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Case Study As Application of T-norm on BL-algebra

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Abstract: *This paper is concerned with the study of t-norm, and demonstrating a conditionality notion via algebraic structures of BL-algebra. On the other hand, we present some relations that explain the relationships between t-norm on BL-algebra and the classical probability space. Also, we illustrate some examples as applications of our work.*

Keyword: T-norm, Probability Space, Conditionality, BL-algebra, State.

Introduction

Basic Logic(BL), was introduced by H**a'**jek in 1998 to construct an algebra which of the theory of basic logic's completeness which has taken a place in the continuous t-norms and the fuzzy logic topic [7].

BL-algebras are certain type of residual lattices, and they have been examined in several papers by Turunen, [13, 14]. The structure $(A, \leq, \odot, \rightarrow, \vee, \wedge, O, I)$ of BL-algebra is said to be an MV-algebra if the complement operation $*: A \rightarrow A$ is involutive, which means that $\psi^{**} = \psi$ or equivalently $(\psi \rightarrow \lambda) \rightarrow \lambda = (\lambda \rightarrow \psi) \rightarrow \psi$ for all $\psi, \lambda \in A$ where $\psi^* = \psi \rightarrow O$, [7]. Also, a BL-algebra is called a G-algebra, if $\psi \odot \psi = \psi$ (i.e idempotent). A Boolean algebra is a BL-algebra which is both an MV-algebra and a G-algebra, for more details see [7].

On the other hand, triangular norms are the operations that are looked to be suitable as well as possible to the notion of conjunction. When also continuity is required to be connectives, then the common part of all possible that have many-valued logics has been defined and called basic logic, see [7].

Triangular norms started by Menger's paper "Statistical metrics"[10], he introduced the notion of a statistical metric space as a natural generalization of the notion of a metric space, in which the distance between p and q from set D was replaced by a distribution function $F_{p,q}(x)$, where $F_{p,q}(x)$ can be interpreted as the probability that the distance between p, and q is less than x. There are four types of t-norms like: The drastic product, the minimum, the product, and the Lukasiewicz.

Thus, the top field where t-norms play a fantastic role was the theory of probabilistic metric spaces (or statistical metric spaces were called after 1964), Schweizer and Sklar [12] have redefined and developed statistical metric spaces.

Triangular norms (for short t-norms) are an important system for version of the conjunction in fuzzy logics and for the intersection of fuzzy sets [7, 12]. It is important to know that the left continuity of t-norm corresponding BL-algebras, and for more interesting details, one can see [7]. It talks about the relationship between continuous t-norms and BL-algebra.

An essential notion that has a main part in our constructions is known by state. Its definition, and properties is equivalent to probability space definition, and properties. Our paper involves three sections, in the first section we set preliminaries of basic notions for this paper, section two shows the construction of one basic notion over the other one and find new concept from this generalization that called s-t-norm, and conditional s-t-norm and explain that by propositions, while the latest section includes application for our paper that describe by two examples.

1 Basic Concepts

In this section, we introduce some basic concepts of t-norm, BL-algebra, state and their properties. Also, we mention measure theory concept and remark that explains how can the structure of probability space satisfies algebraic system BL-algebra.

Definition 1.1 [12] A binary operation T on the unit interval [0,1] such that $T: [0,1]^2 \rightarrow [0,1]$ is said to be triangular norm (t-norm) where 1 is identity element and T satisfies the following conditions for each ψ, λ and $\gamma \in [0,1]$.

(T1) $T(\psi, \lambda) = T(\lambda, \psi)$;

(T2) $T(\psi, T(\lambda, \gamma)) = T(T(\psi, \lambda), \gamma)$;

(T3) $T(\psi, \lambda) \leq T(\psi, \lambda)$ whenever $\lambda \leq \gamma$;

(T4) $T(\psi, 1) = \psi$.

When t-norm is clearly extension of the Boolean conjunction, we can also use the conjunction $\odot$ in [0; 1] to define t-norm [8].

Let us illustrate the following example that explains the properties and conditions of t-norm.

