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On The Rainbow Antimagic Connection Number of Some Wheel Related Graphs

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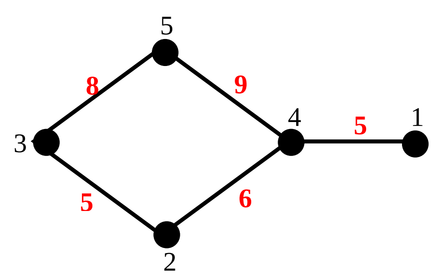
Abstract: *All graphs in this paper is connected and simple. Let $\boldsymbol{G(V,E)}$ be a connected and simple graph with vertices set $\boldsymbol{V}$ and edge set $\boldsymbol{E}$. A bijection function $\boldsymbol{f: V \to \{1,2,3,\dots,|V(G)|\}}$ is called an edge antimagic vertex labelling if for every $\boldsymbol{e(u,v)}$, the edge weight $\boldsymbol{w(e) = f(u) + f(v)}$ are all different. An edge antimagic labeling can generate a rainbow edge antimagic coloring if there is a rainbow path between every pair of vertices. The rainbow edge antimagic connection number of graph $\boldsymbol{G}$, denoted by $\boldsymbol{rc_A(G)}$ is the smallest number of colors that are needed in order to make $\boldsymbol{G}$ rainbow connected under the assignment of edge antimagic labelling. In this paper, we will study the existence of rainbow antimagic coloring its connection number of some wheel related graphs. We have found the values of the rainbow antimagic connection of flower graph and gear graph.*

Keywords *:* rainbow antimagic connection number ; flower ; gear

1. INTRODUCTION

Graph Theory is one of topics that has so many various kinds of gap research. In graph theory, we often heard specific topic about labeling and coloring. An Edge coloring of a graph G is called a rainbow coloring if for any two different vertices of graph are connected by minimally one rainbow path. Rainbow path is a path between two different vertices which every edge has different color . Antimagic labeling of a graph G can be defined as a function $f: V \to \{1,2,3,\dots,|V(G)|\}$ which every vertex has different label. In some last research of graph theory, there are so many incredible topics and research gaps about rainbow coloring and antimagic labeling, one of them is a research which is belong to Dafik et. al. (2019), They collaborate the concept of rainbow coloring and antimagic labeling. According to Dafik et. Al. (2019), rainbow antimagic coloring is a bijection function $f: V \to \{1,2,3,\dots,|V(G)|\}$ which is for any two different vertices, there is minimally one rainbow path and every $\boldsymbol{e(u,v)}$, the edge weight $\boldsymbol{w(e) = f(u) + f(v)}$. The illustration of rainbow antimagic coloring is provided below



The black labels on every vertex is called vertex label, and the red one is called the weights of the edge. Those weights are called the color needed to make that graph rainbow connected. But those weights are not necessarily the minimum colors needed which known as rainbow antimagic connection number. Based on research by Dafik et.al. [ ] , for any graph G , we have $rc(G) \le rc_A(G)$

Proposition 1. *Observation by (Dafik et.al., 2019), for every graph* $G, rc(G) \le rc_A(G)$

Proposition 2. *(Syafrizal et.al., 2014) the rainbow connection number of Gear graph* G_n are

$$rc(G_n) = \begin{cases} 3, & \text{for } n = 3 \\ 4, & \text{for } n \geq 4 \end{cases}$$

Proposition 3. *(Ramya N. dkk., 2012) the rainbow connection number of flower graph* Fl_n is $rc(Fl_n) = 3$ for $n \geq 3$

2. BASIC RESULT

In this section we would like to find the rainbow antimagic connection number of flower graph and gear graph.

Theorem 2.1 The rainbow connection number of flower graph is $rc_A(Fl_n) = 2n$, for $n \geq 3$

Let (Fl_n) be a flower graph with order n . The vertex set of (Fl_n) is $V(Fl_n) = \{z\} \cup \{x_i\,; 1 \leq i \leq n\} \cup \{y_j\,; 1 \leq j \leq n\}$ and the edge set is $E(Fl_n) = \{zx_i\,; 1 \leq i \leq n\} \cup \{zy_j\,; 1 \leq j \leq n\} \cup \{x_i y_j\,; 1 \leq i = j \leq n\} \cup \{x_i x_{i+1}\,; 1 \leq i \leq n-1\} \cup \{x_n x_1\}$. The cardinality of the vertex set is $|V(Fl_n)| = 2n + 1$ and the cardinality of the edge set is $|E(Fl_n)| = 4n$. There are six cases in the rainbow connection number of flower graph, those are for $n = 3$, $n = 4$, $n = 5$, $n = 6$, $n = 7$, and $n \geq 8$

Case 1.

