential equation, which is converted by Galerkin method to three-dimensional ordinary differential equations. The three dimensional free vibration of the beam, are solved analytically and numerically by the multiple time scales perturbation technique and the Runge-Kutta fourth order method. Phase plane technique and frequency response equations are used to investigate he stability of the system and the effects of the parameters of the system, respectively.*

Keywords—Galerkin method; Resonances; Nonlinearities; Runge-Kutta method

1. INTRODUCTION

Problems including nonlinear differential equations are very different, and approaches of solutions or analysis are problem dependent. Nonlinear systems are motivating for engineers, mathematicians and physicists because most physical systems are naturally nonlinear. Yaman [1] studied the sub-combination internal resonance of a uniform cantilever beam of varying orientation with a tip mass under vertical base excitation. The Euler–Bernoulli beam theory used to derive the governing nonlinear partial differential equation. Huang, Fung and Lin [2] studied the dynamic stability of a moving string in three-dimensional vibration. Perngjin and Nayfeh [3] studied three nonlinear integro-differential equations of motion. The analysis focuses on the case of primary resonance of the first in-plane flexural mode when its frequency is approximately twice the frequency of the first out-of-plane flexural-torsional mode. Hegazy [4-5] applied the method of multiple time scales to investigate the response of nonlinear mechanical systems with internal and external resonances. The stability of vibrating systems is investigated by applying both the frequency response equation and the phase plane methods. The numerical solutions are focused on both the effects of the different parameters and the behavior of the system at the considered resonance cases. Srinil, Rega and Chucheepsakul [6] investigated the nonlinear characteristics in the large amplitude three-dimensional free vibrations of inclined sagged elastic cables. Sadri and Younesian [7] analytically studied nonlinear forced vibration of a plate-cavity system. In order to solve the nonlinear equations of plate-cavity system, multiple scales method was employed. Closed form expressions were obtained for the frequency-amplitude relationship in different resonance conditions. Zhang ,et al [8] studied the steady-state periodic response of the forced vibration for an axially moving viscoelastic beam in the supercritical speed range. For this motion, the model is cast in the standard form of continuous gyroscopic systems. Lee and Perkins [9] investigated internal resonances in suspended elastic cables driven by planar excitation. L. Cveticanin [10] developed various approximate analytical methods for obtaining solutions for strong non-linear differential equations in a complex function. The method of harmonic balance, the method of Krylov-Bogoliubov and the elliptic perturbation method were studied.

2. EQUATIONS OF MOTHION

The nonlinear partial differential equation governing the flexural deflection $u(x,t)$ of the beam, subject to harmonic axial excitation $p = p_0 - p_1 \cos\Omega t$, is given by [11, 12]. (1)

$$m\frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} + EI\frac{\partial^4 u}{\partial x^4} + (p_0 + p_1\cos\Omega t)\frac{\partial^2 u}{\partial x^2} + \frac{3}{2}(p_0 + p_1\cos\Omega t)\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial x^2} + EI\begin{bmatrix}\frac{27}{2}\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial^2 u}{\partial x^2}\right)^3 - 3\left(\frac{\partial^2 u}{\partial x^2}\right)^3 \\ -3\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^4 u}{\partial x^4} + \frac{9}{4}\left(\frac{\partial u}{\partial x}\right)^4\frac{\partial^4 u}{\partial x^4}\end{bmatrix} = 0$$

Under the following boundary conditions:
(2)

$$u(x) = 0 \text{ and } \frac{\partial u}{\partial x} = 0 \text{ at } x = 0,\ x = L.$$

Equation (1) can be converted to a three dimensional nonlinear ordinary differential equations applying the method of Galerkins and using the expression (3)

$$u(x,t) = g(t)\sin\left(\frac{\pi x}{L}\right) + h(t)\sin\left(\frac{2\pi x}{L}\right) + k(t)\sin\left(\frac{3\pi x}{L}\right)$$

Into equation (1). Then we have

(4)

$$g'' + \omega_1^2 g + \varepsilon^2(\eta_1 h^2 g + \eta_2 g k^2 + \eta_3 h^2 k + \eta_4 g^3 + \eta_5 g^2 k)$$
$$+\varepsilon(\alpha g' + \eta_6 g k^4 + \eta_7 g^2 k^3 + \eta_8 h^2 k^3 + \eta_9 h^2 g^3 + \eta_{10} h^4 k + \eta_{11} h^4 g$$
$$+\eta_{12} g^3 k^2 + \eta_{13} g^5 + \eta_{14} h^2 k^2 g + \eta_{15} g^4 k + \eta_{16} h^2 g^2 k) = 0$$

(5)

$$h'' + \omega_2^2 h + \varepsilon^2(\lambda_1 hgk + \lambda_2 h^3 + \lambda_3 hk^2 + \lambda_4 hg^2)$$
$$+\varepsilon(\beta h' + \lambda_5 hg^4 + \lambda_6 h^3 k^2 + \lambda_7 h^5 + \lambda_8 hgk^3 + \lambda_9 hk^4$$
$$+\lambda_{10} h^3 g^2 + \lambda_{11} gkh^3 + \lambda_{12} hg^2 k^2 + \lambda_{13} hkg^3) = 0,$$

(6)

$$+\tau_{12} gh^2 k^2 + \tau_{13} kh^2 g^2 + \tau_{14} g^5 + \tau_{15} h^4 k + \tau_{16} h^2 k^3) = 0.$$

3. PURTURBATION SOLUTION

The method of multiple scales is applied to determine an approximate solution for the differential equations (4-6). Assuming that g, h and k are in the forms (7):

$$g(T_0,T_1) = g_0(T_0,T_1) + \varepsilon g_1(T_0,T_1) + \ldots,$$
$$h(T_0,T_1) = h_0(T_0,T_1) + \varepsilon h_1(T_0,T_1) + \ldots,$$
$$k(T_0,T_1) = k_0(T_0,T_1) + \varepsilon k_1(T_0,T_1) + \ldots,$$

Where $T_0 = t,\ T_1 = \varepsilon T_0 = \varepsilon t.$

The time derivatives are written as (8):

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \ldots,$$
$$\frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \ldots,$$

Where (9):

$$D_0 = \frac{\partial}{\partial T_0},\ D_1 = \frac{\partial}{\partial T_1}.$$

Substituting equations. (7-9) into equations. (4-6) and equating coefficients of same powers of ε yields: $o(\varepsilon^0)$:

(10): $\left(D_0^2 + \omega_1^2\right) g_0 = 0$

(11): $\left(D_0^2 + \omega_2^2\right) h_0 = 0.$

(12): $\left(D_0^2 + \omega_1^2\right) k_0 = 0$

$o(\varepsilon^1)$:

(13):

$$\left(D_0^2 + \omega_1^2\right) g_1 = -2D_0 D_1 g_0 - \alpha D_0 g_0 - \eta_6 k_0^4 g_0 - \eta_7 k_0^3 g_0^2 - \eta_8 k_0^3 h_0^2 - \eta_9 g_0^3 h_0^2$$
$$-\eta_{10} h_0^4 k_0 - \eta_{11} h_0^4 g_0 - \eta_{12} g_0^3 k_0^2 - \eta_{13} g_0^5 - \eta_{14} h_0^2 k_0^2 g_0 - \eta_{15} g_0^4 k_0 - \eta_{16} h_0^2 g_0^2 k_0.$$

(14):

International Journal of Academic Multidisciplinary Research (IJAMR)
ISSN: 2643-9670
Vol. 3 Issue 9, September – 2019, Pages: 35-63

$$\left(D_0^{\ 2}+\omega_2^{\ 2}\right)h_1=$$

$$-2D_0D_1h_0-\beta D_0h_0-\lambda_5 g_0^{\ 4}h_0-\lambda_6 h_0^{\ 3}k_0^{\ 2}-$$

$$-\lambda_7 h_0^{\ 5}-\lambda_8 k_0^{\ 3}g_0h_0-\lambda_9 k_0^{\ 4}h_0-\lambda_{10}h_0^{\ 3}g_0^{\ 2}$$

$$-\lambda_{11}h_0^{\ 3}g_0k_0-\lambda_{12}g_0^{\ 2}k_0^{\ 2}h_0-\lambda_{13}g_0^{\ 3}h_0k_0$$

(14):

$$\left(D_0^{\ 2}+\omega_3^{\ 2}\right)k_1=$$

$$-2D_0D_1k_0-\delta D_0k_0-\tau_6 k_0^{\ 5}-\tau_7 g_0^{\ 4}k_0$$

$$-\tau_8 k_0^{\ 3}g_0^{\ 2}-\tau_9 g_0^{\ 3}k_0^{\ 2}$$

(15):

$$-\tau_{10}g_0^{\ 3}h_0^{\ 2}-\tau_{11}h_0^{\ 4}g_0-\tau_{12}h_0^{\ 2}k_0^{\ 2}g_0$$

$$-\tau_{13}g_0^{\ 2}h_0^{\ 2}k_0-\tau_{14}g_0^{\ 5}-\tau_{15}h_0^{\ 4}k_0-\tau_{16}h_0^{\ 2}k_0^{\ 3}$$

The solution of equations (10-12) is expressed as:

(16):

$$g_0(T_0,T_1)=A(T_1)e^{i\omega_1T_0}+\bar{A}(T_1)e^{-i\omega_1T_0}.$$

(17):

$$h_0(T_0,T_1)=B(T_1)e^{i\omega_2T_0}+\bar{B}(T_1)e^{-i\omega_2T_0}.$$

(18):

$$k_0(T_0,T_1)=C(T_1)e^{i\omega_3T_0}+\bar{C}(T_1)e^{-i\omega_3T_0}.$$

Where A,B,C are complex functions in T_1.

Substituting equations (16-18) into equations (13-15), we get:

(19)

$$\left(D_0^{\ 2}+\omega_1^{\ 2}\right)g_1=-2i\,\omega_1A'e^{i\omega_1T_0}-\alpha i\,\omega_1Ae^{i\omega_1T_0}$$
$$-\eta_6C^4Ae^{i(4\omega_3+\omega_1)T_0}-\eta_6\bar{C}^4Ae^{i(-4\omega_3+\omega_1)T_0}$$

$$+\left(-4\eta_6C^3\bar{C}A-2\eta_{14}C^2AB\bar{B}\right)e^{i(2\omega_3+\omega_1)T_0}$$
$$+\left(-4\eta_6\bar{C}^3CA-2\eta_{14}\bar{C}^2AB\bar{B}\right)e^{i(2\omega_3+\omega_1)T_0}$$

$$+\begin{pmatrix}2i\,\omega_1A'-\alpha i\,\omega_1A-6\eta_6AC^2\bar{C}^2-\\6\eta_9B\bar{B}A^2\bar{A}-6\eta_{14}B^2\bar{B}^2A\\-6\eta_{12}B\bar{B}A^2\bar{A}-4\eta_{14}C\bar{C}B\bar{B}A\end{pmatrix}e^{i\omega_1T_0}$$

$$-\eta_7C^3A^2e^{i(3\omega_3+2\omega_1)T_0}$$

$$-\eta_7\bar{C}^3A^2e^{i(-3\omega_3+2\omega_1)T_0}$$

$$+\left(-2\eta_7C^3A\bar{A}-2\eta_8C^3B\bar{B}\right)e^{3i\omega_3T_0}$$

$$+\begin{pmatrix}-3\eta_7C^2\bar{C}A^2-4\eta_{15}CA^3\bar{A}\\-2\eta_{16}CB\bar{B}A^2\end{pmatrix}e^{i(\omega_3+2\omega_1)T_0}$$

$$+\begin{pmatrix}-3\eta_7\bar{C}^2CA^2-4\eta_{15}\bar{C}A^3\bar{A}\\-2\eta_{16}\bar{C}B\bar{B}A^2\end{pmatrix}e^{i(-\omega_3+2\omega_1)T_0}$$

$$+\begin{pmatrix}-6\eta_7C^2\bar{C}A\bar{A}-6\eta_8C^2\bar{C}B\bar{B}-\\6\eta_{10}B^2\bar{B}^2C-6\eta_{15}CA^2\bar{A}^2-4\eta_{16}CB\bar{B}A\bar{A}\end{pmatrix}e^{i\omega_3T_0}$$

$$-\eta_8C^3B^2e^{i(3\omega_3+2\omega_2)T_0}$$

$$-\eta_8\bar{C}^3B^2e^{i(-3\omega_3+2\omega_2)T_0}+$$

$$\begin{pmatrix}-3\eta_8C^2\bar{C}B^2-4\eta_{10}B^3\bar{B}\bar{C}\\-2\eta_{16}CB^2A\bar{A}\end{pmatrix}e^{i(\omega_3+2\omega_2)T_0}$$

$$+\left(-3\eta_8\bar{C}^2CB^2-4\eta_{10}B^3\bar{B}\bar{C}-2\eta_{16}\bar{C}B^2A\bar{A}\right)e^{i(-\omega_3+2\omega_2)T_0}$$

$$+\left(-\eta_9B^2A^3-\eta_{12}B^2A^3\right)e^{i(2\omega_2+3\omega_1)T_0}$$

International Journal of Academic Multidisciplinary Research (IJAMR)
ISSN: 2643-9670
Vol. 3 Issue 9, September – 2019, Pages: 35-63

$$+\left(-\eta_9\bar{B}^2A^3-\eta_{12}\bar{B}^2A^3\right)e^{i(-2\omega_2+3\omega_1)T_0}$$
$$+\begin{pmatrix}-3\eta_9B^2A^2\bar{A}-4\eta_{11}B^3\bar{B}\bar{A}-3\eta_{12}B^2A^2\bar{A}\\-2\eta_{14}C\bar{C}B^2A\end{pmatrix}e^{i(2\omega_2+\omega_1)T_0}$$
$$+\begin{pmatrix}-3\eta_9\bar{B}^2A^2\bar{A}-4\eta_{11}\bar{B}^3B\bar{A}-\\3\eta_{12}\bar{B}^2A^2\bar{A}-2\eta_{14}\bar{C}C\bar{B}^2A\end{pmatrix}e^{i(-2\omega_2+\omega_1)T_0}$$
$$+\left(-2\eta_9A^3B\bar{B}-2\eta_{12}A^3B\bar{B}\right)e^{3i\omega_1T_0}$$
$$-\eta_{10}CB^4e^{i(4\omega_2+\omega_3)T_0}$$
$$-\eta_{10}\bar{C}B^4e^{i(4\omega_2-\omega_3)T_0}$$
$$-\eta_{11}B^4Ae^{i(4\omega_2+\omega_1)T_0}$$
$$-\eta_{11}\bar{B}^4Ae^{i(4\omega_2+\omega_1)T_0}$$
$$-\eta_{13}A^5e^{5i\omega_1T_0}-\eta_{14}C^2B^2Ae^{i(2\omega_3+2\omega_2+\omega_1)T_0}$$
$$-\eta_{14}\bar{C}^2\bar{B}^2Ae^{i(-2\omega_3-2\omega_2+\omega_1)T_0}$$
$$-\eta_{14}C^2\bar{B}^2Ae^{i(2\omega_3-2\omega_2+\omega_1)T_0}$$
$$-\eta_{14}\bar{C}^2B^2Ae^{i(-2\omega_3+2\omega_2+\omega_1)T_0}$$
$$-\eta_{15}CA^4e^{i(\omega_3+4\omega_1)T_0}$$
$$-\eta_{15}\bar{C}A^4e^{i(-\omega_3+4\omega_1)T_0}$$
$$-\eta_{16}CB^2A^2e^{i(\omega_3+2\omega_2+2\omega_1)T_0}$$
$$-\eta_{16}\bar{C}\bar{B}^2A^2e^{i(-\omega_3-2\omega_2+2\omega_1)T_0}$$
$$-\eta_{16}C\bar{B}^2A^2e^{i(\omega_3-2\omega_2+2\omega_1)T_0}$$
$$-\eta_{16}\bar{C}B^2A^2e^{i(-\omega_3+2\omega_2+2\omega_1)T_0}+cc$$