Example 1.1 [9] Let $T: [0,1]^2 \to [0,1]$ be a function defined by $T(\psi, \lambda) = \min(\psi, \lambda)$.
So, we show whether T satisfies t-norm or not.
1. For all $\psi, \lambda \in [0,1], T(\psi, \lambda) = \min(\psi, \lambda) = \min(\lambda, \psi,) = T(\psi, \lambda)$;
2. For all $\psi, \lambda, \gamma \in [0,1], T(\psi, T(\lambda, \gamma)) = \min(\psi, \min(\lambda, \gamma)) = \min(\min(\psi, \lambda,), \gamma)$
$= T(T(\psi, \lambda), \gamma)$;
3. Let $\psi, \lambda, \gamma \in [0,1]$, such that $\lambda \le \gamma$, then $\min(\psi, \lambda) \le \min(\psi, \gamma)$, implies that
$T(\psi, \lambda) \le T(\psi, \gamma)$;
4. Let $\psi \in [0,1]$, then $T(\psi, I) = \min(\psi, I) = \psi$ and $T(\psi, O) = \min(\psi, O) = 0$.
Then T is a t-norm.

Now, we set definition of BL-algebra (Basic logic), its axioms, and tables that satisfies BL-algebra. Since structure of BL-algebra consist of lattice, so that we present at the beginning the definition of lattice and bounded lattice that introduced in [4].

Definition 1.2 A lattice is a partially ordered set in which for every two elements ψ, λ the least upper bound (or join) denoted by $\psi \vee \lambda$ and the greatest lower bound (or meet) denoted by $\psi \wedge \lambda$ are exist.

Definition 1.3 A bounded lattice is an algebraic structure $\mathcal{A} = (A, \vee, \wedge, O, I)$, such that $(A, \vee, \wedge)$ is a lattice, and the constant $O, I \in A$ satisfy
1. For all $\psi \in A$, $\psi \wedge I = \psi$ and $\psi \vee I = I$,
2. For all $\psi \wedge O = O$ and $\psi \vee O = \psi$.

Definition 1.4 [7] An algebra $(A, \le, \odot, \to, \vee, \wedge, O, I)$ of type (2,2,2,2, O, I) is said to be BL-algebra if following conditions hold:

1. $(A, \vee, \wedge, O, I)$ is bounded lattice;
2. $(A, \odot, I)$ is a commutative monoid, such that $\odot$ is an associative and commutative binary operation, and I is a neutral element with respect to $\odot$;
3. $\psi \le \lambda \to \gamma \leftrightarrow \lambda \odot \psi \le \gamma$;
4. $\lambda \wedge \gamma = \lambda \odot (\lambda \to \gamma)$;
5. $(\lambda \to \gamma) \vee (\gamma \to \lambda) = I$.

Proposition 1.1 [7, 13] Let $\mathcal{A} = (A, \le, \odot, \to, \vee, \wedge, O, I)$ be a BL-algebra and consider
$\psi^* = \psi \to O$. Then

1. $\psi \odot (\psi \to \lambda) \le \lambda$;

2. $\psi \le \lambda \ \ iff \ \ \psi \to \lambda = I$;

3. $\psi \vee \lambda = [(\psi \to \lambda) \to \lambda] \wedge [(\lambda \to \psi) \to \psi]$;

4. $I \to \psi = \psi, \psi \to \psi = I, \psi \le \lambda \to \psi, \psi \to I = I$;

5. $\psi^* \odot \psi = O$;

6. $\psi \odot \lambda = O, iff \ \ \psi \le \lambda^* \ and \ \ \psi \le \lambda \ implies \ \ \lambda^* \le \psi^*$;

7. $\psi \vee \lambda = I \ \ implies \ \ \psi \odot \lambda = \psi \wedge \lambda$;

8. $(\psi \to \lambda) \to (\psi \to \gamma) = (\psi \wedge \lambda) \to \gamma$;

9. $\psi \le \psi^{**}, O^* = I, \ I^* = O$;

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10. $(\psi \odot \lambda)^{**} = \psi^{**} \odot \lambda^{**}$.

Example 1.2 [13] Let $A = \{O, \psi, \lambda, I\}$. Then the following operations explain that A is a BL-algebra:

$\odot$	O	ψ	λ	I
O	O	O	O	O
ψ	O	O	ψ	ψ
λ	O	ψ	λ	λ
I	O	ψ	λ	I

$\rightarrow$	O	ψ	λ	I
O	I	I	I	I
ψ	ψ	I	I	I
λ	O	ψ	I	I
I	O	ψ	λ	I

$\wedge$	O	ψ	λ	I
O	O	O	O	O
ψ	O	ψ	ψ	ψ
λ	O	ψ	λ	λ
I	O	ψ	λ	I

$\vee$	O	ψ	λ	I
O	O	ψ	λ	I
ψ	ψ	ψ	ψ	I
λ	λ	λ	λ	I
I	I	I	I	I

Each table of these tables is shown structure of BL-algebra, and how we can construct its binary operations.

