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Elementary Properties of Triangles in the Inner Product Space

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Abstract: *The triangle in the Euclid room is well known. This paper discusses the expansion of triangles in Euclid's space into inner product spaces. It will be proven several properties of triangles in the Euclidean space which are maintained in the inner product space.*

Keywords : Euclid Space, Inner Product Space, Normed Space, Triangeles

1. INTRODUCTION

Triangles in Euclid's space are already very popular, Euclid in his book Element [8,9] has defined a triangle, which is a form that has three angles and three sides. Next in algebra in the inner product space, by utilizing the Cauchy-Swarz theorem has defined angles [6] which allows expanding the understanding of triangles in abstract space [1,2,3,4]. This paper will discuss the expansion of triangles in Euclid's space into inner product spaces. Besides defining triangles in the inner product space, also will be proven elementary properties of the triangle.

Definition 1.1. [7,10] An inner product space (or pre-Hilbert space) is a vector space X with an inner product defined on X. A Hilbert space is a complete inner product space (complete in the metric defined by the inner product). Here, an inner product on X is a mapping of $X \times X$ into the scalar field K of X; that is, with every pair of vectors x and y there is associated a scalar which is written $\langle x, y \rangle$; and is called the inner product of x and y, such that for all vector x, y,z and scalars α we have :

1. $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
2. $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$
3. $\langle x, y\rangle = \overline{\langle y, x\rangle}$
4. $\langle x, x\rangle \geq 0 \; and \; \langle x, x\rangle = 0 \Leftrightarrow \langle x, x\rangle = 0$

Theorem 1.1. [7,10] If x and y are vectors in a real inner product space X, and if α is a scalar, then:
(a) . $\|x\| \geq 0$ with equality if and only if $x = 0$.
(b) . $\|\alpha x\| = |\alpha|\|x\|$
(c) . $d(x, y) = d(y, x)$
(d) . $d(x, y) \geq 0$ with equality if and only if $x = y$.

Definition 1.2. [1,2,6] If X is a real inner product space, then the *norm* (or *length*) of a vector x in X is denoted by and is defined by $\|x\| = \sqrt{\langle x, x\rangle}$ and the distance between two vectors is denoted by $d(x, y)$ and is defined by :

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y\rangle}$$

Theorem 1.2. [6] Cauchy–Schwarz Inequality If x and y are vectors in a real inner product space X, then:

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

Definition 1.3. [6] Let $(X, \langle\cdot, \cdot\rangle)$ be inner product space and for each $x, y, \in X \setminus \{0\}$ defined the angle between two vectors x and y :

$$\angle(x, y) = \arccos\left(\frac{\langle x, y\rangle}{\|x\|\|y\|}\right) \tag{1.1}$$

Because $\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|y - x\|^2$ hence, from (1.1) the cosine rules are as follows :

$$\|x\|^2 = \|y\|^2 + \|z\|^2 - 2\|y\|\|z\| \cos\angle(y, z) \tag{1.2}$$

$$\|y\|^2 = \|x\|^2 + \|z\|^2 - 2\|x\|\|z\| \cos\angle(-x, z) \tag{1.3}$$

$$\|z\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \cos\angle(x, y) \tag{1.4}$$

Furthermore, we can get the rules of the length of the sides of a triangle :

$$\|x\| = \|y\| \cos\angle(x, y) + \|z\| \cos\angle(z, -x) \tag{1.5}$$

$$\|y\| = \|x\| \cos\angle(x, y) + \|z\| \cos\angle(y, z) \tag{1.6}$$

$$\|z\| = \|y\| \cos\angle(y, z) + \|x\| \cos\angle(z, -x) \tag{1.7}$$

Definition 1.3 [1,2,3,4,5] Let $(X, \langle\cdot, \cdot\rangle)$ be inner product space and for each $x, y, z \in X \setminus \{0\}$, defined $\Delta[x, y, z]$ as $\{x, y, z\}$ who fulfills $x + z = y$ which is equipped with an angle $\angle(x, y)$, $\angle(y, z)$, dan $\angle(z, -x)$. Furthermore $\Delta[x, y, z]$ called a triangle in the inner product space.

2. RESULT

This paper will begin by proving the cosine rule to cause the sine rule. Likewise the sine rule causes the cosine rule. This is, as is true in Euclid's space.

Theorem. 2.1. Let $(X, \langle\cdot, \cdot\rangle)$ be inner product space and $\Delta[x, y, z]$ a triangle in the inner product space, then the cosine rule if and only if the sine rule.

Proof.