(20)

$$\left(D_0^2+\omega_2^2\right)h_1=\begin{pmatrix}-2i\omega_2B'-\beta i\omega_2B\\-6\lambda_6BA^2\bar{A}^2\\-6\lambda_6C\bar{C}B^2\bar{B}-10\lambda_7B^3\bar{B}^2\\-6\lambda_9C^2\bar{C}^2B-6\lambda_{10}B^2\bar{B}A\bar{A}\\-4\lambda_{12}C\bar{C}A\bar{A}B\end{pmatrix}e^{i\omega_2T_0}$$

$$-\lambda_5BA^4e^{i(4\omega_1+\omega_2)T_0}$$
$$-\lambda_5\bar{B}A^4e^{i(4\omega_1-\omega_2)T_0}$$

$$+\begin{pmatrix}-4\lambda_5BA^3\bar{A}-3\lambda_{10}B^2\bar{B}A^2\\-2\lambda_{12}\bar{C}^2BA^2\end{pmatrix}e^{i(2\omega_1+\omega_2)T_0}$$
$$+\begin{pmatrix}-4\lambda_5\bar{B}A^3\bar{A}-3\lambda_{10}\bar{B}^2BA^2\\-2\lambda_{12}\bar{C}^2\bar{B}A^2\end{pmatrix}e^{i(2\omega_1-\omega_2)T_0}$$

$$-\left(\lambda_6C^2B^3+\lambda_{10}B^3A^2\right)e^{i(3\omega_2+2\omega_3)T_0}$$
$$-\left(\lambda_6\bar{C}^2B^3+\lambda_{10}\bar{B}^3A^2\right)e^{i(3\omega_2-2\omega_3)T_0}$$

$$-\begin{pmatrix}3\lambda_6C^2B^2\bar{B}+4\lambda_9C^3\bar{C}B\\+2\lambda_{12}C^2BA\bar{A}\end{pmatrix}e^{i(\omega_2+2\omega_3)T_0}$$
$$-\begin{pmatrix}3\lambda_6\bar{C}^2B^2\bar{B}+4\lambda_9\bar{C}^3CB\\+2\lambda_{12}\bar{C}^2BA\bar{A}\end{pmatrix}e^{i(\omega_2+2\omega_3)T_0}$$

$$-\begin{pmatrix} 2\lambda_6 C\bar{C}B^3\bar{B}+5\lambda_7 B^4\bar{B} \\ +2\lambda_{10}B^3A\bar{A} \end{pmatrix} e^{3i\omega_2 T_0}$$

$$-\lambda_7 B^5 e^{5i\omega_2 T_0}-\lambda_8 C^3 BAe^{i(\omega_1+\omega_2+3\omega_3)T_0}$$

$$-\lambda_8\bar{C}^3\bar{B}Ae^{i(\omega_1-\omega_2-3\omega_3)T_0}-\lambda_8 C^3\bar{B}Ae^{i(\omega_1-\omega_2+3\omega_3)T_0}-\lambda_8\bar{C}^3BAe^{i(\omega_1+\omega_2-3\omega_3)T_0}$$

$$-\begin{pmatrix} 3\lambda_8 C^2\bar{C}BA+3\lambda_{11}CB^2\bar{B}A \\ +3\lambda_{13}CBA^2\bar{A} \end{pmatrix} e^{i(\omega_1+\omega_2+\omega_3)T_0}$$

$$-\begin{pmatrix} 3\lambda_8\bar{C}^2C\bar{B}A+3\lambda_{11}\bar{C}B^2\bar{B}A \\ +3\lambda_{13}\bar{C}\bar{B}A^2\bar{A} \end{pmatrix} e^{i(\omega_1-\omega_2-\omega_3)T_0}$$

$$-\begin{pmatrix} 3\lambda_8 C^2\bar{C}\bar{B}A+3\lambda_{11}C\bar{B}^2BA \\ +3\lambda_{13}C\bar{B}A^2\bar{A} \end{pmatrix} e^{i(\omega_1-\omega_2+\omega_3)T_0}$$

$$-\begin{pmatrix} 3\lambda_8\bar{C}^2CBA+3\lambda_{11}\bar{C}B^2\bar{B}A \\ +3\lambda_{13}\bar{C}BA^2\bar{A} \end{pmatrix} e^{i(\omega_1+\omega_2-\omega_3)T_0}$$

$$-\lambda_9 C^4 Be^{i(\omega_2+4\omega_3)T_0}$$
$$-\lambda_9\bar{C}^4 Be^{i(\omega_2-4\omega_3)T_0}$$
$$-\lambda_{10}\bar{B}^3A^2e^{i(2\omega_1-3\omega_2)T_0}$$

$$-\lambda_{11}CB^3Ae^{i(\omega_1+3\omega_2+\omega_3)T_0}$$
$$-\lambda_{11}\bar{C}\bar{B}^3Ae^{i(\omega_1-3\omega_2-\omega_3)T_0}$$
$$-\lambda_{11}C\bar{B}^3Ae^{i(\omega_1-3\omega_2+\omega_3)T_0}$$

$$-\lambda_{11}\bar{C}B^3Ae^{i(\omega_1+3\omega_2-\omega_3)T_0}$$
$$-\lambda_{12}C^2A^2Be^{i(2\omega_1+\omega_2+2\omega_3)T_0}$$
$$-\lambda_{12}\bar{C}^2A^2\bar{B}e^{i(2\omega_1-\omega_2-2\omega_3)T_0}$$

$$-\lambda_{12}C^2A^2\bar{B}e^{i(2\omega_1-\omega_2+2\omega_3)T_0}$$
$$-\lambda_{12}\bar{C}^2A^2Be^{i(2\omega_1+\omega_2-2\omega_3)T_0}$$
$$-\lambda_{13}CA^3Be^{i(3\omega_1+\omega_2+\omega_3)T_0}$$

$$-\lambda_{13}\bar{C}A^3\bar{B}e^{i(3\omega_1-\omega_2-\omega_3)T_0}$$
$$-\lambda_{13}CA^3\bar{B}e^{i(3\omega_1-\omega_2+\omega_3)T_0}$$
$$-\lambda_{13}\bar{C}A^3Be^{i(3\omega_1+\omega_2-\omega_3)T_0}+cc,$$

(21):

$$\left(D_0{}^2+\omega_3{}^2\right)k_1=$$

$$\begin{pmatrix} -2i\,\omega_3 C' - \delta i\,\omega_3 C - 10\tau_6 C^3\bar{C}^3 \\ -6\tau_7 CA^2\bar{A}^2 - 6\tau_8 C^2\bar{C}A\bar{A} \\ -4\tau_{13} CB\bar{B}A\bar{A} \\ -6\tau_{15} CB^2\bar{B}^2 - 6\tau_{16} C^2\bar{C}B\bar{B} \end{pmatrix} e^{i\omega_3 T_0}$$

$$-\tau_6 C^5 e^{5i\omega_3 T_0}$$

$$-\begin{pmatrix} 5\tau_6 C^4\bar{C} + 2\tau_8 C^3 A\bar{A} \\ +2\tau_{16} C^3 B\bar{B} \end{pmatrix} e^{3i\omega_3 T_0}$$

$$-\tau_7 CA^4 e^{i(4\omega_1+\omega_3)T_0}$$

$$-\tau_7 \bar{C}A^4 e^{i(4\omega_1-\omega_3)T_0}$$

$$-\begin{pmatrix} 4\tau_7 CA^4\bar{A} \\ +3\tau_8 C^2\bar{C}A^2 \\ +2\tau_{13} CB\bar{B}A^2 \end{pmatrix} e^{i(2\omega_1+\omega_3)T_0}$$

$$-\begin{pmatrix} 4\tau_7 \bar{C}A^4\bar{A} + 3\tau_8 \bar{C}^2 CA^2 \\ +2\tau_{13} \bar{C}B\bar{B}A^2 \end{pmatrix} e^{i(2\omega_1-\omega_3)T_0}$$

$$-\tau_8 C^3 A^2 e^{i(2\omega_1+3\omega_3)T_0}$$

$$-\tau_8 \bar{C}^3 A^2 e^{i(2\omega_1-3\omega_3)T_0}$$
$$-\tau_9 C^2 A^3 e^{i(3\omega_1+2\omega_3)T_0}$$
$$-\tau_9 \bar{C}^2 A^3 e^{i(3\omega_1-2\omega_3)T_0}$$

$$-\begin{pmatrix} 3\tau_9 C^2 A^2\bar{A} \\ +2\tau_{12} C^2 B\bar{B}A \end{pmatrix} e^{i(\omega_1+2\omega_3)T_0}$$

$$-\begin{pmatrix} 3\tau_9 \bar{C}^2 A^2\bar{A} \\ +2\tau_{12} \bar{C}^2 B\bar{B}A \end{pmatrix} e^{i(\omega_1-2\omega_3)T_0}$$

$$-\tau_{10} B^2 A^3 e^{i(3\omega_1+2\omega_2)T_0}$$

$$-\begin{pmatrix} 3\tau_{10} B^2 A^2\bar{A} + 2\tau_{12} C\bar{C}B^2 A \\ +4\tau_{11} B^3\bar{B}A \end{pmatrix} e^{i(\omega_1+2\omega_2)T_0}$$

$$-\begin{pmatrix} 3\tau_{10} \bar{B}^2 A^2\bar{A} + 2\tau_{12} C\bar{C}\bar{B}^2 A \\ +4\tau_{11} \bar{B}^3 BA \end{pmatrix} e^{i(\omega_1-2\omega_2)T_0}$$

$$-\tau_{10} \bar{B}^2 A^3 e^{i(3\omega_1-2\omega_2)T_0}$$

$$-\tau_{11} B^4 A e^{i(\omega_1+4\omega_2)T_0}$$
$$-\tau_{11} \bar{B}^4 A e^{i(\omega_1-4\omega_2)T_0}$$
$$-\tau_{12} C^2 B^2 A e^{i(\omega_1+2\omega_2+2\omega_3)T_0}$$

International Journal of Academic Multidisciplinary Research (IJAMR)
ISSN: 2643-9670
Vol. 3 Issue 9, September – 2019, Pages: 35-63

$$-\tau_{12}\bar{C}^2\bar{B}^2Ae^{i(\omega_1-2\omega_2-2\omega_3)T_0}$$
$$-\tau_{12}C^2\bar{B}^2Ae^{i(\omega_1-2\omega_2+2\omega_3)T_0}$$
$$-\tau_{12}\bar{C}^2B^2Ae^{i(\omega_1+2\omega_2-2\omega_3)T_0}$$

$$-\begin{pmatrix}6\tau_9C\bar{C}A^2\bar{A}+6\tau_{10}B\bar{B}A^2\bar{A}\\+6\tau_{11}B^2\bar{B}^2A+4\tau_{12}C\bar{C}B\bar{B}A\\+10\tau_{14}A^3\bar{A}^2\end{pmatrix}e^{i\omega_1T_0}$$

$$-\tau_{13}CB^2A^2e^{i(2\omega_1+2\omega_2+\omega_3)T_0}$$
$$-\tau_{13}\bar{C}\bar{B}^2A^2e^{i(2\omega_1-2\omega_2-\omega_3)T_0}$$

$$-\tau_{13}C\bar{B}^2A^2e^{i(2\omega_1-2\omega_2+\omega_3)T_0}$$
$$-\tau_{13}\bar{C}B^2A^2e^{i(2\omega_1+2\omega_2-\omega_3)T_0}$$

$$-\left(2\tau_{13}CB^2A\bar{A}+3\tau_{16}C^2\bar{C}B^2\right)e^{i(2\omega_2+\omega_3)T_0}$$
$$-\left(2\tau_{13}\bar{C}B^2A\bar{A}+3\tau_{16}\bar{C}^2CB^2\right)e^{i(2\omega_2-\omega_3)T_0}$$

$$-\tau_{14}A^5e^{5i\omega_1T_0}-\tau_{15}CB^4e^{i(4\omega_2+\omega_3)T_0}$$
$$-\tau_{15}\bar{C}B^4e^{i(4\omega_2-\omega_3)T_0}$$

$$-\tau_{16}C^3B^2e^{i(2\omega_2+3\omega_3)T_0}$$
$$-\tau_{16}\bar{C}^3B^2e^{i(2\omega_2-3\omega_3)T_0}+cc,$$

Where cc denotes a complex conjugate of the preceding term.