Example 1.3 Let $A = \{O, \psi, \psi^*, \lambda, \lambda^*, I\}$, define $\odot, \rightarrow, \wedge$ and $\vee$ as the following operations explain that A is a BL-algebra where $(\psi \leq \lambda)$

$\odot$	O	ψ	ψ^*	λ	λ^*	I
O	O	O	O	O	O	O
ψ	O	ψ	O	ψ	λ^*	ψ
ψ^*	O	O	ψ^*	λ	λ^*	ψ^*
λ	O	ψ	λ	λ	O	λ
λ^*	O	λ^*	λ^*	O	λ^*	λ^*
I	O	ψ	ψ^*	λ	λ^*	I

$\rightarrow$	O	ψ	ψ^*	λ	λ^*	I
O	I	I	I	I	I	I
ψ	ψ^*	I	ψ^*	ψ^*	ψ^*	I
ψ^*	ψ	ψ	I	λ	ψ	I
λ	λ^*	ψ	ψ^*	I	λ^*	I
λ^*	λ	λ	ψ^*	λ	I	I
I	O	ψ	ψ^*	λ	λ^*	I

$\wedge$	O	ψ	ψ^*	λ	λ^*	I
O	O	O	O	O	O	O
ψ	O	ψ	O	O	O	ψ
ψ^*	O	O	ψ^*	λ	O	ψ^*
λ	O	ψ	λ	λ	O	λ
λ^*	O	O	λ^*	O	λ^*	λ^*
I	O	ψ	ψ^*	λ	λ^*	I

$\vee$	O	ψ	ψ^*	λ	λ^*	I
O	O	ψ	ψ^*	λ	λ^*	I
ψ	ψ	ψ	I	λ	ψ	I
ψ^*	ψ^*	I	ψ^*	I	ψ^*	I
λ	λ	λ	I	λ	I	I
λ^*	λ^*	ψ	ψ^*	I	λ^*	I
I	I	I	I	I	I	I

Remark 1.1 Every Boolean algebra is BL-algebra, such that $\psi \rightarrow \lambda = \psi^* \vee \lambda$, where $\psi^* = \psi \rightarrow O$, and $\psi \odot \lambda = \psi \wedge \lambda$ where $\wedge, \vee$ are standard lattice operations of Boolean algebra[7].

Definition 1.5 [13] Let $\mathcal{A}$ be a BL-algebra. Two elements $\psi, \lambda \in A$ are said to be orthogonal and denoted by $\psi \perp \lambda$, if $\psi \leq \lambda^*$.

Remark 1.2 It is easy to show that, $\psi \perp \lambda$ if and only if $\psi \leq \lambda^*$ and if and only if $\psi \odot \lambda = 0$. It is clear that $\psi \perp \lambda$ if and only if $\lambda \perp \psi$ and $\psi \perp 0$ for each $\psi \in A$.

Definition 1.6 [5, 11] Let $\mathcal{A}$ be a BL-algebra. A function $s: A \rightarrow [0,1]$ is said to be a state if the following conditions hold:

1. $s(O) = 0$;

2. If $\psi \perp \lambda$, then $s(\psi \vee \lambda) = s(\psi) + s(\lambda)$.

Some properties of state:

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1. $s(I) = 1$;

2. $s(\psi^*) = 1 - s(\psi)$ for any $\psi \in A$;

3. $s(\psi) = s(\psi^{**})$ for any $\psi \in A$;

4.If $\psi \leq \lambda$, then

i) $s(\lambda) - s(\psi) = 1 - s(\psi \vee \lambda^*)$

ii) $s(\psi) \leq s(\lambda)$.

The following example shows the properties and behaviour of such state and how it works.

Example 1.4 Let us recall **Example 1.2.** Then according to the table below is a state

.	O	ψ	λ	I
s(.)	0	0.1	0.9	1

Definition 1.7 [6, 8] Let $\mathcal{F}$ be a system of subsets on a nonempty set Ω. Then the system $\mathcal{F}$ is said to be $\sigma -$ algebra on Ω if the following conditions hold.

