International Journal of Academic and Applied Research (IJAAR)
ISSN: 2643-9603
Vol. 3 Issue 10 , October, Pages: 36-41

The Local Fractional Metric Dimension of Corona Product of Complete Graph and Cycle Graph

Siti Aisyah[1], Ratna Dwi C.[2], Mohammad Imam Utoyo[3] and Liliek Susilowati[4]

[1,2]Department of Mathematics, Kaltara University, North Kalimantan, Indonesia
[1]Correspondence author: aisyah_rasyid84@yahoo.com
[3,4]Department of Mathematics, Airlangga University, Surabaya 60115, Indonesia
[2]m.i.utoyo@fst.unair.ac.id, [3]liliek-s@fst.unair.ac.id

Abstract: A vertex x in a connected graph G is said to resolve a pair of vertices $\{u, v\}$ in G if the distance from u to x is not equal to the distance from v to x. A set S of vertices of G is a resolving set for G if every pair of vertices is resolved by some vertices of S. The smallest cardinality of a resolving set for G is called the metric dimension of G, denoted by $\dim(G)$. For the pair of two adjacent vertices $\{u, v\}$ is called the local *resolving neighbourhood* and denoted by $R_l\{u, v\}$. A real valued function $g_l: V(G) \to [0,1]$ is a *local resolving function* of G if $g_l(R_l\{u, v\}) \geq 1$ for every two adjacent vertices $u, v \in V(G)$. The *local fractional metric dimension* of G is defined as

$$dim_{fl}(G) = \min\{|g_l|: g_l\ is\ local\ resolving\ function\ of\ G\},$$

where $|\boldsymbol{g_l}| = \sum_{v \in V} \boldsymbol{g_l}(\boldsymbol{v})$. Let $\boldsymbol{G}$ and $\boldsymbol{H}$ be two graphs of order $\boldsymbol{n_1}$ and $\boldsymbol{n_2}$, respectively. The corona product $\boldsymbol{G \odot H}$ is defined as the graph obtained from $\boldsymbol{G}$ and $\boldsymbol{H}$ by taking one copy of $\boldsymbol{G}$ and $\boldsymbol{n_1}$ copies of $\boldsymbol{H}$ and joining by an edge each vertex from the $\boldsymbol{i^{th}}$-copy of $\boldsymbol{H}$ with the $\boldsymbol{i^{th}}$ -vertex of $\boldsymbol{G}$. In this paper we study the problem of finding exact values for the fractional local metric dimension of corona product of graphs.

Keywords: local fractional metric dimension; resolving function; corona product graphs; complete graph; cycle graph.

1. Introduction

Let $G = (V(G), E(G))$ be a finite, simple, undirected, connected graph. For a graph G, $V(G)$ is the vertex set of G and $E(G)$ is the edge set of G, respectively. For any two vertices x and y of G, $d_G(x, y)$ denotes the distance between x and y, $R_G\{x, y\}$ denotes the set of vertices z such that $d_G(x, z) \neq d_G(y, z)$. A resolving set of G is a subset W of $V(G)$ such that $W \cap R_G\{x, y\} \neq \emptyset$ for any two distinct vertices x and y of G [6]. The metric dimension of G, denoted by $dim(G)$, is the minimum cardinality of the resolving sets of G. Metric dimension was first introduced in the 1970s, independently by Harary and Melter in [1] and by Slater in [2].

Let $g: V(G) \to [0, 1]$ be a real valued function. For $W \subseteq V(G)$, define $g(W) = \sum_{v \in W} g(v)$. We call g is *resolving function* of G if $g(R_G\{x, y\}) \geq 1$ for any two distinct vertices x and y of G. The fractional metric dimension, denoted by $dim_f(G)$, is given by $dim_f(G) = \min\{|g|: g\ is\ a\ resolving\ function\ of\ G\}$, where $|g| = g(V(G))$. Arumugam and Mathew [3], [4] formally introduced the fractional metric dimension of graphs and obtained some basic results. For an ordered set $X = \{x_1,\ x_2, \ldots, x_k\} \subseteq V(G)$ of vertices, we use the ordered k-tuple $r(u|X) = (d(u, x_1), d(u, x_2), \ldots, d(u, x_k))$ as the representation of u with respect to X. If every two adjacent vertices of G has distinct representation with respect to X then X is called as a *local resolving set* for G. A minimum local resolving set is called local basis for G.A local metric dimension for G, denote by $dim_l(G)$, is the number of vertices in a local basis for G by Okamoto *et al*. [5]. From the idea of Arumugam and Mathew [3], [4] Okamoto *et al*.[5] we define the *local resolving neighbourhood* as follow.