$(\Rightarrow)$ Clear that $\cos^2\angle(x, y) + \sin^2\angle(x, y) = 1$

$$\sin^2\angle(x, y) = 1 - \cos^2\angle(x, y)$$

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$$= 1 - \left(\frac{\langle x,y\rangle}{\|x\|\|y\|}\right)^2$$

$$= \frac{\|x\|^2\|y\|^2 - \langle x,y\rangle^2}{\|x\|^2\|y\|^2}$$

$$\|x\|\|y\|\sin\angle(x,y) = \sqrt{\|x\|^2\|y\|^2 - \langle x,y\rangle^2} = K \qquad 2.2a$$

In a similar way obtained :

$$\|y\|\|z\|\sin\angle(y,z) = \sqrt{\|y\|^2\|z\|^2 - \langle y,z\rangle^2} = K \qquad 2.2b$$

$$\|x\|\|z\|\sin\angle(-x,z) = \sqrt{\|x\|^2\|z\|^2 - \langle -x,z\rangle^2} = K \qquad 2.2c$$

or

$$\frac{\sin\angle(x,y)}{\|z\|} = \frac{\sin\angle(y,z)}{\|x\|} = \frac{\sin\angle(-x,z)}{\|y\|} = K \qquad (2.2d)$$

$(\Leftarrow)$ Consider the equation (2,2d), then obtained :

$$K^2(\|x\|^2 + \|y\|^2 - \|z\|^2)$$

$$= \|x\|^2\|y\|^2\|z\|^2(\sin^2\angle(y,z) + \sin^2\angle(z,-x) - \sin^2\angle(x,y))$$

$$= \|x\|^2\|y\|^2\|z\|^2(\sin^2\angle(y,z) + \sin^2\angle(z,-x) - \sin^2(\angle(y,z) + \angle(z,-x)))$$

$$= \|x\|^2\|y\|^2\|z\|^2(\sin^2\angle(y,z) + \sin^2\angle_W(z,-x) -(\sin\angle(y,z)\cos\angle(z,-x) + \cos\angle(y,z)\sin\angle(z,-x))^2)$$

$$= \|x\|^2\|y\|^2\|z\|^2(\sin^2\angle(y,z) + \sin^2\angle(-x,z) - \sin^2\angle(y,z)\cos^2\angle(-x,z) - \cos^2\angle(y,z)\sin^2\angle(-x,z) - 2\sin\angle(y,z)\cos\angle(-x,z)\cos\angle(y,z)\sin\angle(-x,z))$$

$$= \|x\|^2\|y\|^2\|z\|^2(\sin^2\angle(y,z)\,(1 - \cos^2\angle(-x,z)) + \sin^2\angle(-x,z)(1-\cos^2\angle(y,z))$$

$$- 2\sin\angle(y,z)\cos\angle(-x,z)\cos\angle(y,z)\sin\angle(-x,z))$$

$$= \|x\|^2\|y\|^2\|z\|^2(\sin^2\angle(y,z)\,(\sin^2\angle(z,-x))$$

$$+ \sin^2\angle(z,-x)(\sin^2\angle(y,z))$$

$$- 2\sin\angle(y,z)\cos\angle(z,-x)\cos\angle(y,z)\sin\angle(z,-x)$$

$$= 2\|x\|^2\|y\|^2\|z\|^2(\sin^2\angle(y,z)\,(\sin^2\angle(z,-x))$$

$$- \sin\angle(y,z)\cos\angle(z,-x)\cos\angle(y,z)\sin\angle(z,-x)$$

$$= 2\|x\|^2\|y\|^2\|z\|^2(\sin\angle(y,z)\sin\angle(z,-x)\,(\sin\angle(y,z)\sin\angle(z,-x) - \cos\angle(y,z)\cos\angle(z,-x)))$$

$$= 2\|x\|^2\|y\|^2\|z\|^2(\sin\angle(y,z)\sin\angle(z,-a)\cos\angle(x,y))$$

$$= 2\,\|\mathrm{x}\|K\|y\|K\cos\angle(x,y))$$

$$= 2\,K^2\|\mathrm{x}\|\|y\|\cos\angle(x,y)$$

From this we get the cosine rule,

$\|x\|^2 + \|y\|^2 - \|z\|^2 = 2\|x\|\|y\|\cos\angle(x,y)$ or

$$\|z\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos\angle(x,y)$$

Theorem. 2.2. Let $(X,\langle\cdot,\cdot\rangle)$ be inner product space and $\Delta[x,y,z]$ a triangle in the inner product space, then the number of angles $\angle(x,y) + \angle(y,z) + \angle(z,-x) = \pi$.

Proof.

Note that $\angle(x,y) + \angle(y,z) + \angle(z,-x) = \pi$.

$\Leftrightarrow\ \angle(x,y) + \angle(y,z) = \angle(z,-x) - \pi$.

$\Leftrightarrow \cos(\angle(x,y) + \angle(y,z)) = \cos(\angle(z,-x) - \pi)$.

$\Leftrightarrow \cos\angle(x,y)\cos\angle(y,z) - \sin\angle(x,y)\sin\angle(y,z) = -\cos\angle(z,-x)$.