1. $\Omega \in \mathcal{F}$;

2. For all $A \in \Omega$ implies $A^c \in \mathcal{F}$, where $A^c = \Omega - A \in \mathcal{F}$

3. $(A_n)_{n\in N} \in \mathcal{F}$ implies $\bigcup_{n\in N} A_n \in \mathcal{F}$.

Remark 1.3 It has been mentioned that probability space $(\mathcal{F}, P, \vee, \wedge, \odot, \rightarrow, O, I)$ is a BL-algebra [2], where $\vee \equiv \cup$, $\wedge \equiv \cap$, $\odot \equiv \cap$, $I \equiv \Omega$, $O \equiv \emptyset$, and $\rightarrow \equiv A^c \vee B$.

2 Conditional s-t-norm

In this section we present generalization of t-norm on BL-algebra, but by conditional concept. We generalize axioms of t-norm on BL-algebra and we replace boundary condition by adding the interesting property which is state.

Firstly, should know that before conditional s-t-norm, we show a bivariate s-t-norm which we defined it in [1].

Definition 2.1 Let $\mathcal{A}$ be a BL-algebra. A bivariate t-norm on BL-algebra A, is called s-t-norm if a function $BT_s: A \times A \rightarrow [0,1]$, satisfies the following condition

1. For all $\psi \in A, BT_s(\psi, O) = BT_s(O, \psi) = 0$;
2. For all $\psi, \lambda \in A, BT_s(\psi, \lambda) = BT_s(\lambda, \psi)$ (commutative);
3. If $\lambda \leq \gamma$, such that $\psi, \lambda, \gamma \in A$, then $BT_s(\psi, \lambda) \leq BT_s(\psi, \gamma)$ (monotone);
4. $BT_s(\,.\,, I), BT_s(I, .\,)$ are states on A, where I is the greatest element of A.

Connecting with the above definition we can reconstruct its conditionality by the following definition.

Definition 2.2 A function $BT_c: A \times A \rightarrow [0,1]$ is called conditional s-t-norm, if the following conditions hold:
1. For all $\psi \in A, BT_c(\psi, O|\lambda) = BT_c(O, \psi|\lambda) = 0$;

2. $BT_c(\psi_1, \psi_2 |\, \lambda) = BT_c(\psi_2, \psi_1|\lambda)$ (commutative);
3. If $\psi_2 \leq \psi_3$, then $BT_c(\psi_1, \psi_2|\lambda) \leq BT_c(\psi_1, \psi_3|\lambda)$ (monotone);

4. $BT_c(\,.\,, I|\lambda), BT_c(I, .\,|\lambda)$ are states on A, where I is the greatest element of A.

This definition shows that BT_c satisfies the conditions of being s-t-norm.

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The following proposition shows the conditional case of convex property.

Proposition 2.1 Let $\mathcal{A}$ be a BL-algebra. If BT_{c_1} and $BT_{c_2}$are conditional s-t- norms on A, then for any $k \in [0,1]$,

$$BT_c(\psi_1, \psi_2 |\, \lambda) = kBT_{c_1}(\psi_1, \psi_2 |\, \lambda) + (1-k)BT_{c_2}(\psi_1, \psi_2 |\, \lambda) \qquad (1)$$

is a conditional s-t-norm.

proof

1. For all $\psi_1 \in A$, $BT_c(\psi_1, O |\, \lambda) = kBT_{c_1}(\psi_1, O |\, \lambda) + (1-k)BT_{c_2}(\psi_1, O |\, \lambda) = k.0 + (1-k).0 = 0 = BT_c(O, \psi_1|\lambda)$;

2. For all $\psi_1, \psi_2 \in A$, $BT_c(\psi_1, \psi_2 |\, \lambda) = kBT_{c_1}(\psi_1, \psi_2 |\, \lambda) + (1-k)BT_{c_2}(\psi_1, \psi_2 |\, \lambda)$
$= kBT_{c_1}(\psi_2, \psi_1 |\, \lambda) + (1-k)BT_{c_2}(\psi_2, \psi_1 |\, \lambda) = BT_c(\psi_2, \psi_1|\lambda)$ (commutative);

3. If $\psi_2 \le \psi_3$, then we should prove that BT_c is monotone.

Since $kBT_{c_1}(\psi_1, \psi_2 |\, \lambda) \le kBT_{c_2}(\psi_1, \psi_3 |\, \lambda)$,

and $(1-k)BT_c(\psi_1, \psi_2 |\, \lambda) \le (1-k)BT_{c_2}(\psi_1, \psi_3 |\, \lambda)$

thus,

$$kBT_{c_1}(\psi_1, \psi_2 |\, \lambda) + (1-k)BT_{c_2}(\psi_1, \psi_2 |\, \lambda) \le kBT_{c_2}(\psi_1, \psi_3|\lambda) + (1-k)BT_{c_2}(\psi_1, \psi_3|\lambda).$$