For the pair $\{u, v\}$ of two adjacent vertices of G, we define the *local resolving neighbourhood* of the pair $\{u, v\}$ as $R\{u, v\} = \{x \in V(G): d(u, x) \neq d(v, x)\}$. Also, for each vertex $x \in V(G)$, we define the local resolvent neighbourhood of x as $R\{x\} = \{\{u, v\} \in V(G): d(u, v) \neq d(v, x)\}$ is denote by $R_l\{u, v\}$ and is called the *local resolving neighbourhood* of $\{u, v\}$. A real valued function $g_l: V(G) \to [0,1]$ is a *local resolving function* of G if $g_l(R_l\{u, v\}) \geq 1$ for every two adjacent vertices $u, v \in V(G)$. The *local fractional metric dimension* of G, denote $dim_{fl}(G)$ is defined as

$$dim_{fl}(G) = \min\left\{\begin{matrix}|g_l|: g_l\ is\ local\ resolving \\ function\ of\ G\end{matrix}\right\},$$

where $|g_l| = \sum_{v \in V} g_l(v)$. In this paper we study the local fractional metric dimension of corona product graphs. We begin by giving some basic concepts and notations.

2. Corona Product

Let G and H be two graphs of order n and m, respectively. The corona product $G \odot H$ was defined by [6] as the graph obtained from G and H by taking one copy of G and n copies of H and joining by an edge each vertex from the i^{th}-copy of H with the i^{th}-vertex of G. We will denote by $V = \{v_1, v_2, \ldots, v_n\}$ the set of vertices of G and by $H_i = (V_i,\ E_i)$ the copy of H such that $v_i \sim x$ for every $x \in V_i$. Figure 1 shows two examples of corona product graphs where the factors are non-trivial.

International Journal of Academic and Applied Research (IJAAR)
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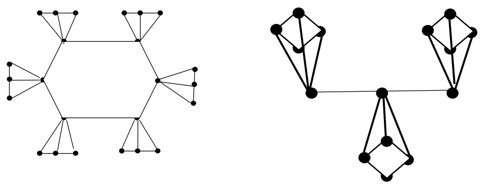

Figure 1: From the left, we show the corona graphs $C_6 \odot P_3$ and $P_3 \odot C_4$.

The join $G \odot H$ is defined as the graph obtained from disjoint graphs G and H by taking one copy of G and one copy of H and joining by an edge each vertex of G with each vertex of H. Notice that the corona graph $K_1 \odot H$ is isomorphic to the join graph $K_1 + H$.

The fractional metric dimension of corona product graphs was studied in [7], besides the dimensions of strong fractional metric have been studied in [8], [9], the fractional metric dimension of permutation graphs was studied in [10]. In this paper we study the local fractional metric dimension of corona product graphs. We determine the local fractional metric dimension of corona product of graphs for complete graph and cycle graph, there are $K_n \odot K_m$, $C_n \odot K_m$, $K_n \odot C_m$ and $C_n \odot C_m$.

Definition 1.1 [11] Let G be a connected graph of order n and H (not necessarily connected) be a graph with $|H| \geq 2$. A graph G *corona* H, $G \odot H$, is defined as a graph which formed by taking n copies of graphs $H_1, H_2, \ldots, H_n$ of H and connecting $i-th$ vertex of G to the vertices of H_i. Throughout this section, we refer to H_i as a $i-th$ copy of H connected to $i-th$ vertex of G in $G \odot H$ for every $i = \{1, 2, \ldots, n\}$.
We begin by giving some basic concepts and notations.
The following is presented notation (index) on the graph $G \odot H$.

a. The set point of $V(G)$ is $U_{0i} = \{v_{0i} : v_{0i} \in V(G)\}$ with $i = 1,2, \ldots, n$.
b. The set point of $V(H)$ is $U_{ji} = \{v_{ij} : v_j \in V(H)\}$ with $j = 1,2, \ldots, m$.