The general solution of equations (19-21) can be written in the following form:

(22)

$$g_1(T_0,T_1)=\frac{\eta_6C^4A}{8\omega_3(2\omega_3+\omega_1)}e^{i(4\omega_3+\omega_1)T_0}$$
$$-\frac{\eta_6\bar{C}^4}{8\omega_3(-2\omega_3+\omega_1)}Ae^{i(-4\omega_3+\omega_1)T_0}$$

$$+\frac{\left(4\eta_6C^3\bar{C}A+2\eta_{14}C^2AB\bar{B}\right)}{4\omega_3(\omega_3+\omega_1)}e^{i(2\omega_3+\omega_1)T_0}$$
$$+\frac{\left(-4\eta_6\bar{C}^3CA-2\eta_{14}\bar{C}^2AB\bar{B}\right)}{4\omega_3(-\omega_3+\omega_1)}e^{i(-2\omega_3+\omega_1)T_0}$$

$$-\frac{\eta_7C^3A^2}{3(\omega_1+3\omega_3)(\omega_1+\omega_3)}e^{i(3\omega_3+2\omega_1)T_0}$$

$$-\frac{\eta_7\bar{C}^3A^2}{3(\omega_1-3\omega_3)(\omega_1-\omega_3)}e^{i(-3\omega_3+2\omega_1)T_0}$$

$$+\frac{\left(-2\eta_7C^3A\bar{A}-2\eta_8C^3B\bar{B}\right)}{(\omega_1-3\omega_3)(\omega_1+3\omega_3)}e^{3i\omega_3T_0}$$

$$+\frac{\left(-3\eta_7C^2\bar{C}A^2-4\eta_{15}CA^3\bar{A}-2\eta_{16}CB\bar{B}A^2\right)}{(3\omega_1+\omega_2)(\omega_1+\omega_2)}e^{i(\omega_3+2\omega_1)T_0}$$
$$+\frac{\left(-3\eta_7\bar{C}^2CA^2-4\eta_{15}\bar{C}A^3\bar{A}-2\eta_{16}\bar{C}B\bar{B}A^2\right)}{(3\omega_1-\omega_2)(\omega_1-\omega_2)}e^{i(-\omega_3+2\omega_1)T_0}$$

$$+\frac{\begin{pmatrix}-6\eta_7C^2\bar{C}A\bar{A}-6\eta_8C^2\bar{C}B\bar{B}-\\6\eta_{10}B^2\bar{B}^2C-6\eta_{15}CA^2\bar{A}^2-\\4\eta_{16}CB\bar{B}A\bar{A}\end{pmatrix}}{(\omega_1-\omega_3)(\omega_1+\omega_3)}e^{i\omega_3T_0}$$

$$-\frac{\eta_8C^3B^2}{(2\omega_2+3\omega_3+\omega_1)(-2\omega_2-3\omega_3+\omega_1)}e^{i(3\omega_3+2\omega_2)T_0}$$

$$-\frac{\eta_8\bar{C}^3B^2}{(2\omega_2-3\omega_3+\omega_1)(-2\omega_2+3\omega_3+\omega_1)}e^{i(-3\omega_3+2\omega_2)T_0}$$

$$+\frac{\left(-3\eta_8C^2\bar{C}B^2-4\eta_{10}B^3\bar{B}\bar{C}-2\eta_{16}CB^2A\bar{A}\right)}{(2\omega_2+\omega_3+\omega_1)(-2\omega_2-\omega_3+\omega_1)}e^{i(\omega_3+2\omega_2)T_0}$$

$$+\frac{\left(-3\eta_8\bar{C}^2CB^2-4\eta_{10}B^3\bar{B}\bar{C}-2\eta_{16}\bar{C}B^2A\bar{A}\right)}{(2\omega_2-\omega_3+\omega_1)(-2\omega_2+\omega_3+\omega_1)}e^{i(-\omega_3+2\omega_2)T_0}$$

$$+\frac{\left(-\eta_9B^2A^3-\eta_{12}B^2A^3\right)}{4(\omega_1+\omega_2)(2\omega_1+\omega_2)}e^{i(2\omega_2+3\omega_1)T_0}$$

$$+\frac{\left(\eta_9\bar{B}^2A^3+\eta_{12}\bar{B}^2A^3\right)}{4(\omega_1-\omega_2)(2\omega_1-\omega_2)}e^{i(-2\omega_2+3\omega_1)T_0}$$

$$+\frac{\begin{pmatrix}-3\eta_9B^2A^2\bar{A}-4\eta_{11}B^3\bar{B}\bar{A}\\-3\eta_{12}B^2A^2\bar{A}-2\eta_{14}C\bar{C}B^2A\end{pmatrix}}{4\omega_2(\omega_1+\omega_2)}e^{i(2\omega_2+\omega_1)T_0}$$

$$+\frac{\begin{pmatrix}-3\eta_9\bar{B}^2A^2\bar{A}\\-4\eta_{11}\bar{B}^3B\bar{A}-3\eta_{12}\bar{B}^2A^2\bar{A}\\-2\eta_{14}\bar{C}C\bar{B}^2A\end{pmatrix}}{4\omega_2(\omega_1-\omega_2)}e^{i(-2\omega_2+\omega_1)T_0}$$

$$+\frac{\left(2\eta_9A^3B\bar{B}+2\eta_{12}A^3B\bar{B}\right)}{8\omega_1^2}e^{3i\omega_1T_0}$$

$$-\frac{\eta_{10}CB^4}{(4\omega_2+\omega_3+\omega_1)(-4\omega_2-\omega_3+\omega_1)}e^{i(4\omega_2+\omega_3)T_0}$$

$$-\frac{\eta_{10}\bar{C}B^4}{(4\omega_2-\omega_3+\omega_1)(-4\omega_2+\omega_3+\omega_1)}e^{i(4\omega_2-\omega_3)T_0}$$

$$-\frac{\eta_{11}B^4A}{8\omega_2(\omega_1+\omega_2)}e^{i(4\omega_2+\omega_1)T_0}$$

$$-\frac{\eta_{11}\bar{B}^4A}{8\omega_2(\omega_1-\omega_2)}e^{i(4\omega_2+\omega_1)T_0}$$

$$+\frac{\eta_{13}A^5}{24\omega_1^2}e^{5i\omega_1T_0}$$

$$+\frac{\eta_{14}C^2B^2A}{4(\omega_2+\omega_3)(\omega_1+\omega_2+\omega_3)}e^{i(2\omega_3+2\omega_2+\omega_1)T_0}$$

$$+\frac{\eta_{14}\bar{C}^2\bar{B}^2A}{4(\omega_2+\omega_3)(\omega_1-\omega_2-\omega_3)}e^{i(-2\omega_3-2\omega_2+\omega_1)T_0}$$

$$+\frac{\eta_{14}C^2\bar{B}^2A}{4(\omega_2-\omega_3)(\omega_1-\omega_2+\omega_3)}e^{i(2\omega_3-2\omega_2+\omega_1)T_0}$$

$$+\frac{\eta_{14}\bar{C}^2B^2A}{4(\omega_2-\omega_3)(\omega_1+\omega_2-\omega_3)}e^{i(-2\omega_3+2\omega_2+\omega_1)T_0}$$

$$\frac{\eta_{15}CA^4}{(3\omega_1+\omega_3)(5\omega_1+\omega_3)}e^{i(\omega_3+4\omega_1)T_0}$$

$$+\frac{\eta_{15}\bar{C}A^4}{(3\omega_1-\omega_3)(5\omega_1-\omega_3)}e^{i(-\omega_3+4\omega_1)T_0}$$

$$+\frac{\eta_{16}CB^2A^2}{(2\omega_2+\omega_3+3\omega_1)(2\omega_2+\omega_3+\omega_1)}e^{i(\omega_3+2\omega_2+2\omega_1)T_0}$$

$$+\frac{\eta_{16}CB^2A^2}{(-2\omega_2-\omega_3+3\omega_1)(-2\omega_2-\omega_3+\omega_1)}e^{i(-\omega_3-2\omega_2+2\omega_1)T_0}$$

$$+\frac{\eta_{16}CB^2A^2}{(-2\omega_2+\omega_3+3\omega_1)(-2\omega_2+\omega_3+\omega_1)}e^{i(\omega_3-2\omega_2+2\omega_1)T_0}$$

$$+\frac{\eta_{16}CB^2A^2}{(2\omega_2-\omega_3+3\omega_1)(2\omega_2-\omega_3+\omega_1)}e^{i(-\omega_3+2\omega_2+2\omega_1)T_0}+cc$$

(23):

$$h_1(T_0,T_1)=$$

$$\frac{\lambda_5BA^4}{8\omega_1(2\omega_1+\omega_2)}e^{i(4\omega_1+\omega_2)T_0}$$

$$+\frac{\lambda_5\bar{B}A^4}{8\omega_1(-2\omega_1+\omega_2)}e^{i(4\omega_1-\omega_2)T_0}$$

$$+\frac{\left(-4\lambda_5BA^3\bar{A}-3\lambda_{10}B^2\bar{B}A^2-2\lambda_{12}\bar{C}^2BA^2\right)}{4\omega_1(\omega_1+\omega_2)}e^{i(2\omega_1+\omega_2)T_0}$$

$$+\frac{\left(-4\lambda_5\bar{B}A^3\bar{A}-3\lambda_{10}\bar{B}^2BA^2-2\lambda_{12}\bar{C}^2\bar{B}A^2\right)}{4\omega_1(-\omega_1+\omega_2)}e^{i(2\omega_1-\omega_2)T_0}$$

$$+\frac{\left(\lambda_6C^2B^3+\lambda_{10}B^3A^2\right)}{4(\omega_2+\omega_3)(2\omega_2+\omega_3)}e^{i(3\omega_2+2\omega_3)T_0}$$

$$+\frac{\left(\lambda_6\bar{C}^2B^3+\lambda_{10}\bar{B}^3A^2\right)}{4(\omega_2-\omega_3)(2\omega_2-\omega_3)}e^{i(3\omega_2-2\omega_3)T_0}$$

$$-\frac{\left(3\lambda_6C^2B^2\bar{B}+4\lambda_9C^3\bar{C}B+2\lambda_{12}C^2BA\bar{A}\right)}{4\omega_3(\omega_2+\omega_3)}e^{i(\omega_2+2\omega_3)T_0}$$

$$+\frac{\left(3\lambda_6\bar{C}^2B^2\bar{B}+4\lambda_9\bar{C}^3CB+2\lambda_{12}\bar{C}^2BA\bar{A}\right)}{4\omega_3(\omega_2-\omega_3)}e^{i(\omega_2-2\omega_3)T_0}$$

$$+\frac{\left(2\lambda_6C\bar{C}B^3\bar{B}+5\lambda_7B^4\bar{B}+2\lambda_{10}B^3A\bar{A}\right)}{8\omega_2^2}e^{3i\omega_2T_0}$$

$$+\frac{\lambda_7B^5}{24\omega_2^2}e^{5i\omega_2T_0}$$

$$+\frac{\lambda_8C^3BA}{(\omega_1+3\omega_3)(\omega_1+2\omega_2+3\omega_3)}e^{i(\omega_1+\omega_2+3\omega_3)T_0}$$

$$+\frac{\lambda_8\bar{C}^3\bar{B}A}{(\omega_1-3\omega_3)(\omega_1-2\omega_2-3\omega_3)}e^{i(\omega_1-\omega_2-3\omega_3)T_0}$$

$$+\frac{\lambda_8C^3\bar{B}A}{(\omega_1+3\omega_3)(\omega_1-2\omega_2+3\omega_3)}e^{i(\omega_1-\omega_2+3\omega_3)T_0}$$

$$+\frac{\lambda_8\bar{C}^3BA}{(\omega_1-3\omega_3)(\omega_1+2\omega_2-3\omega_3)}e^{i(\omega_1+\omega_2-3\omega_3)T_0}$$

$$+\frac{\left(3\lambda_8C^2\bar{C}BA+3\lambda_{11}CB^2\bar{B}A+3\lambda_{13}CBA^2\bar{A}\right)}{(\omega_1+\omega_3)(\omega_1+2\omega_2+\omega_3)}e^{i(\omega_1+\omega_2+\omega_3)T_0}$$

$$+\frac{\left(3\lambda_8\bar{C}^2C\bar{B}A+3\lambda_{11}\bar{C}B^2\bar{B}A+3\lambda_{13}\bar{C}\bar{B}A^2\bar{A}\right)}{(\omega_1-\omega_3)(\omega_1-2\omega_2-\omega_3)}e^{i(\omega_1-\omega_2-\omega_3)T_0}$$

$$+\frac{\left(3\lambda_8C^2\bar{C}\bar{B}A+3\lambda_{11}C\bar{B}^2BA+3\lambda_{13}C\bar{B}A^2\bar{A}\right)}{(\omega_1+\omega_3)(\omega_1-2\omega_2+\omega_3)}e^{i(\omega_1-\omega_2+\omega_3)T_0}$$

$$+\frac{\left(3\lambda_8\bar{C}^2CBA+3\lambda_{11}\bar{C}B^2\bar{B}A+3\lambda_{13}\bar{C}BA^2\bar{A}\right)}{(\omega_1-\omega_3)(\omega_1+2\omega_2-\omega_3)}e^{i(\omega_1+\omega_2-\omega_3)T_0}$$

$$+\frac{\lambda_9C^4B}{8\omega_3(\omega_2+2\omega_3)}e^{i(\omega_2+4\omega_3)T_0}$$

$$+\frac{\lambda_9\bar{C}^4B}{8\omega_3(\omega_2-2\omega_3)}e^{i(\omega_2-4\omega_3)T_0}$$

$$+\frac{\lambda_{10}\bar{B}^3A^2}{(\omega_1-\omega_2)(\omega_1-2\omega_2)}e^{i(2\omega_1-3\omega_2)T_0}$$

$$+\frac{\lambda_{11}CB^3A}{(\omega_1+2\omega_2+\omega_3)(\omega_1+4\omega_2+\omega_3)}e^{i(\omega_1+3\omega_2+\omega_3)T_0}$$

$$+\frac{\lambda_{11}\bar{C}\bar{B}^3A}{(\omega_1-2\omega_2-\omega_3)(\omega_1-4\omega_2-\omega_3)}e^{i(\omega_1-3\omega_2-\omega_3)T_0}$$

$$+\frac{\lambda_{11}C\bar{B}^3A}{(\omega_1-2\omega_2+\omega_3)(\omega_1-4\omega_2+\omega_3)}e^{i(\omega_1-3\omega_2+\omega_3)T_0}$$

$$+\frac{\lambda_{11}\bar{C}B^3A}{(\omega_1+2\omega_2-\omega_3)(\omega_1+4\omega_2-\omega_3)}e^{i(\omega_1+3\omega_2-\omega_3)T_0}$$

$$+\frac{\lambda_{12}C^2A^2B}{4(\omega_1+\omega_3)(\omega_1+\omega_2+\omega_3)}e^{i(2\omega_1+\omega_2+2\omega_3)T_0}$$

$$+\frac{\lambda_{12}\bar{C}^2A^2\bar{B}}{4(\omega_1-\omega_3)(\omega_1-\omega_2-\omega_3)}e^{i(2\omega_1-\omega_2-2\omega_3)T_0}$$

$$+\frac{\lambda_{12}C^2A^2\bar{B}}{4(\omega_1+\omega_3)(\omega_1-\omega_2+\omega_3)}e^{i(2\omega_1-\omega_2+2\omega_3)T_0}$$

$$+\frac{\lambda_{12}\bar{C}^2A^2B}{4(\omega_1-\omega_3)(\omega_1+\omega_2-\omega_3)}e^{i(2\omega_1+\omega_2-2\omega_3)T_0}$$

$$+\frac{\lambda_{13}CA^3B}{(3\omega_1+\omega_3)(3\omega_1+2\omega_2+\omega_3)}e^{i(3\omega_1+\omega_2+\omega_3)T_0}$$

$$+\frac{\lambda_{13}\bar{C}A^3\bar{B}}{(3\omega_1-\omega_3)(3\omega_1-2\omega_2-\omega_3)}e^{i(3\omega_1-\omega_2-\omega_3)T_0}$$

$$+\frac{\lambda_{13}CA^3\bar{B}}{(3\omega_1+\omega_3)(3\omega_1-2\omega_2+\omega_3)}e^{i(3\omega_1-\omega_2+\omega_3)T_0}$$

$$+\frac{\lambda_{13}\bar{C}A^3B}{(3\omega_1-\omega_3)(3\omega_1+2\omega_2-\omega_3)}e^{i(3\omega_1+\omega_2-\omega_3)T_0}+cc.$$