But, $BT_c(\psi_1, \psi_2|\lambda) = kBT_{c_1}(\psi_1, \psi_2 |\, \lambda) + (1-k)BT_{c_2}(\psi_1, \psi_2 |\, \lambda)$, and
$BT_c(\psi_1, \psi_3|\lambda) = kBT_{c_1}(\psi_1, \psi_2 |\, \lambda) + (1-k)BT_{c_2}(\psi_1, \psi_2 |\, \lambda)$.

Therefore, BT_c is monotone;

4. To prove that $BT_c(.,I|\lambda)$ and $BT_c(I,.|\lambda)$ are states on A. For $BT_c(.,I|\lambda)$ we have

i) $BT_c(I,I|\lambda) = kBT_{c_1}(I,I|\, \lambda) + (1-k)BT_{c_2}(I,I|\, \lambda) = k + 1 - k = 1$;
ii) $BT_c(O,I|\lambda) = kBT_{c_1}(O,I|\, \lambda) + (1-k)BT_{c_2}(O,I|\, \lambda) = k.0 + (1-k).0 = 0$;
iii) Let $\psi_1 \perp \psi_2$, such that $\psi_1, \psi_2 \in A$. Then we should prove

$$BT_c(\psi_1 \vee \psi_2, I|\lambda) = BT_c(\psi_1, I|\lambda) + BT_c(\psi_2, I|\lambda)$$

For $BT_c(\psi_1 \vee \psi_2, I|\lambda) = kBT_{c_1}(\psi_1 \vee \psi_2, I|\lambda) + (1-k)BT_{c_2}(\psi_1 \vee \psi_2, I|\lambda)$

$$= k\left(BT_{c_1}(\psi_1, I|\lambda) + BT_{c_1}(\psi_2, I|\lambda)\right) + (1-k)(BT_{c_2}(\psi_1, I|\lambda) + BT_{c_2}(\psi_2, I|\lambda))$$

$= kBT_{c_1}(\psi_1, I|\lambda) + (1-k)BT_{c_2}(\psi_1, I|\lambda) + kBT_{c_1}(\psi_2, I|\lambda) + (1-k)BT_{c_2}(\psi_2, I|\lambda))$.

Conversely, $BT_c(\psi_1, I|\lambda) = kBT_{c_1}(\psi_1, I|\lambda) + (1-k)BT_{c_2}(\psi_1, I|\lambda)$, and

$BT_c(\psi_2, I|\lambda) = kBT_{c_1}(\psi_2, I|\lambda) + (1-k)BT_{c_2}(\psi_2, I|\lambda)$.

Thus,
$BT_c(\psi_1, I|\lambda) + BT_c(\psi_2, I|\lambda) = kBT_{c_1}(\psi_1, I|\lambda) + (1-k)BT_{c_2}(\psi_1, I|\lambda) + kBT_{c_1}(\psi_2, I|\lambda)$

$$+(1-k)BT_{c_2}(\psi_2, I|\lambda).$$

Hence, $BT_c(.,I|\lambda)$ is a state on A.

Similarly, for $BT_c(I,.|\lambda)$.

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Therefore, BT_c is a conditional s-t-norm.

It is important to explain conditional s-t-norm and other spaces like conditional probability space. From Remark 1.2 we said that probability space satisfies BL-algebra, so the following proposition proves fact of this relationship.

Proposition 2.2 Let a BL-algebra $\mathcal{A}$ be a probability space $(\Omega, \mathcal{F}, P)$. Then BT_c in terms of conditional probability such that,

$$BT_c(H_1, H_2|G) = P(H_1 \cap H_2|G), \quad H_1, H_2, G \in \mathcal{F} \tag{2}$$

hold the conditions of conditional s-t-norm.