Figure 1 shows example of corona product graphs.

3. RESULT

In this section, we investigate the value of $dim_{fl}(G \odot H)$ when G and H is a complete graph and cycle graph. Previously we called the results of local fractional metric dimensions on special graphs.

Theorem 2.1. For any graph G of order n,

a. *For the path graph* (P_n), $dim_{fl}(P_n) = 1$.
b. *Forcomplete grah* (K_n), $dim_{fl}(K_n) = \frac{n}{2}$.
c. *For the cycle graph* (C_n) *,we have*
$$dim_{fl}(C_n) = \begin{cases} \dfrac{n}{n-1}, & if\ n\ is\ odd \\ 1, & if\ n\ is\ even \end{cases}$$
d. *For the star graph*(S_n), $dim_{fl}(S_n) = 1$.
e. *For the bipartite graph*$(K_{n,m})$, $dim_{fl}(K_{n,m}) = 1$
f. *Let* G *be a connected graphs of order* n*. Then* $dim_{fl}(G) = 1$ *if and only if* G *is bipartite graph.*
g. *Let* G *be a connected graphs of order* n*. Then* $dim_{fl}(G) = \frac{n}{2}$ *if and only if* G *is complete graph.*

Theorem 2.2. Let $K_n \odot K_m$ be a connected graph with $n, m \geq 3$, then
$$dim_{fl}(K_n \odot K_m) = |V(K_n)| dim_{fl}(K_m)$$
Proof. The graph $K_n \odot K_m$ is a connected graph. Let $f_l : V(K_n \odot K_m) \to [0,1]$ is a local resolving function. Any two adjacent vertices $u, v \in V(K_n \odot K_m)$. There are three possibilities u and v.

i. If u, v in the same leaf, then there is $i \in \{1,2, \ldots, n\}$ and $j, k \in \{1,2,3, \ldots, m\}$ with $j \neq k$ such that $u = v_{ji}$ and $v = v_{ki}$, we get $R_l\{u, v\} = \{v_{ji}, v_{ki}\}$ local resolving function are $f_l(v_{ji}) + f_l(v_{ki}) \geq 1$. Therefore, obtained
$$(m-1) \sum_{v \in U_i} f_l(v) \geq \binom{m}{2}$$
because the number of points in the parent is n then,
$$(m-1)\left(\sum_{v \in U_1} f_l(u) + \sum_{v \in U_2} f_l(u) + \cdots + \sum_{v \in U_n} f_l(u)\right) \geq n\binom{m}{2},$$
$$\Leftrightarrow (m-1)\left(\sum_{z \in V(K_n \odot K_m)} f_l(z) - \sum_{v \in U} f_l(v)\right) \geq n\binom{m}{2}$$
because $\sum_{v \in U} f_l(v) \geq 0$, then
$$\sum_{z \in V(K_n \odot K_m)} f_l(z) \geq n\frac{m}{2} \quad (1)$$
ii. If u, v in parent, then there are $i, j \in \{1,2, \ldots, n\}$ with $i \neq j$, such that $u = v_{0i}$ and $v = v_{0j}$. Local resolving neighborhood $R_l\{u, v\} = \{v_{0i}, v_{1i}, \ldots, v_{mi}, v_{0j,} v_{1j}, \ldots, v_{mj}\}$ so that $f_l(R_l\{u, v\}) = \sum_{u \in U_i} f_l(u) + \sum_{u \in U_j} f_l(u) + f_l(v_{0i}) + f_l(v_{0j}) \geq 1$. Therefore, obtained
(5.9)

$$(n-1)\sum_{v\in U} f_l(v) + (n-1)\sum_{v\in U_i} f_l(u) \geq \binom{n}{2}$$

$$\Leftrightarrow (n-1)\sum_{z\in V(K_n\odot K_m)} f_l(z) \geq \frac{n!}{(n-2)!.2!}$$

$$\Leftrightarrow (n-1)\sum_{z\in V(K_n\odot K_m)} f_l(z) \geq \frac{n(n-1)}{2}$$