(24)

$$k_1(T_0,T_1)=\frac{\tau_6C^5}{24\omega_3^2}e^{5i\omega_3T_0}$$

$$+\frac{\left(5\tau_6C^4\bar{C}+2\tau_8C^3A\bar{A}+2\tau_{16}C^3B\bar{B}\right)}{8\omega_3^2}e^{3i\omega_3T_0}$$

$$+\frac{\tau_7CA^4}{8\omega_1(\omega_1+\omega_3)}e^{i(4\omega_1+\omega_3)T_0}$$

$$+\frac{\tau_7\bar{C}A^4}{8\omega_1(-\omega_1+\omega_3)}e^{i(4\omega_1-\omega_3)T_0}$$

$$+\frac{\begin{pmatrix}4\tau_7CA^4\bar{A}+3\tau_8C^2\bar{C}A^2\\+2\tau_{13}CB\bar{B}A^2\end{pmatrix}}{4\omega_1(\omega_1+\omega_3)}e^{i(2\omega_1+\omega_3)T_0}$$

$$+\frac{\left(4\tau_7\bar{C}A^4\bar{A}+3\tau_8\bar{C}^2CA^2+2\tau_{13}\bar{C}B\bar{B}A^2\right)}{4\omega_1(-\omega_1+\omega_3)}e^{i(2\omega_1-\omega_3)T_0}$$

$$+\frac{\tau_8C^3A^2}{4(\omega_1+2\omega_3)(\omega_1+\omega_3)}e^{i(2\omega_1+3\omega_3)T_0}$$

$$+\frac{\tau_8\bar{C}^3A^2}{4(\omega_1-2\omega_3)(\omega_1-\omega_3)}e^{i(2\omega_1-3\omega_3)T_0}$$

$$+\frac{\tau_9C^2A^3}{3(\omega_1+\omega_3)(3\omega_1+\omega_3)}e^{i(3\omega_1+2\omega_3)T_0}$$

$$+\frac{\tau_9\bar{C}^2A^3}{3(\omega_1-\omega_3)(3\omega_1-\omega_3)}e^{i(3\omega_1-2\omega_3)T_0}$$

$$+\frac{\left(3\tau_9C^2A^2\bar{A}+2\tau_{12}C^2B\bar{B}A\right)}{(\omega_1+3\omega_3)(\omega_1+\omega_3)}e^{i(\omega_1+2\omega_3)T_0}$$

$$+\frac{\left(3\tau_9\bar{C}^2A^2\bar{A}+2\tau_{12}\bar{C}^2B\bar{B}A\right)}{(\omega_1-3\omega_3)(\omega_1-\omega_3)}e^{i(\omega_1-2\omega_3)T_0}$$

$$+\frac{\tau_{10}B^2A^3}{(3\omega_1+2\omega_2+\omega_3)(3\omega_1+2\omega_2-\omega_3)}e^{i(3\omega_1+2\omega_2)T_0}$$

$$+\frac{\begin{pmatrix}3\tau_{10}B^2A^2\bar{A}+2\tau_{12}C\bar{C}B^2A\\+4\tau_{11}B^3\bar{B}A\end{pmatrix}}{(3\omega_1-2\omega_2+\omega_3)(3\omega_1-2\omega_2-\omega_3)}e^{i(\omega_1+2\omega_2)T_0}$$

$$+\frac{\begin{pmatrix}3\tau_{10}\bar{B}^2A^2\bar{A}+2\tau_{12}C\bar{C}\bar{B}^2A\\+4\tau_{11}\bar{B}^3BA\end{pmatrix}}{(\omega_1-2\omega_2+\omega_3)(\omega_1-2\omega_2-\omega_3)}e^{i(\omega_1-2\omega_2)T_0}$$

$$+\frac{\tau_{10}\bar{B}^2A^3}{(3\omega_1-2\omega_2+\omega_3)(3\omega_1-2\omega_2-\omega_3)}e^{i(3\omega_1-2\omega_2)T_0}$$

$$+\frac{\tau_{11}B^4A}{(\omega_1+4\omega_2+\omega_3)(\omega_1+4\omega_2-\omega_3)}e^{i(\omega_1+4\omega_2)T_0}$$

$$+\frac{\tau_{11}\bar{B}^4A}{(\omega_1-4\omega_2+\omega_3)(\omega_1-4\omega_2-\omega_3)}e^{i(\omega_1-4\omega_2)T_0}$$

$$+\frac{\tau_{12}C^2B^2A}{(\omega_1+2\omega_2+\omega_3)(\omega_1+2\omega_2+3\omega_3)}e^{i(\omega_1+2\omega_2+2\omega_3)T_0}$$

$$+\frac{\tau_{12}\bar{C}^2\bar{B}^2A}{(\omega_1-2\omega_2-\omega_3)(\omega_1-2\omega_2-3\omega_3)}e^{i(\omega_1-2\omega_2-2\omega_3)T_0}$$

$$+\frac{\tau_{12}C^2\bar{B}^2A}{(\omega_1-2\omega_2+\omega_3)(\omega_1-2\omega_2+3\omega_3)}e^{i(\omega_1-2\omega_2+2\omega_3)T_0}$$

$$+\frac{\tau_{12}\bar{C}^2B^2A}{(\omega_1+2\omega_2-\omega_3)(\omega_1+2\omega_2-3\omega_3)}e^{i(\omega_1+2\omega_2-2\omega_3)T_0}$$

$$+\frac{\begin{pmatrix}6\tau_9C\bar{C}A^2\bar{A}+6\tau_{10}B\bar{B}A^2\bar{A}\\+6\tau_{11}B^2\bar{B}^2A+4\tau_{12}C\bar{C}B\bar{B}A\\+10\tau_{14}A^3\bar{A}^2\end{pmatrix}}{(-\omega_1+\omega_3)(\omega_1+\omega_3)}e^{i\omega_1T_0}$$

$$+\frac{\tau_{13}CB^2A^2}{4(\omega_1+\omega_2)(\omega_1+\omega_2+\omega_3)}e^{i(2\omega_1+2\omega_2+\omega_3)T_0}$$

$$+\frac{\tau_{13}\bar{C}\bar{B}^2A^2}{4(\omega_1-\omega_2)(\omega_1-\omega_2-\omega_3)}e^{i(2\omega_1-2\omega_2-\omega_3)T_0}$$

$$+\frac{\tau_{13}C\bar{B}^2A^2}{4(\omega_1-\omega_2)(\omega_1-\omega_2+\omega_3)}e^{i(2\omega_1-2\omega_2+\omega_3)T_0}$$

$$+\frac{\tau_{13}\bar{C}B^2A^2}{4(\omega_1+\omega_2)(\omega_1+\omega_2-\omega_3)}e^{i(2\omega_1+2\omega_2-\omega_3)T_0}$$

International Journal of Academic Multidisciplinary Research (IJAMR)
ISSN: 2643-9670
Vol. 3 Issue 9, September – 2019, Pages: 35-63

$$+\frac{\left(2\tau_{13}CB^{2}A\bar{A}+3\tau_{16}C^{2}\bar{C}B^{2}\right)}{4\omega_2(\omega_2+\omega_3)}e^{i(2\omega_2+\omega_3)T_0}$$

$$+\frac{\left(2\tau_{13}\bar{C}B^{2}A\bar{A}+3\tau_{16}\bar{C}^{2}CB^{2}\right)}{4\omega_2(-\omega_2+\omega_3)}e^{i(2\omega_2-\omega_3)T_0}$$

$$+\frac{\tau_{14}A^{5}}{(-5\omega_1+\omega_3)(5\omega_1+\omega_3)}e^{5i\omega_1T_0}$$

$$+\frac{\tau_{15}CB^{4}}{8\omega_2(\omega_2+\omega_3)}e^{i(4\omega_2+\omega_3)T_0}$$

$$+\frac{\tau_{15}\bar{C}B^{4}}{8\omega_2(-\omega_2+\omega_3)}e^{i(4\omega_2-\omega_3)T_0}$$

$$+\frac{\tau_{16}C^{3}B^{2}}{4(\omega_2+2\omega_3)(\omega_2+\omega_3)}e^{i(2\omega_2+3\omega_3)T_0}$$

$$+\frac{\tau_{16}\bar{C}^{3}B^{2}e^{i(2\omega_2-3\omega_3)T_0}}{4(\omega_2-2\omega_3)(\omega_2-\omega_3)}+cc$$

From equations (22-24), the following resonance cases are extracted:

- Internal Resonance:

$$(\omega_1=\omega_2=\omega_3=5.5),$$

$$(\omega_1=\omega_2=\omega_3=2.2),$$

$$(\omega_1=3\omega_3),$$

$$(\omega_1=\omega_2=2.2),$$

4. STABILITY ANALYSIS

We shall study the stability of the nonlinear system by considering the following relations between the natural frequencies of the three modes of the system using the detuning parameter σ.

(25)

$$\omega_1=3\omega_3+\varepsilon\sigma_1 \qquad \omega_2=3\omega_1+\varepsilon\sigma_2 \qquad \omega_3=2\omega_2+\varepsilon\sigma_3$$

Substituting eq. (25) into eqs. (19-21) and eliminating terms that produce secular term then performing some algebraic manipulations, we obtain the following modulation equations:

(26):

$$\begin{pmatrix}-6\eta_6AC^{2}\bar{C}^{2}-6\eta_9B\bar{B}A^{2}\bar{A}\\-6\eta_{11}B^{2}\bar{B}^{2}A-6\eta_{12}B\bar{B}A^{2}\bar{A}\\-4\eta_{14}C\bar{C}B\bar{B}A\end{pmatrix}e^{i\omega_1T_0}$$

$$-\left(2\eta_7C^{3}A\bar{A}+2\eta_8C^{3}B\bar{B}\right)e^{3i\omega_3T_0}=0.$$

(27):

$$\begin{pmatrix}-2i\omega_2B'-\beta i\omega_2B-6\lambda_6BA^{2}\bar{A}^{2}\\-6\lambda_6C\bar{C}B^{2}\bar{B}-10\lambda_7B^{3}\bar{B}^{2}\\-6\lambda_9C^{2}\bar{C}^{2}B-6\lambda_{10}B^{2}\bar{B}A\bar{A}\\-4\lambda_{12}C\bar{C}A\bar{A}B\end{pmatrix}e^{i\omega_2T_0}$$

$$-\left(\lambda_6\bar{C}^{2}B^{3}+\lambda_{10}\bar{B}^{3}A^{2}\right)e^{i(3\omega_2-2\omega_3)T_0}$$

$$-\lambda_{11}\bar{C}\bar{B}^{3}Ae^{i(\omega_1-3\omega_2-\omega_3)T_0}=0$$

(28):

$$\begin{pmatrix} -2i\,\omega_3 C' - \delta i\,\omega_3 C - 10\tau_6 C^3 \bar{C}^3 \\ -6\tau_7 C A^2 \bar{A}^2 - 6\tau_8 C^2 \bar{C} A \bar{A} \\ -4\tau_{13} C B \bar{B} A \bar{A} - 6\tau_{15} C B^2 \bar{B}^2 \\ -6\tau_{16} C^2 \bar{C} B \bar{B} \end{pmatrix} e^{i\omega_3 T_0}$$

$$-\left(3\tau_9 \bar{C}^2 A^2 \bar{A} + 2\tau_{12} \bar{C}^2 B \bar{B} A\right) e^{i(\omega_1 - 2\omega_3)T_0}$$

$$-\tau_{11} \bar{B}^4 A e^{i(\omega_1 - 4\omega_2)T_0}$$

$$-\left(2\tau_{13} \bar{C} B^2 A \bar{A} + 3\tau_{16} \bar{C}^2 C B^2\right) e^{i(2\omega_2 - \omega_3)T_0}$$
$$-\tau_{15} \bar{C} B^4 e^{i(4\omega_2 - \omega_3)T_0} = 0$$

Letting

$$A = \frac{1}{2} a_1 e^{i\theta_1},$$
$$B = \frac{1}{2} a_2 e^{i\theta_2},$$
$$C = \frac{1}{2} a_3 e^{i\theta_3}.$$

Where $a_1, a_2, a_3, \theta_1, \theta_2, \theta_3$ are functions of T_1.. Separating real and imaginary parts gives the following six equations governing the amplitude and phase modulations.