1. For all $H_1, G \in \mathcal{F}$, $BT_c(H_1, \emptyset|\, G) = \frac{P(H_1 \cap \emptyset \cap G)}{P(\Omega \cap \Omega \cap G)} = \frac{P(\emptyset)}{P(\Omega \cap \Omega \cap G)} = 0 = BT_c(\emptyset, H_1|G)$;

2. For all $H_1, H_2, G \in \mathcal{F}$, $BT_c(H_1, H_2|\, G) = P(H_1 \cap H_2|G) = P(H_2 \cap H_1|G)$
$= BT_c(H_2, H_1|G)$ (commutative);

3. Let $H_2 \subseteq H_3$. To prove monotonicity of BT_c we need to show that

$BT_c(H_1, H_2|G) \leq BT_c(H_1, H_3|G)$. Thus

$BT_c(H_1, H_2|G) = P(H_1 \cap H_2|G) = \frac{P(H_1 \cap H_2 \cap G)}{P(\Omega \cap \Omega \cap G)}$.

Hence, $BT_c(H_1, H_3|G) = P(H_1 \cap H_3|G) = \frac{P(H_1 \cap H_3 \cap G)}{P(\Omega \cap \Omega \cap G)}$.

Thus, $\frac{P(H_1 \cap H_2 \cap G)}{P(\Omega \cap \Omega \cap G)} \leq \frac{P(H_1 \cap H_3 \cap G)}{P(\Omega \cap \Omega \cap G)}$. Therefore, BT_c is monotone;

4. We need To show that $BT_c(., \Omega|G)$ and $BT_c(\Omega, .|G)$ are states on A. Thus, for $BT_c(., \Omega|G)$ we have

i) $BT_c(\Omega, \Omega|G) = \frac{P(\Omega \cap \Omega \cap G)}{P(\Omega \cap \Omega \cap G)} = 1$;

ii) $BT_c(\emptyset, \Omega|G) = \frac{P(\emptyset \cap \Omega \cap G)}{P(\Omega \cap \Omega \cap G)} = \frac{P(\emptyset)}{P(\Omega \cap \Omega \cap G)} = 0$;

iii) Let $H_1, H_2 \in \mathcal{F}$, if $H_1 \cap H_2 = \emptyset$. Then we should prove

$$BT_c(H_1 \cup H_2, \Omega|G) = BT_c(H_1, \Omega|G) + BT_c(H_2, \Omega|G)$$

$$BT_c(H_1 \cup H_2, \Omega|G) = P\big((H_1 \cup H_2) \cap \Omega\big|G\big) = \frac{P((H_1 \cup H_2) \cap G)}{P(\Omega \cap \Omega \cap G)}$$

$$= \frac{P\big((H_1 \cap G) \cup (H_2 \cap G)\big)}{P(\Omega \cap \Omega \cap G)}$$

$$= \frac{P(H_1 \cap G) + P(H_2 \cap G) - P((H_1 \cap G) \cap (H_2 \cap G))}{P(\Omega \cap \Omega \cap G)}$$

since, $P\big((H_1 \cap G) \cap (H_2 \cap G)\big) = P\big((H_1 \cap H_2) \cap G\big) = P(\emptyset \cap G) = P(\emptyset) = 0$.

$$P\big((H_1 \cup H_2) \cap \Omega\big|G\big) = \frac{P(H_1 \cap G) + P(H_2 \cap G)}{P(\Omega \cap \Omega \cap G)} = \frac{P(H_1 \cap \Omega \cap G)}{P(\Omega \cap \Omega \cap G)} + \frac{P(H_2 \cap \Omega \cap G)}{P(\Omega \cap \Omega \cap G)}.$$

Conversely, $BT_c(H_1,\Omega|G) = \frac{P(H_1 \cap \Omega \cap G)}{P(\Omega \cap \Omega \cap G)}$, $BT_c(H_2,\Omega|G) = \frac{P(H_2 \cap \Omega \cap G)}{P(\Omega \cap \Omega \cap G)}$

$$BT_c(H_1,\Omega|G) + BT_c(H_2,\Omega|G) = \frac{P(H_1 \cap \Omega \cap G)}{P(\Omega \cap \Omega \cap G)} + \frac{P(H_2 \cap \Omega \cap G)}{P(\Omega \cap \Omega \cap G)}.$$

Hence,

$$BT_c(H_1 \cup H_2,\Omega|G) = BT_c(H_1,\Omega|G) + BT_c(H_2,\Omega|G).$$

Thus, $BT_c(.,\Omega|G)$ is state.

Similarly, $BT_c(\Omega,.|G)$ is state.

Therefore, BT_c is a conditional s-t-norm.

This complete the proof, and show that the relation between a conditional s-t-norm and a conditional probability space.