$$\Leftrightarrow \sum_{z\in V(K_n\odot K_m)} f_l(z) \geq \frac{n}{2}. \quad (2)$$

iii. If u in parent and v in leaf of u, then there are $i \in \{1,2,\dots,n\}$ and $j \in \{1,2,3,\dots,m\}$, such that $u = v_{0i}$ and $v = v_{ji}$. Local resolving neighborhood $R_l\{u,v\} = V(K_n \odot K_m) - (V(U_i)\backslash\{v_{ji}\})$ so that $f_l(R_l\{u,v\}) = \sum_{z\in V(K_n\odot K_m)} f(z) - (\sum_{u\in U_i} f_l(u) - f_l(v_{ji})) \geq 1$. Therefore, obtained

$$m\sum_{z\in V(K_n\odot K_m)} f_l(z) - (m-1)\sum_{j=1}^{m} f_l(v_{ji}) \geq m$$

because the number of points in the parent is n, then

$$nm\sum_{z\in V(K_n\odot K_m)} f_l(z) - (m-1)\sum_{i=1}^{n}\sum_{j=1}^{m} f_l(v_{ji}) \geq nm.$$

because $\sum_{i=1}^{n}\sum_{j=1}^{m} f_l(v_{ji}) \geq 0$, then

$$nm\sum_{z\in V(K_n\odot K_m)} f_l(z) \geq nm\sum_{z\in V(K_n\odot K_m)} f_l(z) - (m-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{m} f_l(v_{ji})\right) \geq nm.$$

therefore,

$$\sum_{z\in V(K_n\odot K_m)} f_l(z) \geq 1. \quad (3)$$

Based on the results of the description above, in order to all conditions the maximum values taken from equations $(1)-(3)$ are:

$$\sum_{z\in V(K_n\odot K_m)} f_l(z) \geq n\frac{m}{2}.$$

As a result:

$$dim_{fl}(K_n \odot K_m) = min\left\{\sum_{z\in V(K_n\odot K_m)} f_l(z) : f_l \text{ local resolving function}\right\} = n\frac{m}{2}.$$

Because ordo of K_n is n and $dim_{fl}(K_m) = \frac{m}{2}$ then

$$dim_{fl}(K_n \odot K_m) = |V(K_n)| dim_{fl}(K_m). \blacksquare$$

Theorem 2.3. Let $C_n \odot K_m$ be a connected graph with $n, m \geq 3$, then $dim_{fl}(C_n \odot K_m) = |V(C_n)| dim_{fl}(K_m)$.

Proof. The graph $C_n \odot K_m$ is a connected graph. Let $f_l: V(C_n \odot K_m) \to [0,1]$ is a local resolving function. Any two adjacent vertices $u, v \in V(C_n \odot K_m)$. There are three possibilities u and v.

i. If u, v in the same leaf, then there is $i \in \{1,2,\dots,n\}$ and $j,k \in \{1,2,3,\dots,m\}$ with $j \neq k$ such that $u = v_{ji}$ and $v = v_{ki}$, we get $R_l\{u,v\} = \{v_{ji}, v_{ki}\}$ local resolving function are $f_l(v_{ji}) + f_l(v_{ki}) \geq 1$. Therefore, obtained

$$(m-1)\sum_{v\in U_i} f_l(v) \geq \binom{m}{2}$$

because the number of points in the parent is n then,

$$(m-1)\left(\sum_{v\in U_1} f_l(u) + \sum_{v\in U_2} f_l(u) + \cdots + \sum_{v\in U_n} f_l(u)\right) \geq n\binom{m}{2},$$

$$\Leftrightarrow (m-1)\left(\sum_{z\in V(C_n\odot K_m)} f_l(z) - \sum_{v\in U} f_l(v)\right) \geq n\binom{m}{2}$$

because $\sum_{v\in U} f_l(v) \geq 0$, then

$$\sum_{z\in V(C_n\odot K_m)} f_l(z) \geq n\frac{m}{2}. \quad (4)$$

ii. If u, v in parent, then there are $i, j \in \{1,2,\dots,n-1\}$ with $i \neq j$, such that $u = v_{0i}$ and $v = v_{0(i+1)}$ or $u = v_{0n}$ and $v = v_{01}$. Local resolving neighborhood

$$R_l\{u,v\} = \begin{cases} V(C_n \odot K_m), & n \text{ genap} \\ V(C_n \odot K_m) - \{v_k\} \cup U_k, & n \text{ gasal} \end{cases},$$