(29):

$$\frac{a_1}{2}\left(\nu_5' - \nu_1'\right) = a_1 \sigma_1 - \frac{3}{16\omega_1} \eta_6 a_1 a_3^4$$

$$-\frac{3}{16\omega_1} \eta_9 a_2^2 a_1^3 - \frac{3}{16\omega_1} \eta_{11} a_1 a_2^4$$

$$-\frac{3}{16\omega_1} \eta_{12} a_2^2 a_1^3 - \frac{1}{8\omega_1} \eta_{14} a_1 a_2^2 a_3^2$$

$$-\left(\frac{1}{16\omega_1} \eta_7 a_1^2 a_3^3 + \frac{1}{16\omega_1} \eta_8 a_2^2 a_3^3\right) \cos \nu_1,$$

(30):

$$a_1' = -\frac{1}{2} \alpha a_1 - \left(\frac{1}{16\omega_1} \eta_7 a_1^2 a_3^3 + \frac{1}{16\omega_1} \eta_8 a_2^2 a_3^3\right) \sin \nu_1.$$

(31):

$$a_2\left(\frac{\nu_4'}{5} - \frac{\nu_3'}{10} - \frac{\nu_1'}{10}\right) = \frac{a_2}{5} \sigma_3 - \frac{3}{16\omega_2} \lambda_5 a_1^4 a_2$$

$$-\frac{3}{16\omega_2} \lambda_6 a_3^2 a_2^3 - \frac{5}{16\omega_2} \lambda_7 a_2^5$$

$$-\frac{3}{16\omega_2} \lambda_9 a_3^4 a_2 - \frac{3}{16\omega_2} \lambda_{10} a_1 a_2^4$$

$$-\frac{1}{4\omega_2} \lambda_{12} a_1^2 a_3^2 a_2 - \frac{1}{32\omega_2} \lambda_8 a_1 a_2 a_3^3 \cos \nu_2$$

$$-\frac{1}{32\omega_2} \lambda_{11} a_1 a_3 a_2^3 \cos \nu_3 - \frac{1}{32\omega_2} \lambda_6 a_2^3 a_3^2 \cos \nu_4$$

$$-\frac{1}{32\omega_2} \lambda_8 a_1 a_2 a_3^3 \cos \nu_5,$$

(32):

$$a_2{}' = -\frac{1}{2}\beta a_2 - \frac{1}{32\omega_2}\lambda_8 a_1 a_2 a_3{}^3 \sin\nu_2$$

$$-\frac{1}{32\omega_2}\lambda_{11} a_1 a_3 a_2{}^3 \sin\nu_3$$

$$-\frac{1}{32\omega_2}\lambda_6 a_2{}^3 a_3{}^2 \sin\nu_4 - \frac{1}{32\omega_2}\lambda_8 a_1 a_2 a_3{}^3 \sin\nu_5,$$

(33):

$$a_3\left(\frac{\nu_4{}'}{10} - \frac{\nu_3{}'}{5}\right) = -\frac{a_3}{5}\sigma_3 - \frac{5}{16\omega_3}\tau_6 a_3{}^5$$

$$-\frac{3}{16\omega_3}\tau_7 a_3 a_1{}^4 - \frac{3}{16\omega_3}\tau_{10} a_3{}^3 a_1{}^2 - \frac{1}{4\omega_3}\tau_{13} a_3 a_1{}^2 a_2{}^2$$

$$-\frac{3}{16\omega_3}\tau_{15} a_3 a_2{}^4 - \frac{3}{16\omega_3}\tau_{16} a_3{}^3 a_2{}^2$$

$$-\left(\frac{3}{32\omega_3}\tau_9 a_3{}^2 a_1{}^3 + \frac{1}{16\omega_3}\tau_{12} a_3{}^2 a_2{}^2 a_1\right)\cos\nu_6,$$

(34):

$$a_3{}' = -\frac{1}{2}\delta a_3 - \left(\frac{3}{32\omega_3}\tau_9 a_3{}^2 a_1{}^3 + \frac{1}{16\omega_3}\tau_{12} a_3{}^2 a_2{}^2 a_1\right)\sin\nu_6,$$

$$\nu_1 = -\theta_1 + 3\theta_3 - \sigma_1 T,$$

$$\nu_2 = \theta_1 - 3\theta_3 + \sigma_1 T_1, \nu_3 = \theta_1 + 2\theta_2 + \theta_3 + \sigma_1 T_1,$$

$$\nu_4 = -4\theta_2 + 2\theta_3 + 2\sigma_3 T_1, \nu_5 = \theta_1 + 3\theta_3 + \sigma_1 T_1, \nu_6 = -\nu_1.$$

The steady-state solutions of eqs. (29-34) are obtained by setting $a_1' = a_2' = a_3' = \nu_5' = \nu_8' = \nu_{12}' = 0.$ into eqs. (29-34). This results in the following nonlinear algebraic equations, which are called the frequency response equations:

(35):

$$\Lambda_1 a_1{}^8 + \Lambda_2 a_1{}^7 + \Lambda_3 a_1{}^6 + \Lambda_4 a_1{}^5$$
$$+\Lambda_5 a_1{}^4 + \Lambda_6 a_1{}^2 + \Lambda_7 = 0.$$

(36):

$$\Gamma_1 a_2{}^{10} + \Gamma_2 a_2{}^9 + \Gamma_3 a_2{}^8 + \Gamma_4 a_2{}^7 + \Gamma_5 a_2{}^6$$
$$+\Gamma_6 a_2{}^5 + \Gamma_7 a_2{}^4 + \Gamma_8 a_2{}^2 + \Gamma_9 = 0.$$

(37):

$$\Delta_1 a_3{}^{10} + \Delta_2 a_3{}^8 + \Delta_3 a_3{}^7 + \Delta_4 a_3{}^6$$
$$+\Delta_5 a_3{}^5 + \Delta_6 a_3{}^4 + \Delta_7 a_3{}^3 + \Delta_8 a_3{}^2 = 0.$$

The coefficients $\Lambda_i, i = 1, 2, ..., 7$ are given in Appendix A.

The coefficients $\Gamma_i, i = 1, 2, ..., 9$ are given in Appendix A.

The coefficients $\Delta_i, i = 1, 2, ..., 8$ are given in Appendix A.

5. NUMERICAL RESULTS AND DISCUSSION

In this section, the Runge-Kutta fourth order method is applied to determine the numerical time series solutions (t, g), (t, h), and (t, k) and the phase planes (g, v), (h, v), (k, v), respectively, for the three modes of the nonlinear system (4-6). Moreover, the fixed points of the model is obtained by solving the frequency response equations (35-37) numerically.

- **Time response solution**

A non-resonant time response and the phase plane of the three modes of vibration of the system is shown in Fig (1). In Fig (2), different resonance cases are investigated and an approximate percentage of increase, if exists, in maximum steady-state amplitude compared to that in the non-resonant case is indicated.

Internal resonance cases:

$(\omega_1 = \omega_2 = \omega_3 = 5.5), (150\%, 125\%, 150\%)$, Fig (2) (a)

$(\omega_1=\omega_2=\omega_3=2.2),(150\%,125\%,150\%)$, Fig (2) (b)

$(\omega_1=3\omega_3),(125\%,125\%,125\%)$, Fig (2) (c)

$(\omega_1=\omega_2=2.2),(250\%,200\%,\text{None})$, Fig (2) (d)

Note: Figures are in Appendix B

- **Theoretical Frequency Response solution**

The numerical results are presented graphically in Figs. (3-5) as the amplitudes a_1, a_2, a_3 against the detuning parameters $\sigma_1, \sigma_2, \sigma_3$ for different values of other parameters, each curve in these figures consists of two branches.

Considering Fig (3) (a) as basic case to compare with, it can be seen from Fig (3) (d, g) that the steady-state amplitude a_1 increases as each of the nonlinear coefficient η_{14}, and natural frequency ω_1 is increasing.

However, in Figs (3) (b, h) the steady-state amplitude a_1 decreases as each of the second mode amplitude a_2, and the nonlinear coefficients η_{11} increases.

In Figs (4), each curve consists of two branches that are bent to the right.

Considering. Fig (4) (a) as basic case to compare with, it can be seen from Figs (4) (b, d, e, f), that the steady-state amplitude a_2 increases as each of the first mode amplitude a_1, the natural frequency ω_2 and the nonlinear coefficients λ_6, λ_8 and λ_{11} are increasing.

In Fig (5), each curve consists of two branches that are bent to the left.

Considering. Fig (5) (a) as basic case to compare with, it can be seen from Figs (5) (b, d, f) steady- state amplitude a_3 decreases as each of the first mode amplitude a_1, the Linear damping coefficient δ. and the nonlinear coefficient τ_9 increases.

But in Figs (5) (e) steady-state amplitude a_3 increases as the nonlinear coefficient τ_7 increases.

Whereas the frequency response curves are shifting to the left if a_2 and τ_9 are increasing, and to the right if τ_7 increases.

Note: Figures are in Appendix C.

6. CONCLUSION

We have studied the analytic and numerical solutions of three-dimensional nonlinear differential equations that describe the oscillations of a beam subjected to external forces. The multiple scales method and Runge-Kutta fourth order numerical method are utilized to investigate the system behavior and its stability. All possible resonance cases were be extracted and effect of different parameters on system behavior at resonant condition were studied. We may conclude the following:

1- The steady-state amplitude of the first mode increases as each of the external force amplitude F and nonlinear coefficients η_3, η_5 are increased.

2- The steady-state amplitude of the first mode decreases as each of the linear damping coefficient α and the nonlinear coefficient η_4 and the natural frequency ω_1 are increased.

3- The steady-state amplitude of the second and third mode increase as the external force amplitude F increases.

4- The steady-state amplitude of the second mode decreases as each of the linear damping coefficient β, nonlinear coefficient λ_2, the second mode amplitude a_2 and the natural frequency ω_2 are increased.

5- The steady-state amplitude of the third mode increases as each of the external force amplitude F and the first mode amplitude a_1 are increased.

REFERENCES

[1] Mustafa Yaman, Analysis of subcombination internal resonances in a non- linear cantilever beam of varying orientation with tip mass, International Journal ofnonlinear Mechanics 58 (2014) 22–29.

[2] Huang , Fung and Lin, dynamic stability of amoving string undergoing three-dimensional vibation, Int. J. Mech. Sci.Vol. 37(1995),No. 2, pp. 145-160.

[3] Pai and Nayfeh, Three-Dimensional Nonlinear Vibrations of Composite Beams Chordwise Excitations, Nonlinear Dynamics (1991), 2:137- 156.

[4] Hegazy, 3:1 Internal Resonance of a String-Beam Coupled System with Cubic Nonlinearities, Communications in Nonlinear Scienceand Numerical Simulation, Vol. 15(2010) (12) pp. 4219-4229.

[5] Hegazy, Nonlinear Vibrations of a Thin Plate Under Simultaneous Internal and External Resonances, ASME Journal of Vibration and Acoustics, Vol 132(2010), 051004, 9 pages.

[6] Srinil,Rega and Chuchheepsaku, large amplitude three-dimensional free vibrations of inclined sagged elastic cables, Nonlinear Dynamics33: 129–154, 2003.

[7] Mehran Sadri and DavoodYounesian, Nonlinear harmonic vibration analysis of a plate-cavity system, Nonlinear Dyn (2013), 74:1267–1279.

[8] Guo-Ce Zhang, Hu Ding, Li-Qun Chen and Shao-Pu Yang, Galerkin method for steady-state response of nonlinear forced vibration of axially moving beams at supercritical speeds, Journal of Sound and Vibration 331 (2012), 1612–1623.

[9] Christopher Lee and Noel C. perkins, three-dimensional oscillations of suspended cables involving simultaneous internal resonances, Nonlinear Dynamics 8: 45-63,1995.

[10] L. Cveticanin , analytical methods for solving strongly non-linear differential equations, Journal of Sound and Vibration 214(2) (1998), 325-338.

[11] U. H. Hegazy, "Single-Mode Response and Control of a Hinged-Hinged Flexible Beam" Archive of Applied Mechanics (Springer), Vol. 79, pp. 335-345, 2009.

[12] Usama H. Hegazy and Mousa A. ALshaweesh (2016) , Harmonic Oscillations and Resonances in 3-D Nonlinear Dynamical System ,International Journal of Partial Differential Equations and Applications,4(1) ,7-15. (ISI Cited Publication).

7. APPENDICIES

Appendix A:

The coefficients presented in (35), are as follow:

$$\Lambda_1 = \frac{9}{256\omega_1^2}\eta_6^2 a_3^2$$

$$\Lambda_2 = \frac{9}{128\omega_1^2}\eta_6\eta_9 a_3 a_2^2 + \frac{9}{128\omega_1^2}\eta_6\eta_{12} a_3 a_2^2$$

$$\Lambda_3 = \frac{9}{128\omega_1^2}\eta_9\eta_{12} a_2^4 + \frac{9}{256\omega_1^2}\eta_{12}^2 a_2^4 + \frac{9}{256\omega_1^2}\eta_9^2 a_2^4$$

$$\Lambda_4 = \frac{3}{16\omega_1^2}\eta_6\eta_{14} a_2^2 a_3^3 + \frac{9}{128\omega_1^2}\eta_6\eta_{11} a_2^4 a_3 + \frac{3}{8\omega_1}\sigma_1\eta_6 a_3$$

$$\Lambda_5 = \frac{9}{128\omega_1^2}\eta_9\eta_{11} a_2^6 + \frac{9}{128\omega_1^2}\eta_9\eta_{12} a_2^6 + \frac{3}{64\omega_1^2}\eta_9\eta_{14} a_2^4 a_3^2 + \frac{3}{64\omega_1^2}\eta_{12}\eta_{14} a_2^4 a_3^2 - \frac{1}{256\omega_1^2}\eta_7^2 a_3^6 + \frac{3}{8\omega_1}\sigma_1\eta_9 a_2^2 + \frac{3}{8\omega_1}\sigma_1\eta_{12} a_2^2$$

$$\Lambda_6 = \frac{1}{4\omega_1}\sigma_1\eta_{14} a_3^2 a_2^2 + \frac{3}{8\omega_1}\sigma_1\eta_{11} a_2^4 + \frac{9}{256\omega_1^2}\eta_{11}^2 a_2^8 + \sigma_1^2 + \frac{1}{64\omega_1^2}\eta_{14}^2 a_3^4 a_2^4, + \frac{3}{64\omega_1^2}\eta_{11}\eta_{14} a_2^6 a_3^2 + \frac{1}{4}\alpha^2$$

$$\Lambda_7 = -\frac{1}{256\omega_1^2}\eta_8^2 a_3^6 a_2^4$$

The coefficients presented in (36), are as follow:

$$\Gamma_1 = \frac{25}{256\omega_2^2}\lambda_7^2$$

$$\Gamma_2 = \frac{15}{128\omega_2^2}\lambda_7\lambda_{10} a_1$$

$$\Gamma_3 = \frac{15}{128\omega_2^2}\lambda_6\lambda_7 a_3^2 + \frac{9}{256\omega_2^2}\lambda_{10}^2 a_1^2$$

$$\Gamma_4 = \frac{9}{128\omega_2^2}\lambda_6\lambda_{10} a_1 a_3^2$$

$$\Gamma_5 = \frac{9}{256\omega_2^2}\lambda_6^2 a_3^4 + \frac{5}{32\omega_2^2}\lambda_7\lambda_{12} a_1^2 a_3^2 + \frac{15}{128\omega_2^2}\lambda_5\lambda_7 a_1^4 - \frac{1}{8\omega_2}\sigma_3\lambda_7 + \frac{15}{128\omega_2^2}\lambda_7\lambda_9 a_3^4 - \frac{1}{512\omega_2^2}\lambda_6\lambda_{11} a_1 a_3^3$$

$$\Gamma_6=\frac{9}{128\omega_2^{\,2}}\lambda_9\lambda_{10}a_1a_3^{\,4}-\frac{9}{128\omega_2^{\,2}}\lambda_5\lambda_{10}a_1^{\,5}$$

$$-\frac{3}{40\omega_2}\sigma_3\lambda_{10}a_1+\frac{3}{32\omega_2^{\,2}}\lambda_{10}\lambda_{12}a_1^{\,3}a_3^{\,2}$$

$$\Gamma_7=\frac{9}{128\omega_2^{\,2}}\lambda_5\lambda_6a_1^{\,4}a_3^{\,2}-\frac{3}{32\omega_2^{\,2}}\lambda_6\lambda_{12}a_1^{\,2}a_3^{\,4}$$

$$+\frac{9}{128\omega_2^{\,2}}\lambda_6\lambda_9a_3^{\,6}-\frac{3}{40\omega_2}\sigma_3\lambda_6a_3^{\,2}$$

$$-\frac{2}{512\omega_2^{\,2}}\lambda_8\lambda_{11}a_1^{\,2}a_3^{\,4}-\frac{2}{512\omega_2^{\,2}}\lambda_6\lambda_8a_1a_3^{\,5}$$