3 Case Study As Applications

Our case study involves two different examples of real life that we have randomly collected and reconstructed. The first chosen problem concerned with weather conditions and their effect on train runs within one year we reconstructed it to use it with our notion conditional s-t-norm. Afterwords, we reconstruct and recall an example that represents an experiment that have been shown by [3] in order to study and examine causes that effect on diabetes disease, or side effect of diabetes. Indeed, diabetes were derived to two different types that we explain them later in detail.

Example 3.1 Someone called James is interested in weather conditions and its effect on train movements within one year. James has recorded data in Table 1 that related to the weather conditions like sunny, cloudy, rainy, and snowy, and as show in the table below

Weather	On time	Delayed	Total
Sunny	167	3	170
Cloudy	115	5	120
Rainy	40	15	55
Snowy	8	12	20
Total	330	35	365

Table 1: Data of train runs

Now, let us answer the questions of James that could explain the relationship of interest subject (train runs) to weather condition. This is of course a situation relevant to the conditional probability space. Consequently, we explain this situation in terms of conditional s-t-norm rather that using of direct conditionality. First question arise from the constructed **Table 1** is:
What is the chance that the train is delayed?
Second question associated with the train and one of the given conditions, that is: What is the chance that train is delayed given that it is snow?
To answer these questions, suppose that A_i be the event of delayed, B_i be the event that the weather condition, and C_i be the event of total to the conditions of weather. Then by means
of s-t-norm and its conditionality, we have

$$BT_s(A,C) = \frac{A}{C} = \frac{35}{365} = 0.096 \qquad (3)$$

$$BT_c(A_4,C_4|B_4) = \frac{A_4}{C_4} = \frac{12}{20} = 0.6 \qquad (4)$$

Note that this situation is not the only one that we could conclude, so there are several different situations related to the weather conditions as a given cause to our interest (train run), consider another situation in order to compare with results in (3), and (4).

Suppose that D be an event of sunny then

$$BT_c(A_1, C_1|B_1) = \frac{A_1}{C_1} = \frac{3}{20} = 0.15 \qquad (5)$$

we can see that according to the concepts of dependency and non -dependency, we conclude that the magnitude in (3), and (5) are quite close to each other. By means of conditional Probability measure, one can see that the first situation represents a type of dependent random events, while the second situation is nothing more than independent random events when the probability of event A is far away from the probability of A_4 given B_4, so it means that A and B_4 are dependent and vice versa.

As a conclusion of what we have examined in the previous situations, we can say that sunny days does not effect on arriving time of the train, while snowy days are certainly cause a delay time of the train.
Next, let us build and discuss another example that shows a more complicated situations with medical treatment.

Example 3.2 We have samples of 186 patients of diabetes. These patients were chosen randomly and classified to two main types according to their ages. So, the patients of age from 2-32 are called type I, while the patients from 32-85 are called type II.
The observables that represent the causes that effect on each type of diabetes or side effects of each type of it on patients were derived to several different parts like gender, place of living, and so on. The obtained data that combines the type of diabetes to those causes or side effects is chosen in the following tables.

Gender	Type I	Type II	Total
Male	36	42	78
Female	37	71	108
Total	73	113	186

Table 2: Data of diabetic with respect to gender

Place of living	Type I	Type II	Total
Ruler	18	22	40
Urban	55	91	146
Total	73	113	186

Table 3: Data of diabetic with respect to place of living

Marital status	Type I	Type II	Total
Unmarried	45	2	47
Separated	1	0	1
Widowed	0	12	12
Married	27	99	126
Total	73	113	186

Table 4: Data of diabetic with respect to marital status

Type of drug	Type I	Type II	Total
Injection needles	73	0	73
Oral	0	62	62
Both	0	51	51
Total	73	113	186

Table 5: Data of diabetic with respect to type of drug

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The symptoms	Type I	Type II	Total
Poly urea	43	80	123
Extreme thirst	48	74	122
Hunger	48	72	120
Blurred version	32	84	116
Weight loss	31	47	78
Tiredness	45	93	138
Total	247	450	697

Table 6: Data of diabetic with respect to the symptoms

In the same way of **Example 3.1**, we want to discuss and answer the following questions upon s-t-norm and conditional s-t-norm.

What is the chance of being injured with diabetes of type I, or type II?
What is the chance of a patient of type I or type II given gender, place of living, and other causes or side effects, or in other word what is dependency between diabetes type I or type II and the causes injured diabetes or side effects of it?