Therefore, obtained

$$f_l(R_l\{u,v\}) = \begin{cases} \sum_{z\in V(C_n\odot K_m)} f_l(z) \geq 1, & n \text{ genap} \\ \sum_{z\in V(C_n\odot K_m)} f_l(z) - \left(f_l(v_k) + \sum_{v\in V(U_k)} f_l(v)\right) \geq 1, & n \text{ gasal} \end{cases}. \quad (5)$$

iii. If u in parent and v in leaf of u, then there are $i \in \{1,2,\dots,n\}$ and $j \in \{1,2,3,\dots,m\}$, such that $u = v_{0i}$

and $v = v_{ji}$. Local resolving neighborhood $R_l\{u,v\} = V(C_n \odot K_m) - \left(V(U_i)\backslash\{v_{ji}\}\right)$ so that $f_l\left(R_l\{u,v\}\right) = \sum_{z\in V(C_n\odot K_m)} f(z) - \left(\sum_{u\in U_i} f_l(u) - f_l(v_{ji})\right) \geq 1$.
Therefore, obtained

$$m \sum_{z\in V(C_n\odot K_m)} f_l(z) - (m-1)\sum_{j=1}^{m} f_l(v_{ji}) \geq m$$

because the number of points in the parent is n, then

$$nm\sum_{z\in V(C_n\odot K_m)} f_l(z) - (m-1)\sum_{i=1}^{n}\sum_{j=1}^{m} f_l(v_{ji}) \geq nm.$$

because $\sum_{i=1}^{n}\sum_{j=1}^{m} f_l(v_{ji}) \geq 0$, then

$$nm\sum_{z\in V(C_n\odot K_m)} f_l(z) \geq nm\sum_{z\in V(C_n\odot K_m)} f_l(z) - (m-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{m} f_l(v_{ji})\right) \geq nm.$$

therefore,

$$\sum_{z\in V(C_n\odot K_m)} f_l(z) \geq 1. \quad (6)$$

Based on the results of the description above, in order to all conditions the maximum values taken from equations $(4)-(6)$ are:

$$\sum_{z\in V(C_n\odot K_m)} f_l(z) \geq n\frac{m}{2}.$$

As a result:

$$dim_{fl}(C_n \odot K_m) = min\left\{\sum_{z\in V(C_n\odot K_m)} f_l(z): f_l \text{ local resolving function}\right\} = n\frac{m}{2}.$$

Because ordo of C_n is n and $dim_{fl}(K_m) = \frac{m}{2}$ then

$$dim_{fl}(C_n \odot K_m) = |V(C_n)|dim_{fl}(K_m). \quad \blacksquare$$

Theorem 2.4. Let $K_n \odot C_m$ be a connected graph with $n \geq 3$ and $m > 4$, then

$$dim_{fl}(K_n \odot C_m) = \frac{1}{4}|V(K_n)||V(C_m)|.$$

Proof. The graph $K_n \odot C_m$ is a connected graph. Let $f_l: V(K_n \odot C_m) \to [0,1]$ is a local resolving function. Any two adjacent vertices $u, v \in V(K_n \odot C_m)$. There are three possibilities u and v.

i. If u, v in the same leaf, then there is $i \in \{1,2,\dots,n\}$ and $j \in \{1,2,3,\dots,m-1\}$ such that $u = v_{ji}$ and $v = v_{(j+1)i}$, or $u = v_{mi}$ and $u = v_{1i}$. Local resolving neighborhood

$$R_l\{v_{ji}, v_{(j+1)i}\} = \begin{cases} \{v_{mi}, v_{ji}, v_{(j+1)i}, v_{(j+2)i}\}, & j = 1 \\ \{v_{(j-1)i}, v_{ji}, v_{(j+1)i}, v_{(j+2)i}\}, & j = 2,3,\dots,m-2 \\ \{v_{(j-1)i}, v_{ji}, v_{(j+1)i}, v_{1i}\}, & j = m-1 \\ \{v_{(j-1)i}, v_{ji}, v_{1i}, v_{2i}\}, & j = m \end{cases}$$