$$-\lambda_{11}^{\,2}a_1^{\,2}-\lambda_6^{\,2}a_3^{\,2}$$

$$\Gamma_8=\frac{1}{4}\beta^2-\frac{1}{10\omega_2}\sigma_3\lambda_{12}a_1^{\,2}a_3^{\,2}$$

$$+\frac{1}{25}\sigma_3^{\,2}+\frac{9}{128\omega_2^{\,2}}\lambda_5\lambda_9a_1^{\,4}a_3^{\,4}$$

$$+\frac{3}{32\omega_2^{\,2}}\lambda_5\lambda_{12}a_1^{\,6}a_3^{\,2}+\frac{3}{32\omega_2^{\,2}}\lambda_9\lambda_{12}a_1^{\,2}a_3^{\,6}$$

$$-\frac{3}{40\omega_2}\sigma_3\lambda_9a_3^{\,4}+\frac{9}{256\omega_2^{\,2}}\lambda_5^{\,2}a_1^{\,8}$$

$$-\frac{3}{40\omega_2}\sigma_3\lambda_5a_1^{\,4}+\frac{9}{256\omega_2^{\,2}}\lambda_9^{\,2}a_3^{\,8}$$

$$+\frac{1}{16\omega_2^{\,2}}\lambda_{12}^{\,2}a_1^{\,4}a_3^{\,4}-\frac{1}{512\omega_2^{\,2}}\lambda_8^{\,2}a_1^{\,2}a_3^{\,6}$$

$$\Gamma_9=-2\lambda_8^{\,2}a_1^{\,2}a_3^{\,4}$$

The coefficients presented in (37), are as follow:

$$\Delta_1=\frac{25}{256\omega_3^{\,2}}\tau_6^{\,2},$$
$$\Delta_2=\frac{15}{128\omega_3^{\,2}}\tau_6\tau_{16}a_2^{\,2}+\frac{15}{128\omega_3^{\,2}}\tau_6\tau_{10}a_1^{\,2},$$
$$\Delta_3=\frac{15}{128\omega_3^{\,2}}\tau_6\tau_7a_1^{\,4}$$

$$\Delta_4=\frac{9}{256\omega_3^{\,2}}\tau_{16}^{\,2}a_2^{\,4}+\frac{9}{256\omega_3^{\,2}}\tau_{10}^{\,2}a_1^{\,4}$$

$$+\frac{5}{32\omega_3^{\,2}}\tau_6\tau_{13}a_1^{\,2}a_2^{\,2}+\frac{9}{128\omega_3^{\,2}}\tau_{10}\tau_{13}a_1^{\,2}a_2^{\,2}$$

$$+\frac{15}{128\omega_3^{\,2}}\tau_6\tau_{15}a_2^{\,4}+\frac{1}{8\omega_3}\sigma_3\tau_6$$

$$\Delta_5=\frac{9}{128\omega_3^{\,2}}\tau_7\tau_{10}a_1^{\,6}+\frac{9}{128\omega_3^{\,2}}\tau_7\tau_{16}a_1^{\,4}a_2^{\,2},$$

$$\Delta_6=\frac{3}{40\omega_3}\sigma_3\tau_{16}a_2^{\,2}+\frac{9}{128\omega_3^{\,2}}\tau_{15}\tau_{16}a_2^{\,6}$$

$$+\frac{3}{32\omega_3^{\,2}}\tau_{10}\tau_{13}a_1^{\,4}a_2^{\,2}+\frac{9}{128\omega_3^{\,2}}\tau_{10}\tau_{15}a_1^{\,2}a_2^{\,4}$$

$$+\frac{9}{256\omega_3^{\,2}}\tau_7^{\,2}a_1^{\,8}$$

$$+\frac{3}{32\omega_3^{\,2}}\tau_{13}\tau_{16}a_1^{\,2}a_2^{\,4}+\frac{3}{40\omega_3}\sigma_3\tau_{10}a_1^{\,2}$$

$$-\frac{9}{256\omega_3^{\,2}}\tau_9^{\,2}a_1^{\,6}-\frac{6}{256\omega_3^{\,2}}\tau_9\tau_{12}a_1^{\,4}a_2^{\,2}$$

$$-\frac{9}{256\omega_3^{\,2}}\tau_{12}^{\,2}a_1^{\,2}a_2^{\,4}$$

$$\Delta_7=\frac{3}{32\omega_3^{\ 2}}\tau_7\tau_{13}a_1^{\ 6}a_2^{\ 2}+\frac{9}{128\omega_3^{\ 2}}\tau_7\tau_{15}a_1^{\ 4}a_2^{\ 4}$$

$$+\frac{3}{40\omega_3}\sigma_3\tau_7a_1^{\ 4}$$

$$\Delta_8=\frac{3}{32\omega_3^{\ 2}}\tau_{13}\tau_{15}a_1^{\ 2}a_2^{\ 6}+\frac{1}{25}\sigma_3^{\ 2}+\frac{3}{40\omega_3}\sigma_3\tau_{15}a_2^{\ 4}$$

$$+\frac{9}{256\omega_3^{\ 2}}\tau_{15}^{\ 2}a_2^{\ 8}+\frac{1}{16\omega_3^{\ 2}}\tau_{13}^{\ 2}a_1^{\ 4}a_2^{\ 4}$$

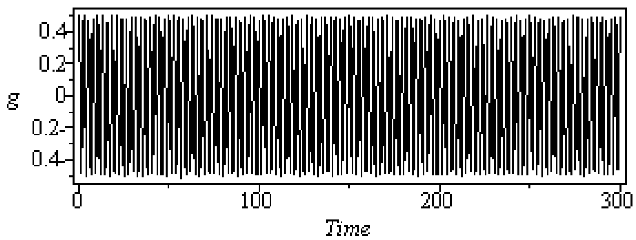
$$+\frac{1}{10\omega_3}\sigma_3\tau_{13}a_1^{\ 2}a_2^{\ 2}+\frac{1}{4}\delta^2$$

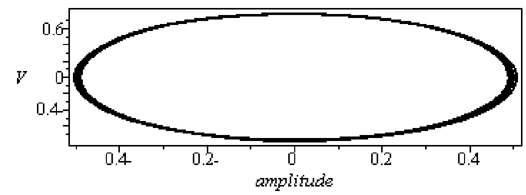
International Journal of Academic Multidisciplinary Research (IJAMR)
ISSN: 2643-9670
Vol. 3 Issue 9, September – 2019, Pages: 35-63

Appendix B:

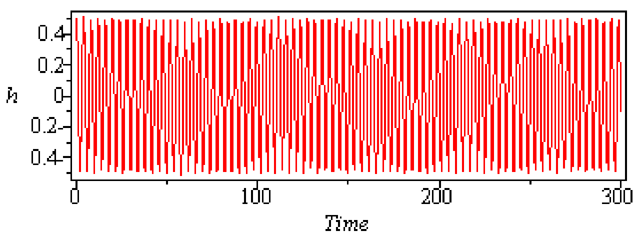
1- Fig (1): Non-resonance time solution of the 3-D free motion beam

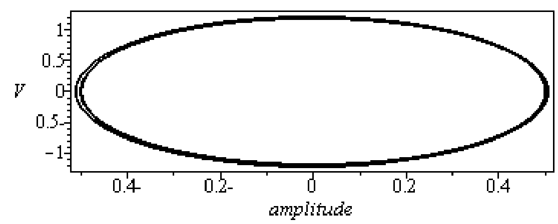
(a) the first mode:



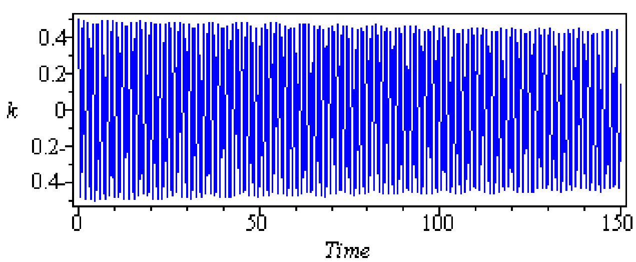


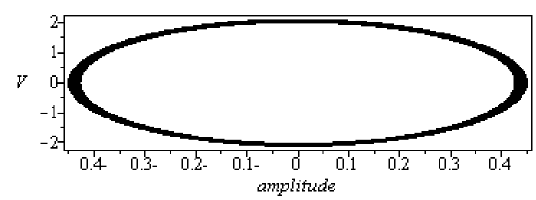
(b) the second mode





(a) the third mode



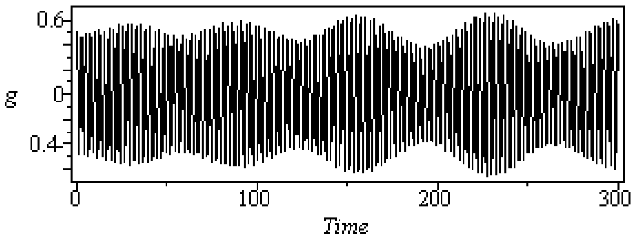


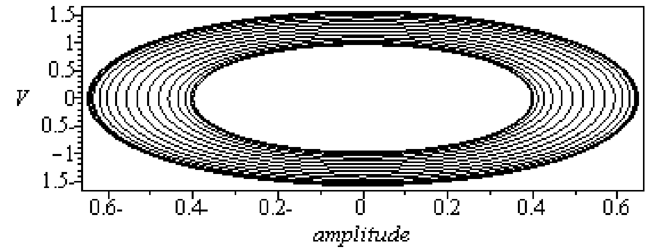
International Journal of Academic Multidisciplinary Research (IJAMR)
ISSN: 2643-9670
Vol. 3 Issue 9, September – 2019, Pages: 35-63

2- Fig (2) (a): The internal resonance condition

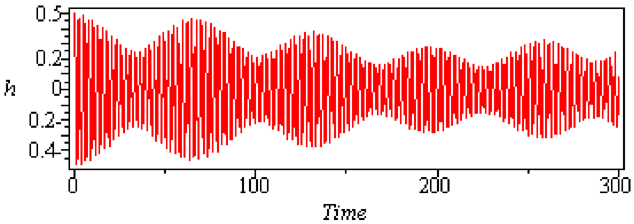
$$\omega_1 = \omega_2 = \omega_3 = 5.5$$

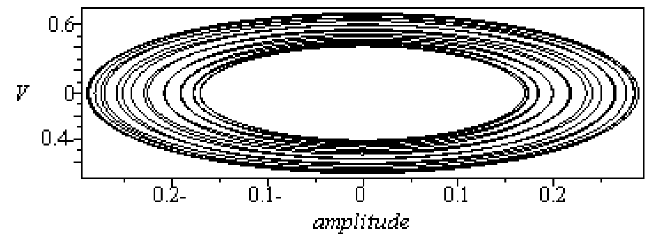
The first mode:



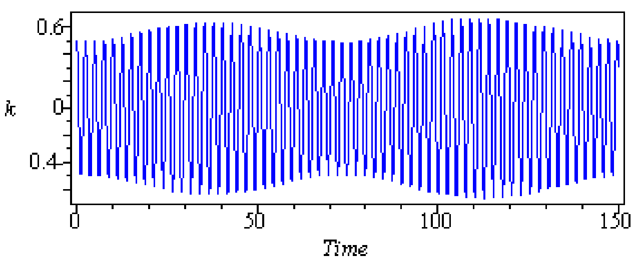


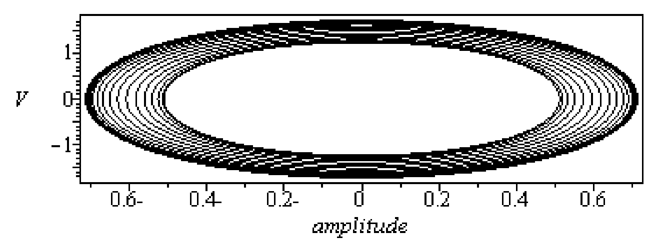
The second mode:





The third mode:



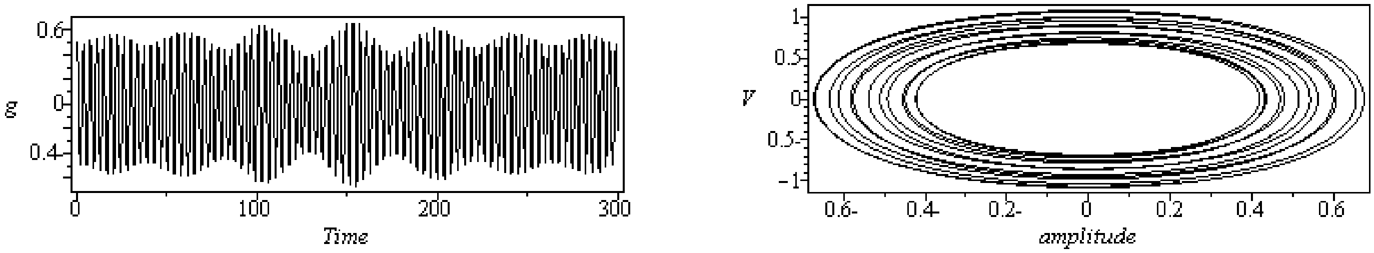


International Journal of Academic Multidisciplinary Research (IJAMR)
ISSN: 2643-9670
Vol. 3 Issue 9, September – 2019, Pages: 35-63

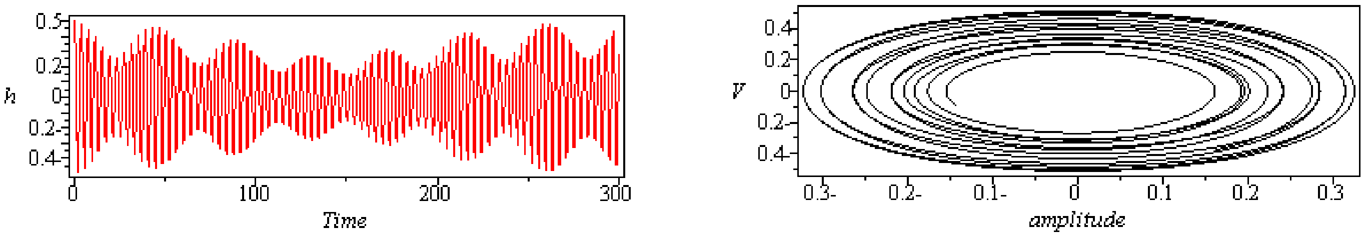
3- Fig (2) (b): The internal resonance condition

$$\omega_1 = \omega_2 = \omega_3 = 2.2$$

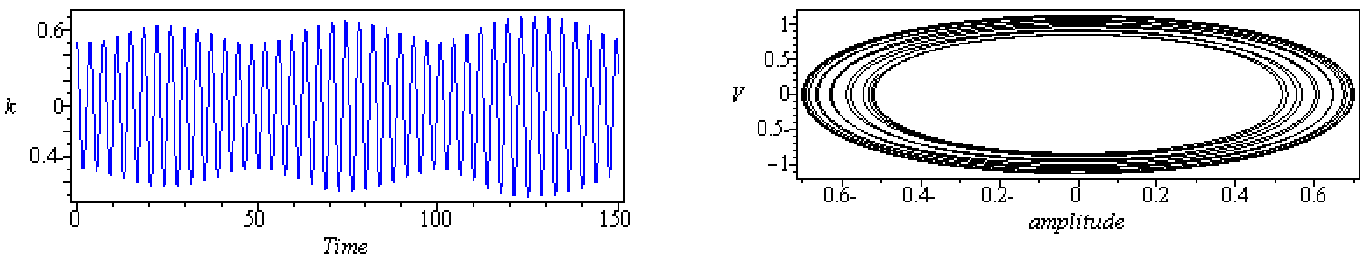
The first mode:



The second mode:



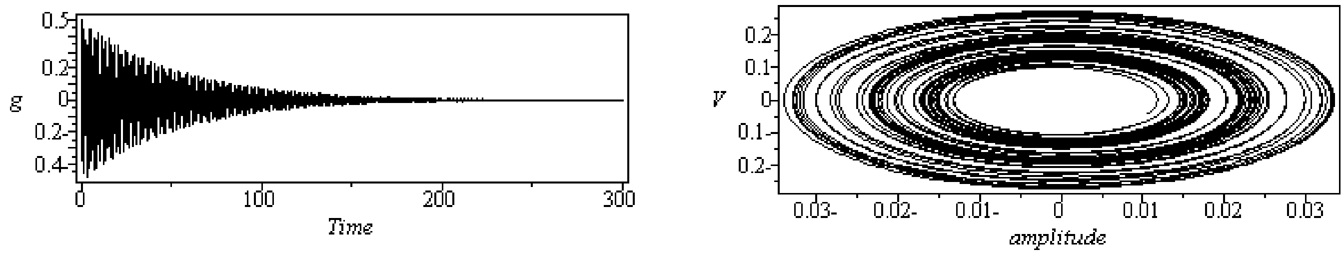
The third mode:



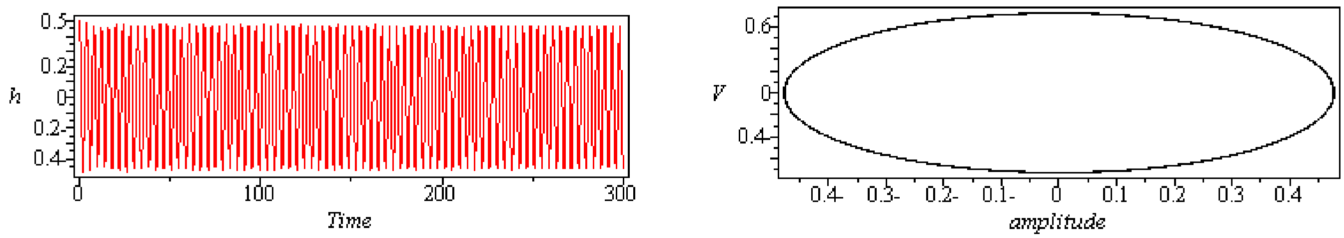
4- Fig (2) (c): The internal resonance condition

$$\omega_1 = 3\omega_3$$

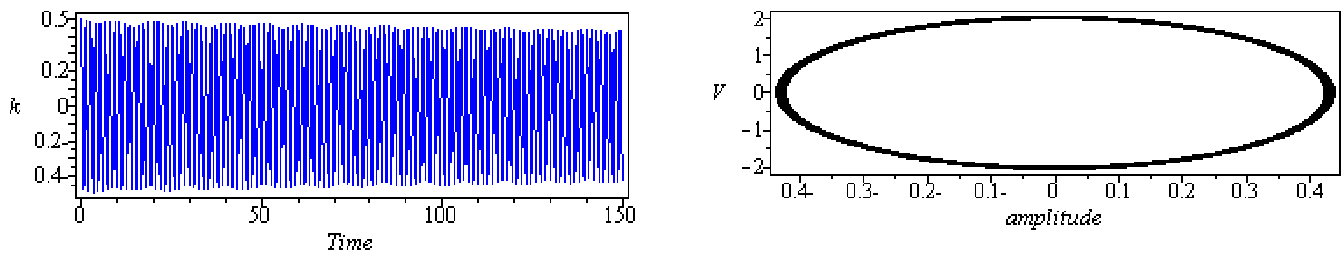
The first mode:



The second mode:



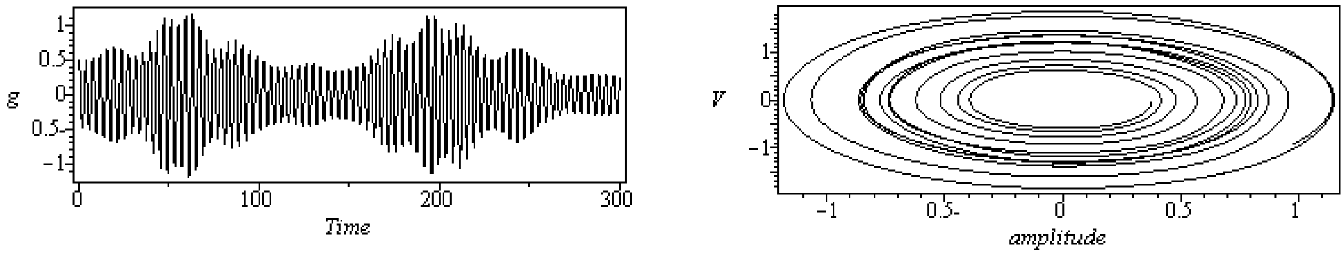
The third mode:



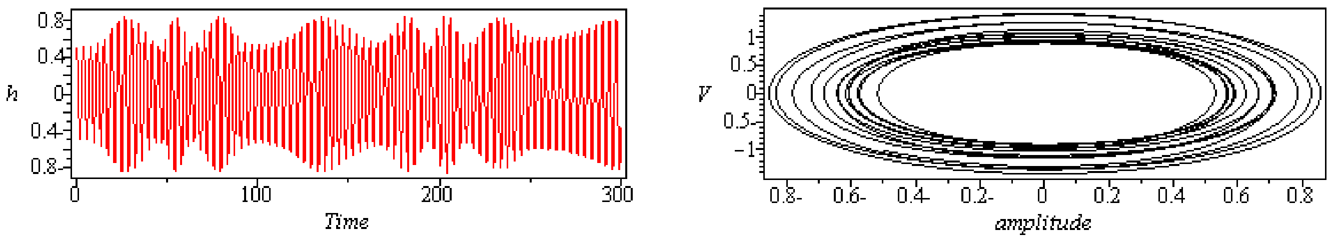
International Journal of Academic Multidisciplinary Research (IJAMR)
ISSN: 2643-9670
Vol. 3 Issue 9, September – 2019, Pages: 35-63

5- Fig (2) (d): The internal resonance condition
$\omega_1 = \omega_2 = 2.2$

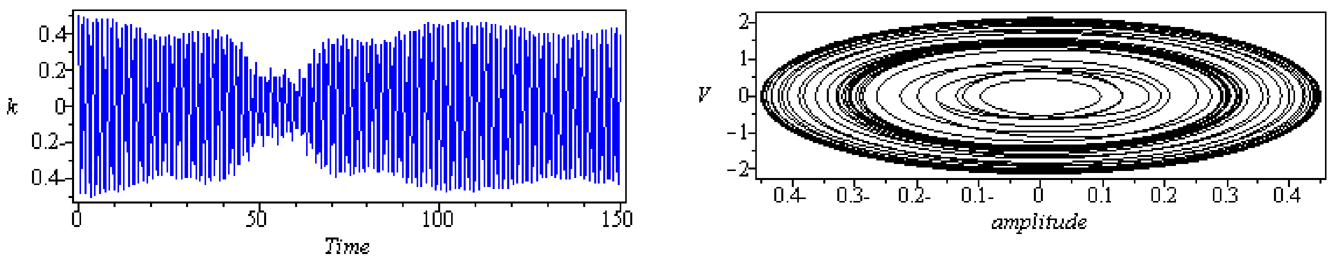
The first mode:



The second mode:



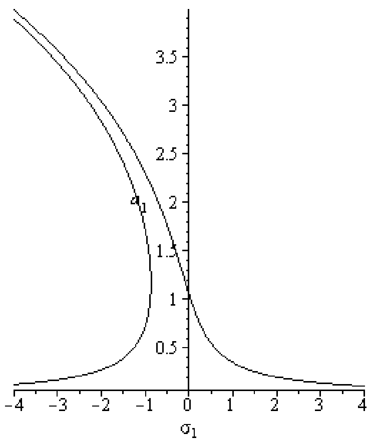
The third mode:



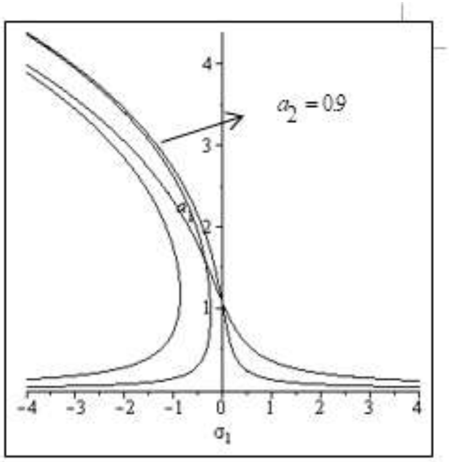
International Journal of Academic Multidisciplinary Research (IJAMR)
ISSN: 2643-9670
Vol. 3 Issue 9, September – 2019, Pages: 35-63

Appendix C:

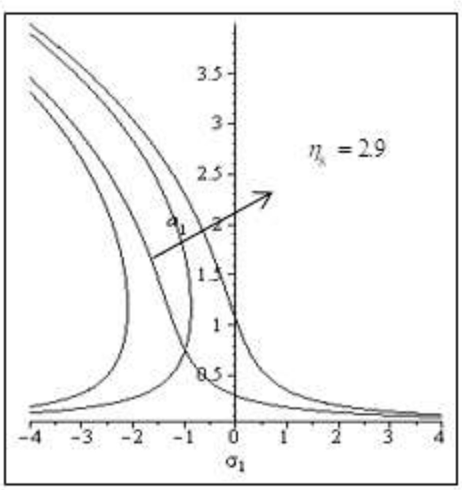
1- Fig (3) (a - f):



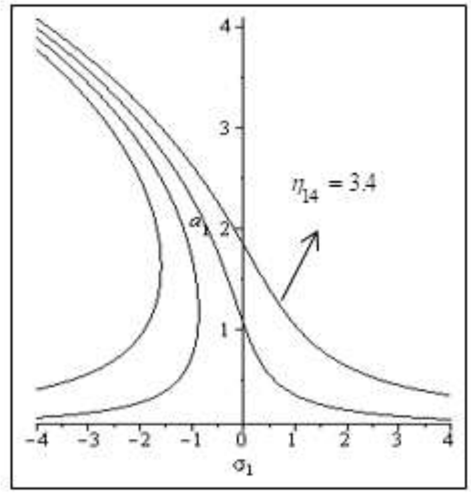
a: Basic Case



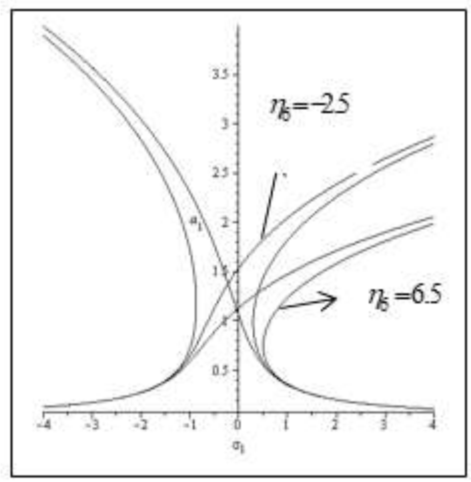
b: Steady-state amplitude of second mode



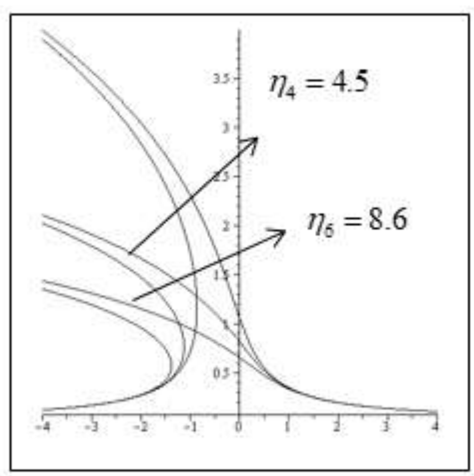
c: Non-linear parameter



d: Non-linear parameter



e: Non-linear parameters



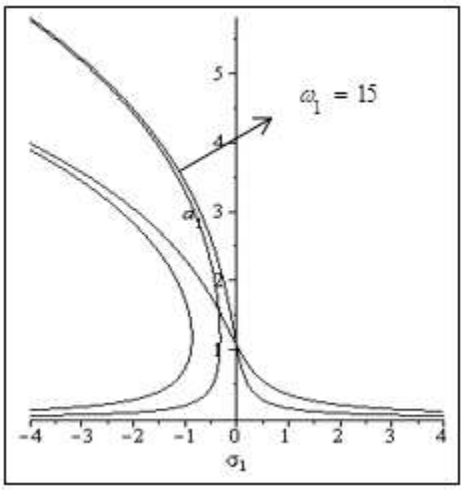
f: Non-linear parameters

2- Fig (3) (g, h):

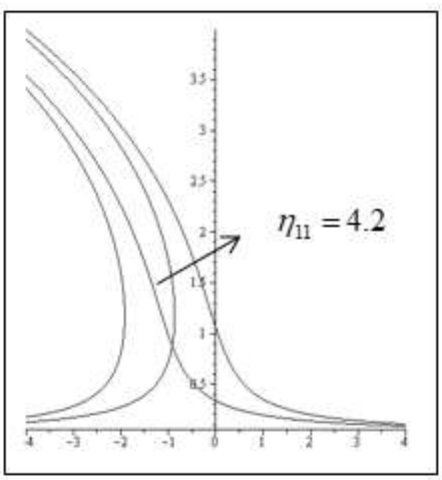
Frequency response curves of the first mode of the system at resonance

$\eta_6=0.5,\eta_7=0.1,\eta_8=0.9,\eta_9=0.2,\eta_{11}=0.5,\eta_{12}=0.5,\eta_{14}=0.4$