The calculations with respect to our notion s-t-norm and conditional s-t-norm yield the following The chance of event A which represents diabetes type I calculated by our construction

$$BT_s(A, C) = \frac{A}{C} = \frac{73}{186} = 0.392\ .$$

Suppose C event of total . If we want know calculations of diabetes type I given causes or side effects of diabetes we suppose that $BT_c(A_i, C_i|B_i) = \frac{A_i}{C_i}$
such that A_i represents event of diabetes type I, C_i be event of total of diabetics, and B_i is causes and side effects of diabetes. At first we begin by gender by using our notion conditional s-t-norm, then diabetes type I given gender is

let as start with male cell $BT_c(A_1, C_1|B_1) = \frac{A_1}{C_1} = \frac{36}{78} = 0.461$

now we turn to female cell $BT_c(A_2, C_2|B_2) = \frac{A_2}{C_2} = \frac{37}{108} = 0.342.$

While, in classification of place of living conditionality of s-t-norm is

$$BT_c(A_1, C_1|B_1) = \frac{A_1}{C_1} = \frac{18}{40} = 0.45$$

$$BT_c(A_2, C_2|B_2) = \frac{A_2}{C_2} = \frac{55}{146} = 0.376$$

Now, we test by our notion BT_c another causes which is marital status.

$$BT_c(A_1, C_1|B_1) = \frac{A_1}{C_1} = \frac{45}{47} = 0.957$$

$$BT_c(A_2, C_2|B_2) = \frac{A_2}{C_2} = \frac{1}{1} = 1$$

$$BT_c(A_3, C_3|B_3) = \frac{0}{12} = 0$$

$$BT_c(A_4, C_4|B_4) = \frac{27}{126} = 0.214$$

On the other hand, side effects or observables of diabetes like type of drug by using conditional s-t-norm is

$$BT_c(A_1, C_1|B_1) = \frac{A_1}{C_1} = \frac{73}{73} = 1$$

$$BT_c(A_2, C_2|B_2) = \frac{0}{62} = 0$$

$$BT_c(A_3, C_3|B_3) = \frac{0}{51} = 0$$

And when we test another side effect of diabetes which is symptoms we get the following results

$$BT_c(A_1, C_1|B_1) = \frac{43}{123} = 0.349$$

$$BT_c(A_2, C_2|B_2) = \frac{48}{122} = 0.393$$

$$BT_c(A_3, C_3|B_3) = \frac{48}{120} = 0.4$$

$$BT_c(A_4, C_4|B_4) = \frac{32}{116} = 0.275$$

$$BT_c(A_5, C_5|B_5) = \frac{31}{78} = 0.397$$

$$BT_c(A_6, C_6|B_6) = \frac{45}{138} = 0.326.$$

But in this classification of data result of diabetes type I is $BT_s(A, C) = \frac{247}{697} = 0.354.$

Then we can comment on the obtained calculations by the following way comparing $BT_s(A, C)$ with $BT_c(A_i, C_i|B_i)$ shows that gender does not effects on diabetes type I because the values of BT_s and BT_c are too close to each other. So this means diabetes is independent to the cause gender. Similarly, we can obtained from the other computations that diabetes is independent to the cause that is place of living because the values of their BT_s and BT_c are closed to each other. While the values of given others causes and side effects of diabetes like marital status, and type of drug with respect to diabetes type I show that there is effect of these causes on this disease because the value of s-t-norm is not close to the conditional one.

Further, the symptoms which they independent or there is no connection between them and diabetes is (poly urea, thirst, hunger, weight, tiredness). We found by our calculations there is weak effect of side effect blurred vision. This means that diabetes type I is dependent event to the events (marital status, and type of drug).
In fact, we can similarly repeat our calculations with respect to diabetes type II and the same assigned causes of this experiment. The results show that various aspects of dependent and even independent events, and how our s-t-norm with its conditional s-t-norm can be employed so that we obtain some similar and equivalent results to the results that we could obtained by classical ways.

4 Conclusion

This study shows applications of conditional s-t-norm that generalized on complicated algebraic structure which is called BL-algebra. As shown we could apply the conditional case of s-t-norm on weather conditions to predict arrival of the train by mathematical way. Also, from data of diabetic type I and type II, we showed how we can separate between causes and non-causes of diabetes through classification data to gender, place of living, marital status, type of drug, and the symptoms.

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