from here obtained

$$f_l(R_l\{v_{ji}, v_{(j+1)i}\}) = \begin{cases} f_l(v_{mi}) + f_l(v_{ji}) + f_l(v_{(j+1)i}) + f_l(v_{(j+2)i}) \geq 1, & j = 1 \\ f_l(v_{(j-1)i}) + f_l(v_{ji}) + f_l(v_{(j+1)i}) + f_l(v_{(j+2)i}) \geq 1, & j = 2,3,\dots,m-2 \\ f_l(v_{(j-1)i}) + f_l(v_{ji}) + f_l(v_{(j+1)i}) + f_l(v_{1i}) \geq 1, & j = m-1 \\ f_l(v_{(j-1)i}) + f_l(v_{ji}) + f_l(v_{(j+1)i}) + f_l(v_{(j+2)i}) \geq 1, & j = m \end{cases}.$$

based on the description above,

$$4\sum_{j=1}^{m} f_l(v_{ji}) \geq m$$

$$\Leftrightarrow \sum_{j=1}^{m} f_l(v_{ji}) \geq \frac{m}{4}.$$

Because the number of points in the parent is n, then

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f_l(v_{ji}) \geq \frac{m}{4}n. \quad (7)$$

ii. If u, v in parent, then there are $i, j \in \{1,2,\dots,n\}$ with $i \neq j$, such that $u = v_{0i}$ and $v = v_{0j}$. Local resolving neighborhood $R_l\{u,v\} = \{v_{0i}, v_{1i}, \dots, v_{mi}, v_{0j}, v_{1j}, \dots, v_{mj}\}$ so that $f_l\left(R_l\{u,v\}\right) = \sum_{u\in U_i} f_l(u) + \sum_{u\in U_j} f_l(u) + f_l(v_{0i}) + f_l(v_{0j}) \geq 1$. Therefore,

$$(n-1)\sum_{v\in U} f_l(v) + (n-1)\sum_{v\in U_i} f_l(u) \geq \binom{n}{2}$$

$$\Leftrightarrow (n-1)\sum_{z\in V(K_n\odot C_m)} f_l(z) \geq \frac{n!}{(n-2)!.2!}$$

$$\Leftrightarrow (n-1)\sum_{z\in V(K_n\odot C_m)} f_l(z) \geq \frac{n(n-1)}{2}$$

$$\Leftrightarrow \sum_{z\in V(K_n\odot C_m)} f_l(z) \geq \frac{n}{2}. \quad (8)$$

iii. If u in parent and v in leaf of u, then there are $i \in \{1,2,\dots,n\}$ and $j \in \{1,2,3,\dots,m\}$, such that $u = v_{0i}$ and $v = v_{ji}$. Local resolving neighborhood

$$R_l\{v_{0i}, v_{ji}\} = \begin{cases} V(K_n \odot C_m) - \{v_{(j+1)i}, v_{mi}\}, & j = 1 \\ V(K_n \odot C_m) - \{v_{(j-1)i}, v_{(j+1)i}\}, & j = 2,3,\dots,m-1. \\ V(K_n \odot C_m) - \{v_{(j-1)i}, v_{1i}\}, & j = m \end{cases}$$

so that

$$f_l(R_l\{v_{0i}, v_{ji}\}) = \begin{cases} \sum_{z\in V(K_n\odot C_m)} f(z) - \left(f_l(v_{(j+1)}) + f_l(v_{mi})\right) \geq 1, & j = 1 \\ \sum_{z\in V(K_n\odot C_m)} f(z) - \left(f_l(v_{(j-1)}) + f_l(v_{(j+1)i})\right) \geq 1, & j = 2,3,\dots,m-1 \\ \sum_{z\in V(K_n\odot C_m)} f(z) - \left(f_l(v_{(j-1)}) + f_l(v_{1i})\right) \geq 1, & j = m \end{cases}.$$