$\Leftrightarrow \cos\angle(x,y)\cos\angle(y,z) - \sin\angle(x,y)\sin\angle(y,z) + \cos\angle(z,-x) = 0$

$$\Leftrightarrow \frac{\langle x,y\rangle}{\|\mathrm{x}\|\|y\|}\frac{\langle y,z\rangle}{\|y\|\|z\|} + \frac{\langle z,-x\rangle}{\|z\|\|x\|} - \frac{\mathrm{K}}{\|\mathrm{x}\|\|\mathrm{y}\|}\frac{\mathrm{K}}{\|\mathrm{y}\|\|\mathrm{z}\|}$$

$$\Leftrightarrow \frac{\langle x,y\rangle\langle y,z\rangle}{\|\mathrm{x}\|\|\mathrm{y}\|^2\|\mathrm{z}\|} + \frac{\langle z,-x\rangle}{\|z\|\|x\|} - \frac{\mathrm{K}^2}{\|\mathrm{x}\|\|\mathrm{y}\|^2\|\mathrm{z}\|}$$

$$\Leftrightarrow \frac{4\mathrm{K}^2 - 4\mathrm{K}^2}{4\|\mathrm{x}\|\|\mathrm{y}\|^2\|\mathrm{z}\|} = 0 \quad , \qquad \text{as a result}$$

$\angle(x,y) + \angle(y,z) + \angle(z,-x) = \pi$. ■

Example. 2.1

Let $(M^{2\times2},\langle\cdot,\cdot\rangle)$ be the inner product space . Choose a triangle $\Delta[x,y,z]$, with $x = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}$, $y = \begin{pmatrix}-1 & 5\\6 & 8\end{pmatrix}$, $z = \begin{pmatrix}-2 & 3\\3 & 4\end{pmatrix}$.

$$\|x\| = \sqrt{\langle x,x\rangle} = \sqrt{30}$$
$$\|y\| = \sqrt{\langle y,y\rangle} = \sqrt{126}$$
$$\|z\| = \sqrt{\langle z,z\rangle} = \sqrt{38}$$

$$\angle(-x,z) = \arccos\left(\frac{\langle -x,z\rangle}{\|-x\|\|z\|}\right)$$
$$= \arccos\left(\frac{-29}{\sqrt{30}\sqrt{38}}\right) = 149,19^0$$

$$\angle(y,z) = \arccos\left(\frac{\langle y,z\rangle}{\|y\|\|z\|}\right)$$
$$= \arccos\left(\frac{67}{\sqrt{126}\sqrt{38}}\right) = 14,47^0$$

$$\angle(x,y) = \arccos\left(\frac{\langle x,y\rangle}{\|x\|\|y\|}\right)$$
$$= \arccos\left(\frac{59}{\sqrt{30}\sqrt{126}}\right) = 16,34^0$$

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So obtained :

$\angle(x,y) + \angle(y,z) + \angle(z,-x) = 16{,}34^0 + 14{,}47^0 + 149,19^0 = 180^0 = \pi.$

Example 2.2.

Let $(\mathbf{C}[1{,}0], \langle\cdot\,,\ \cdot\rangle)$ be inner product space. Choose triangle $\Delta[x,y,z]$, $\{x(t), y(t), z(t)\}$ with :

$x(t) = 2t^3 - 3t^2 + 2t + 1$

$y(t) = 2t^3 + 4$

$z(t) = 3t^2 - 2t + 3$

Then obtained :

$$\|x(t)\|^2 = \int_0^1 (4t^6 - 12t^5 + 17t^4 - 8t^3 + 2t^2 + 4t + 1)dt$$
$$= \left[\frac{4}{7}t^7 - 2t^6 + \frac{17}{5}t^5 - 2t^4 + \frac{2}{3}t^3 + 2t^2 + t\right]_0^1$$
$$= 3{,}64$$
$$\|x(t)\| = 1{,}9$$

$$\|y(t)\|^2 = \int_0^1 (4t^6 + 4t^3 + 16)dt$$
$$= \left[\frac{4}{7}t^7 + t^4 + 16t\right]_0^1$$
$$= 17{,}57$$
$$\|y(t)\| = 4{,}19$$

$$\|z(t)\|^2 = \int_0^1 (9t^4 - 12t^3 + 22t^2 - 6t + 9)dt$$
$$= \left[\frac{9}{5}t^5 - 3t^4 + \frac{22}{3}t^3 - 3t^2 + 9t\right]_0^1$$
$$= 12{,}1\ 3$$
$$\|z(t)\| = 3{,}48$$

$$\langle -x(t), z(t)\rangle = \int_0^1 (-6t^5 + 13t^4 - 18t^3 + 10t^2 - 4t - 3)dt$$
$$= \left[-t^6 + \frac{13}{5}t^5 - \frac{18}{4}t^4 + \frac{10}{3}t^2 - 2\,t^2 - 3t\right]_0^1$$
$$= -4{,}57$$