The rainbow antimagic connection number of graph Fl_3 is 6

Proof. Based on Proposition 1 and 3, we have $3 \leq rc_A(Fl_3)$. We prove that the lower bound of the rainbow antimagic connection number of Fl_3 is $6 \leq rc_A(Fl_3)$. Assume that $rc_A(Fl_3) < 6$, then we take $rc_A(Fl_3) = 5$. It means we should color all the edges of graph Fl_3 with 5 colors. As we know that graph Fl_3 has 6 spokes, so if it is colored with 5 colors, at least there are two spokes have the same color. For example,

$$\begin{aligned} & w(x_k z) = w(zx_l) \qquad ; 1 \leq k \neq l \leq 2n \\ \leftrightarrow \quad & f(x_k) + f(z) = f(z) + f(x_l) \\ \leftrightarrow \quad & f(x_k) = f(x_l) \end{aligned}$$

There is a contradiction, based on antimagic labeling, every vertex has different label, so the assumption is wrong. It means $6 \leq rc_A(Fl_3)$.

Furthermore, we prove the upper bound of the rainbow antimagic connection number of Fl_3 is $6 \geq rc_A(Fl_3)$. Define a function $f: V(Fl_3) \to \{1, 2, 3, \dots, 7\}$ as a bijective function of all vertices in graph Fl_3 to a set of natural number $\{1, 2, 3, \dots, 7\}$ as provided below,

$$f(z) = 3$$
$$f(x_1) = 6\ , \quad f(x_2) = 1\ , \quad f(x_3) = 4$$
$$f(x_4) = 2\ , \quad f(x_5) = 7\ , \quad f(x_6) = 5$$

Based on the function above, for every path $x_k x_l$ where $k \neq l$ through the points $(x_k - z - x_l)$, there two colors that must be passed $w(x_k z)$ and $w(zx_l)$. And for every path $y_k y_l$ where $k \neq l$ through the points $(y_k - z - y_l)$, there two colors that must be passed $w(y_k z)$ and $w(zy_l)$. We know that the weight of every spokes are different $w(x_k z) \neq w(zx_l)$ and $w(y_k z) \neq w(zy_l)$. It means for every two different vertices in graph Fl_3, at least there exist one rainbow path connecting them. So, $6 \geq rc_A(Fl_3)$. We have $6 \leq rc_A(Fl_3) \leq 6$, it means that $(Fl_3) = 6$.

Case 2.

The rainbow antimagic connection number of graph Fl_4 is 8

Proof. Based on Proposition 1 and 3, we have $3 \leq rc_A(Fl_4)$. We prove that the lower bound of the rainbow antimagic connection number of Fl_4 is $8 \leq rc_A(Fl_4)$. Assume that $rc_A(Fl_4) < 8$, then we take $rc_A(Fl_4) = 7$. It means we should color all the edges of graph Fl_4 with 7 colors. As we know that graph Fl_4 has 8 spokes, so if it is colored with 7 colors, at least there are two spokes have the same color. For example,

$$\begin{aligned} & w(x_k z) = w(zx_l) \qquad ; 1 \leq k \neq l \leq 2n \\ \leftrightarrow \quad & f(x_k) + f(z) = f(z) + f(x_l) \\ \leftrightarrow \quad & f(x_k) = f(x_l) \end{aligned}$$

There is a contradiction, based on antimagic labeling, every vertex has different label, so the assumption is wrong. It means $8 \leq rc_A(Fl_4)$.

Furthermore, we prove the upper bound of the rainbow antimagic connection number of Fl_4 is $8 \geq rc_A(Fl_4)$. Define a function $f: V(Fl_4) \to \{1, 2, 3, \dots, 9\}$ as a bijective function of all vertices in graph Fl_3 to a set of natural number $\{1, 2, 3, \dots, 9\}$ as provided below,

$$f(z) = 3$$
$$f(x_1) = 1\ , \ f(x_2) = 6\ , \ f(x_3) = 2, \ f(x_4) = 7$$
$$f(x_5) = 9\ , \ f(x_6) = 5\ , \ f(x_7) = 8, \ f(x_8) = 4$$