Therefore,

$$m \sum_{z\in V(K_n\odot C_m)} f_l(z) - 2\left(\sum_{j=1}^{m} f_l(v_{ji})\right) \geq m.$$

because the number of points in the parent is n, then

$$nm \sum_{z\in V(K_n\odot C_m)} f_l(z) - 2\left(\sum_{i=1}^{n}\sum_{j=1}^{m} f_l(v_{ji})\right) \geq nm.$$

because $\sum_{i=1}^{n}\sum_{j=1}^{m} f_l(v_{ji}) \geq 0 \geq 0$, then

$$n.m \sum_{z\in V(K_n\odot C_m)} f_l(z) \geq n.m \sum_{z\in V(K_n\odot C_m)} f_l(z) - 2\left(\sum_{i=1}^{n}\sum_{j=1}^{m} f_l(v_{ji})\right) \geq nm.$$

therefore,

$$\sum_{z\in V(K_n\odot C_m)} f_l(z) \geq 1. \qquad (9)$$

Based on the results of the description above, in order to all conditions the maximum values taken from equations $(7)-(9)$ are:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f_l(v_{ji}) \geq \frac{m}{4} n.$$

As a result:

$$dim_{fl}(K_n \odot C_m) = min\left\{\sum_{z\in V(K_n\odot C_m)} f_l(z) : f_l \text{ fungsi pembeda lokal}\right\} = \frac{m}{4} n.$$

Because ordo of K_n is n and ordo of C_m is m then $dim_{fl}(K_n \odot C_m) = \frac{1}{4}|V(C_n)||K_m|$. ■

Theorem 2.5. Let $C_n \odot C_m$ be a connected graph with $n \geq 3$ and $m > 4$, then

$$dim_{fl}(C_n \odot C_m) = \frac{1}{4}|V(C_n)||V(C_m)|.$$

Proof. The graph $C_n \odot C_m$ is a connected graph. Let $f_l: V(C_n \odot C_m) \to [0,1]$ is a local resolving function. Any two adjacent vertices $u, v \in V(C_n \odot C_m)$. There are three possibilities u and v.

i. If u, v in the same leaf, then there is $i \in \{1,2,\dots,n\}$ and $j \in \{1,2,3,\dots,m-1\}$ such that $u = v_{ji}$ and $v = v_{(j+1)i}$, or $u = v_{mi}$ and $u = v_{1i}$. Local resolving neighborhood

$$R_l\{v_{ji}, v_{(j+1)i}\} = \begin{cases} \{v_{mi}, v_{ji}, v_{(j+1)i}, v_{(j+2)i}\}, & j = 1 \\ \{v_{(j-1)i}, v_{ji}, v_{(j+1)i}, v_{(j+2)i}\}, & j = 2,3,\dots,m-2 \\ \{v_{(j-1)i}, v_{ji}, v_{(j+1)i}, v_{1i}\}, & j = m-1 \\ \{v_{(j-1)i}, v_{ji}, v_{1i}, v_{2i}\}, & j = m \end{cases}.$$

Therefore,

$$f_l(R_l\{v_{ji}, v_{(j+1)i}\}) = \begin{cases} f_l(v_{mi}) + f_l(v_{ji}) + f_l(v_{(j+1)i}) + f_l(v_{(j+2)i}) \geq 1, & j = 1 \\ f_l(v_{(j-1)i}) + f_l(v_{ji}) + f_l(v_{(j+1)i}) + f_l(v_{(j+2)i}) \geq 1, & j = 2,3,\dots,m-2 \\ f_l(v_{(j-1)i}) + f_l(v_{ji}) + f_l(v_{(j+1)i}) + f_l(v_{1i}) \geq 1, & j = m-1 \\ f_l(v_{(j-1)i}) + f_l(v_{ji}) + f_l(v_{(j+1)i}) + f_l(v_{(j+2)i}) \geq 1, & j = m \end{cases}.$$

based on the description above,

$$4\sum_{j=1}^{m} f_l(v_{ji}) \geq m$$

$$\Leftrightarrow \sum_{j=1}^{m} f_l(v_{ji}) \geq \frac{m}{4}.$$

Because the number of points in the parent is n, then

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f_l(v_{ji}) \geq \frac{m}{4} n. \qquad (10)$$