$a_2=1.7,a_3=2.5,\alpha=0.00008,\omega_1=5.5$



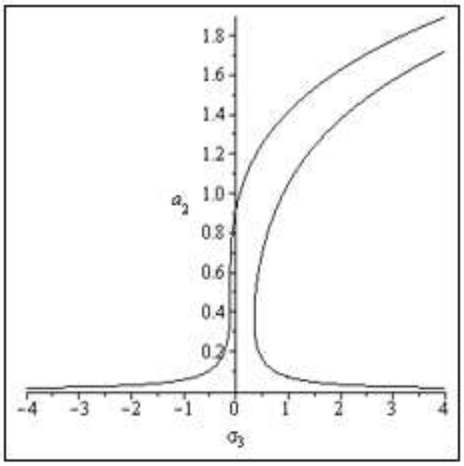
g: Natural frequency



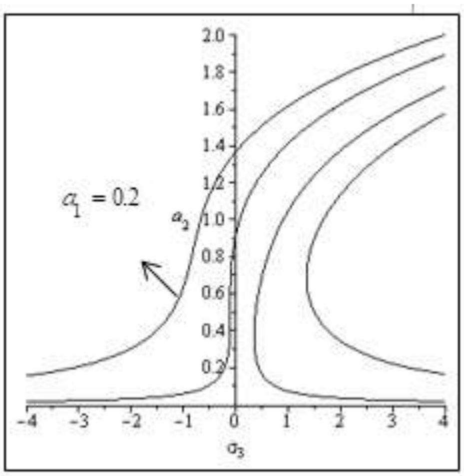
h: Non-linear parameter

International Journal of Academic Multidisciplinary Research (IJAMR)
ISSN: 2643-9670
Vol. 3 Issue 9, September – 2019, Pages: 35-63

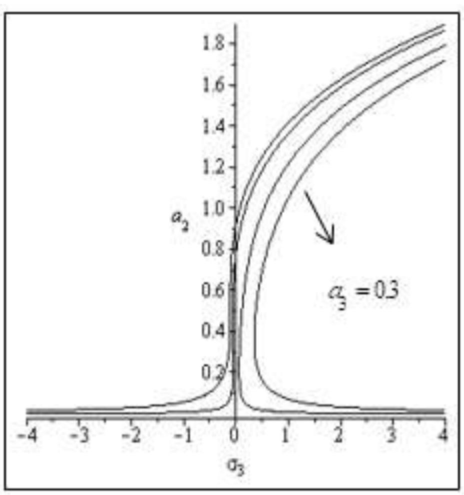
3- Fig (4) (a - f):



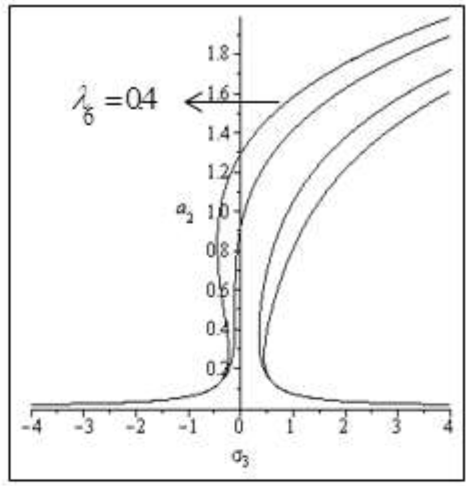
a: Basic Case



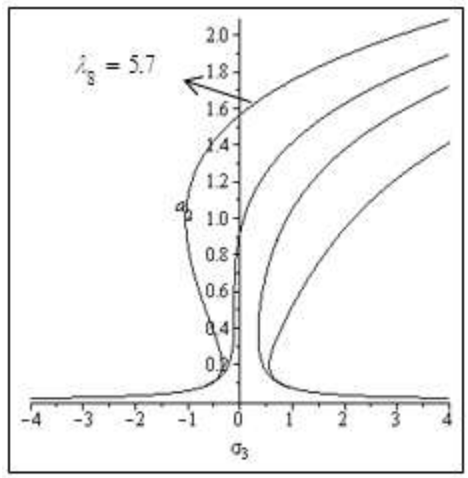
b: Steady-state amplitude of first mode



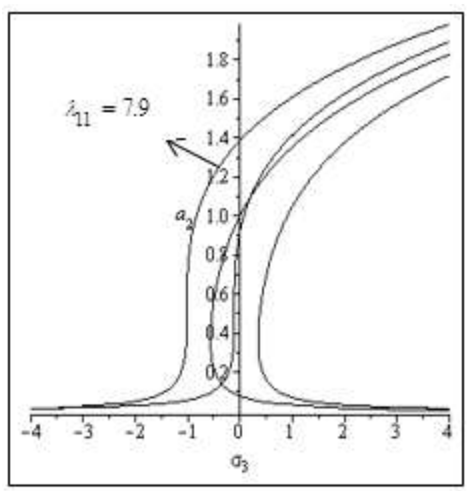
c: Steady-state amplitude of third mode



d: Non-linear parameter



e: Non-linear parameter



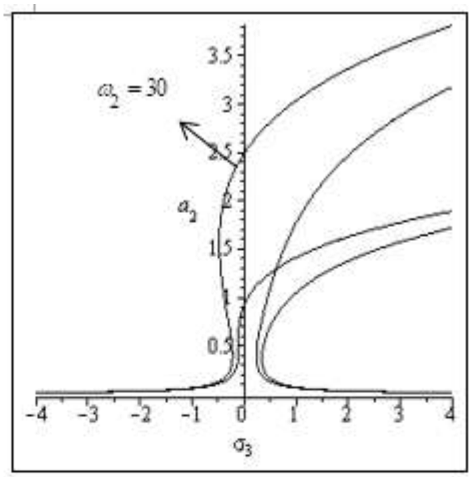
f: Non-linear parameter

4- Fig (4) (g - i):

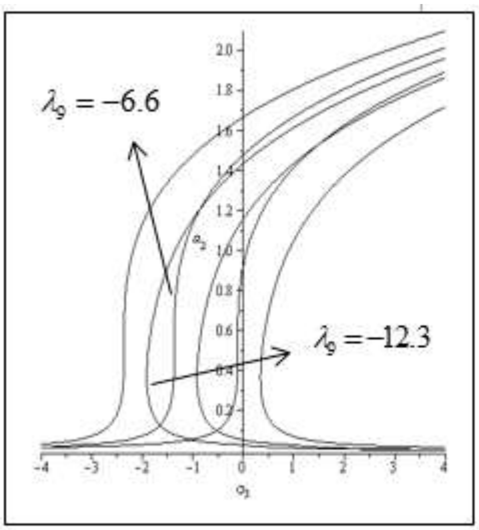
Frequency response curves of the second mode of the system at resonance

$$\lambda_5=0.2,\lambda_6=0.1,\lambda_7=0.5,\lambda_8=0.7,\lambda_9=0.6,\lambda_{10}=0.1,\lambda_{11}=0$$

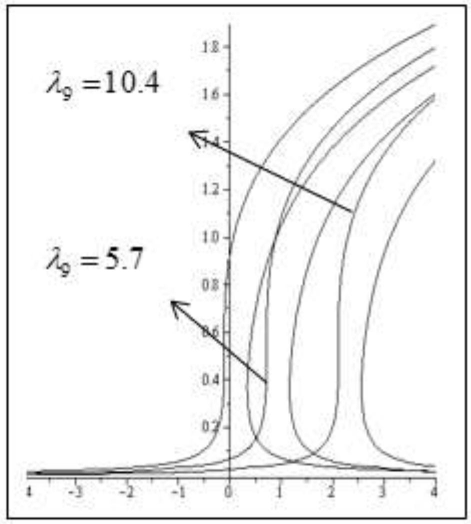
$$a_1=0.02,a_3=0.8,\beta=0.00008,\omega_2=2.2$$



g: Natural frequency



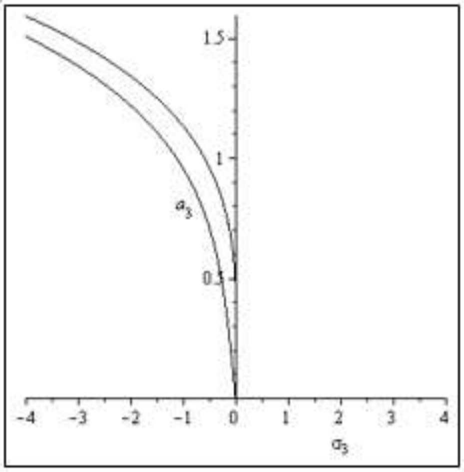
h: Non-linear parameters



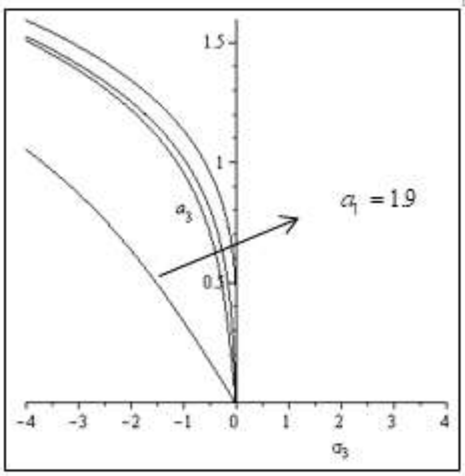
i: Non-linear parameters

International Journal of Academic Multidisciplinary Research (IJAMR)
ISSN: 2643-9670
Vol. 3 Issue 9, September – 2019, Pages: 35-63

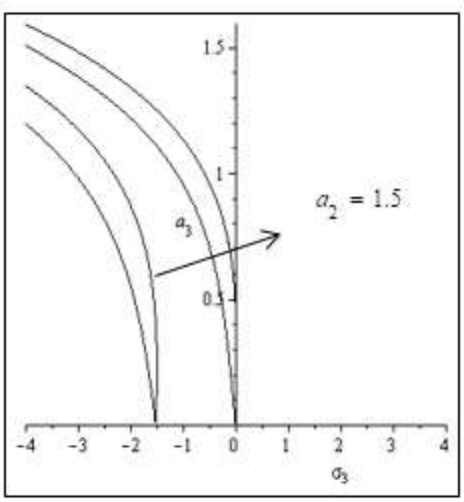
5- Fig (5) (a – f):



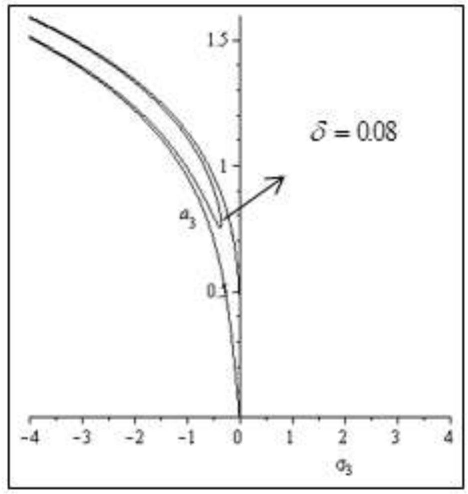
a: Basic Case



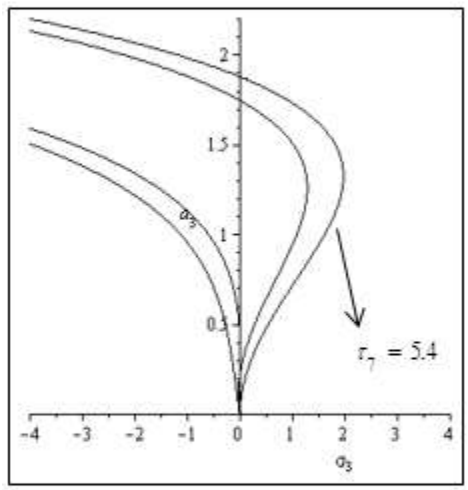
b: Steady-state amplitude of first mode



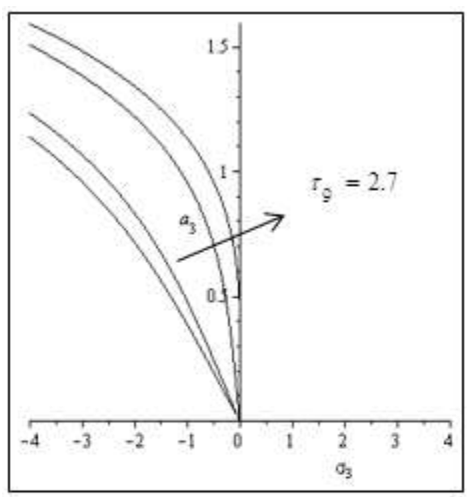
c: Steady-state amplitude second mode



d: Linear damping coefficient



e: Non-linear parameter



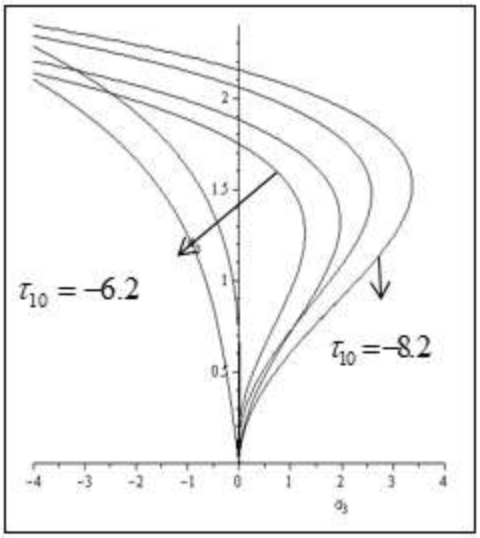
f: Non-linear parameter

6- Fig (5) (g, h):

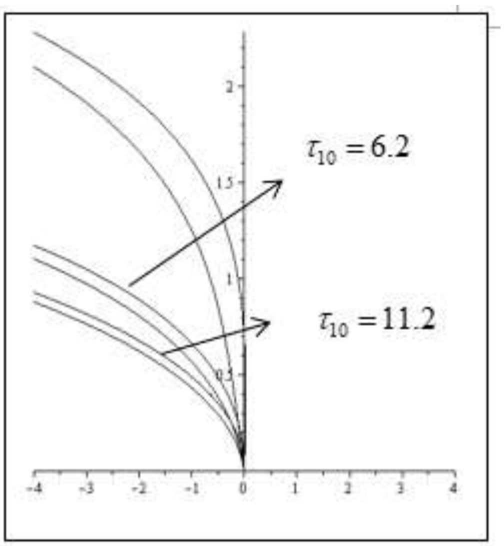
Frequency response curves of the third mode of the system at resonance

$$\tau_6 = 0.3, \tau_7 = 0.4, \tau_9 = 0.7, \tau_{10} = 0.2, \tau_{12} = 0.45, \tau_{13} = 0.5, \tau_{15} = 0.7, \tau_{16} = 0.5$$

$$a_1 = 1.1, a_2 = 0.1, \delta = 0.00008, \omega_3 = 2.2$$



g: Non-linear parameter



h: Non-linear parameter