$$\langle y(t), z(t)\rangle = \int_0^1 (6t^5 - 4t^4 + 6t^3 + 12t^2 - 8t + 12)dt$$
$$= \left[-t^6 - \frac{4}{5}t^5 + \frac{6}{4}t^4 + 4t^3 - 4\,t^2 + 12t\right]_0^1$$
$$= 11{,}7$$

$$\langle x(t), y(t)\rangle = \int_0^1 (4t^6 - 6t^5 + 4t^4 + 10t^3 - 12t^2 + 8t + 4)dt$$
$$= \left[\frac{4}{7}t^7 - t^6 + \frac{4}{5}t^5 + \frac{10}{4}t^4 - 3\,t^3 + 4t^2 + 4t\right]_0^1$$
$$= 7{,}87$$

$$\angle(-x,z) = \arccos\left(\frac{\langle -x, z\rangle}{\|-x\|\|z\|}\right)$$
$$= \arccos\left(\frac{-4{,}57}{1{,}9.3{,}48}\right)$$
$$= 133{,}70^0$$
$$\angle(y,z) = \arccos\left(\frac{\langle y, z\rangle}{\|y\|\|z\|}\right)$$
$$= \arccos\left(\frac{11{,}7}{4{,}19.3{,}48}\right)$$
$$= 36{,}64^0$$
$$\angle(x,y) = \arccos\left(\frac{\langle x, y\rangle}{\|x\|\|y\|}\right)$$
$$= \arccos\left(\frac{7{,}87}{1{,}9\ .4{,}19}\right)$$
$$= 9{,}66^0$$

$\angle(x,y) + \angle(y,z) + \angle(z,-x) = 9{,}66^0 + 36{,}64^0 + 133{,}70^0 = 180^0 = \pi.$

Definition 2.1. Let $(X, \langle\cdot\,,\cdot\,\rangle)$ be inner product space and $\Delta[a,b,c]$ a triangle in the inner product space. Triangle $\Delta[a,b,c]$ named an isosceles triangle if two sides have the same norm.

Example 2.3.

Let $(M^{2\times 2}, \langle\cdot\,,\cdot\rangle)$ be the inner product space . Choose a triangle $\Delta[x,y,z]$, with $x = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $y = \begin{pmatrix} -1 & 3 \\ 7 & 7 \end{pmatrix}$, $z = \begin{pmatrix} -2 & 1 \\ 4 & 3 \end{pmatrix}$.

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{30}$$
$$\|y\| = \sqrt{\langle y, y\rangle} = \sqrt{108}$$
$$\|z\| = \sqrt{\langle z, z\rangle} = \sqrt{30}$$

Theorem 2.3. Let $(X, \langle\cdot\,,\cdot\,\rangle)$ be inner product space and $\Delta[x,y,z]$ a triangle in the inner product space. The opposite angles of the isosceles triangle are equal.

Proof.

Let $\|x\| = \|z\|$ be from the side length rule then obtained

$$\|y\|\cos\angle(x,y) + \|z\|\cos\angle(z,-x) = \|y\|\cos\angle(y,z) + \|x\|\cos\angle(z,-x)$$
$$\Leftrightarrow\ \|y\|\cos\angle(x,y) = \|y\|\cos\angle(y,z)$$
$$\Leftrightarrow\ \cos\angle(x,y) = \cos\angle(y,z)$$
$$\Leftrightarrow\ \angle(x,y) = \angle(y,z) \qquad \blacksquare$$

Example 2.4.

Let $(M^{2\times 2}, \langle\cdot,\cdot\rangle)$ be the inner product space . Choose a triangle $\Delta[x,y,z]$, with $x = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $y = \begin{pmatrix} -1 & 3 \\ 7 & 7 \end{pmatrix}$, $z = \begin{pmatrix} -2 & 1 \\ 4 & 3 \end{pmatrix}$.

$$\|x\| = \sqrt{\langle x,x\rangle} = \sqrt{30}$$
$$\|y\| = \sqrt{\langle y,y\rangle} = \sqrt{108}$$
$$\|z\| = \sqrt{\langle z,z\rangle} = \sqrt{30}$$
$$\angle(x,y) = \arccos\left(\frac{\langle x,y\rangle}{\|x\|\|y\|}\right)$$
$$= \arccos\left(\frac{54}{\sqrt{30}\sqrt{108}}\right)$$
$$= 18{,}44^0$$
$$\angle(y,z) = \arccos\left(\frac{\langle y,z\rangle}{\|y\|\|z\|}\right)$$
$$= \arccos\left(\frac{54}{\sqrt{108}\sqrt{30}}\right)$$
$$= 18{,}44^0$$

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