ii. If u, v in parent, then there are $i, j \in \{1,2,\dots,n-1\}$ such that $u = v_{0i}$ and $v = v_{0(i+1)}$ or $u = v_n$ and $v = v_{01}$. Local resolving neighborhood

$$R_l\{u, v\} = \begin{cases} V(C_n \odot C_m), & n \text{ genap} \\ V(C_n \odot C_m) - \{v_k\} \cup U_k, & n \text{ gasal} \end{cases}$$

so that

$$f_l(R_l\{u, v\}) = \begin{cases} \sum_{z\in V(C_n\odot C_m)} f_l(z) \geq 1 & n \text{ genap} \\ \sum_{z\in V(C_n\odot C_m)} f_l(z) - \left(f_l(v_k) + \sum_{v\in V(U_k)} f_l(v)\right) \geq 1 & n \text{ gasal} \end{cases}. \qquad (11)$$

iii. If u in parent and v in leaf of u, then there are $i \in \{1,2,\dots,n\}$ and $j \in \{1,2,3,\dots,m\}$, such that $u = v_{0i}$ and $v = v_{ji}$. Local resolving neighborhood

$$R_l\{v_{0i}, v_{ji}\} = \begin{cases} V(C_n \odot C_m) - \{v_{(j+1)i}, v_{mi}\}, & j = 1 \\ V(C_n \odot C_m) - \{v_{(j-1)i}, v_{(j+1)i}\}, & j = 2,3,\dots,m-1 \\ V(C_n \odot C_m) - \{v_{(j-1)i}, v_{1i}\}, & j = m \end{cases}$$

so that

$$f_l(R_l\{v_{0i}, v_{ji}\}) = \begin{cases} \sum_{z\in V(C_n\odot C_m)} f(z) - \left(f_l(v_{(j+1)}) + f_l(v_{mi})\right) \geq 1, & j = 1 \\ \sum_{z\in V(C_n\odot C_m)} f(z) - \left(f_l(v_{(j-1)}) + f_l(v_{(j+1)i})\right) \geq 1, & j = 2,3,\dots,m-1 \\ \sum_{z\in V(C_n\odot C_m)} f(z) - \left(f_l(v_{(j-1)}) + f_l(v_{1i})\right) \geq 1, & j = m \end{cases}.$$

Therefore,

$$m \sum_{z\in V(C_n\odot C_m)} f_l(z) - 2\sum_{j=1}^{m} f_l(v_{ji}) \geq m.$$

because the number of points in the parent is n, then

$$nm \sum_{z\in V(C_n\odot C_m)} f_l(z) - 2\sum_{i=1}^{n}\sum_{j=1}^{m} f_l(v_{ji}) \geq nm.$$

because $\sum_{i=1}^{n}\sum_{j=1}^{m} f_l(v_{ji}) \geq 0$, then

$nm \sum_{z\in V(C_n\odot C_m)} f_l(z) \geq n.m \sum_{z\in V(C_n\odot C_m)} f_l(z) - 2\left(\sum_{i=1}^{n}\sum_{j=1}^{m} f_l(v_{ji})\right) \geq nm.$

therefore,

$$\sum_{z\in V(C_n\odot C_m)} f_l(z) \geq 1. \qquad (12)$$

Based on the results of the description above, in order to all conditions the maximum values taken from equations $(10) - (12)$ are:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f_l(v_{ji}) \geq \frac{m}{4}n.$$

As a result:

$$dim_{fl}(C_n \odot C_m) = min\left\{\sum_{z\in V(C_n\odot C_m)} f_l(z) : f_l \text{ } local\ resolving\ function\right\} = \frac{m}{4}n.$$

Because ordo of C_n is n and ordo of C_m is m then

$$dim_{fl}(C_n \odot C_m) = \frac{1}{4}|V(C_n)||C_m|. \blacksquare$$

3. Conclusion

In this paper we have given local fractional metric dimension number of corona product of complate graphs and cycle graph, namely complete corona complete, cycle corona complete, complete corona cycle and cycle corona cycle. We conclude this paper with open problems on corona product of other graphs.

4. Acklowledgment

We gratefully acknowledge the support from DIKTI Indonesia trough the "Penelitian Dosen Pemula" RISTEKDIKTI 2019 research project.

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