Based on the function above, for every path $x_k x_l$ where $k \neq l$ through the points $(x_k - z - x_l)$, there two colors that must be passed $w(x_k z)$ and $w(zx_l)$. And for every path $y_k y_l$ where $k \neq l$ through the points $(y_k - z - y_l)$, there two colors that must be passed $w(y_k z)$ and $w(zy_l)$. We know that the weight of every spokes are different $w(x_k z) \neq w(zx_l)$ and $w(y_k z) \neq w(zy_l)$. It means for every two different vertices in graph Fl_4, at least there exist one rainbow path connecting them. So, $8 \geq rc_A(Fl_4)$. We have $8 \leq rc_A(Fl_4) \leq 8$, it means that $(Fl_4) = 8$.

Case 3.

The rainbow antimagic connection number of graph Fl_5 is 10

Proof. Based on Proposition 1 and 3, we have $3 \leq rc_A(Fl_5)$. We prove that the lower bound of the rainbow antimagic connection number of Fl_5 is $10 \leq rc_A(Fl_5)$. Assume that $rc_A(Fl_5) < 10$, then we take $rc_A(Fl_5) = 9$. It means we should color all the edges of graph Fl_5 with 9 colors. As we know that graph Fl_5 has 10 spokes, so if it is colored with 9 colors, at least there are two spokes have the same color. For example,

$$\begin{aligned} & w(x_k z) = w(z x_l) \qquad ;1 \leq k \neq l \leq 2n \\ \leftrightarrow \quad & f(x_k) + f(z) = f(z) + f(x_l) \\ \leftrightarrow \quad & f(x_k) = f(x_l) \end{aligned}$$

There is a contradiction, based on antimagic labeling, every vertex has different label, so the assumption is wrong. It means $10 \leq rc_A(Fl_5)$.

Furthermore, we prove the upper bound of the rainbow antimagic connection number of Fl_5 is $10 \geq rc_A(Fl_5)$. Define a function $f: V(Fl_5) \to \{1, 2, 3, \dots, 11\}$ as a bijective function of all vertices in graph Fl_3 to a set of natural number $\{1, 2, 3, \dots, 11\}$ as provided below,

$$\begin{gathered} f(z) = 3, \quad f(x_1) = 1, \quad f(x_2) = 6 \\ f(x_3) = 2, \ f(x_4) = 5, \ f(x_5) = 4, \ f(x_6) = 11 \\ f(x_7) = 7, \ f(x_8) = 10, \ f(x_9) = 8, \ f(x_{10}) = 9 \end{gathered}$$

Based on the function above, for every path $x_k x_l$ where $k \neq l$ through the points $(x_k - z - x_l)$, there two colors that must be passed $w(x_k z)$ and $w(z x_l)$. And for every path $y_k y_l$ where $k \neq l$ through the points $(y_k - z - y_l)$, there two colors that must be passed $w(y_k z)$ and $w(z y_l)$. We know that the weight of every spokes are different $w(x_k z) \neq w(z x_l)$ and $w(y_k z) \neq w(z y_l)$. It means for every two different vertices in graph Fl_5, at least there exist one rainbow path connecting them. So, $10 \geq rc_A(Fl_5)$. We have $10 \leq rc_A(Fl_5) \leq 10$, it means that $(Fl_5) = 10$

Case 4.

The rainbow antimagic connection number of graph Fl_6 is 12

Proof. Based on Proposition 1 and 3, we have $3 \leq rc_A(Fl_6)$. We prove that the lower bound of the rainbow antimagic connection number of Fl_6 is $12 \leq rc_A(Fl_6)$. Assume that $rc_A(Fl_6) < 12$, then we take $rc_A(Fl_6) = 11$. It means we should color all the edges of graph Fl_6 with 11 colors. As we know that graph Fl_6 has 12 spokes, so if it is colored with 11 colors, at least there are two spokes have the same color. For example,

$$\begin{aligned} & w(x_k z) = w(z x_l) \qquad ;1 \leq k \neq l \leq 2n \\ \leftrightarrow \quad & f(x_k) + f(z) = f(z) + f(x_l) \\ \leftrightarrow \quad & f(x_k) = f(x_l) \end{aligned}$$

There is a contradiction, based on antimagic labeling, every vertex has different label, so the assumption is wrong. It means $12 \leq rc_A(Fl_6)$.

Furthermore, we prove the upper bound of the rainbow antimagic connection number of Fl_6 is $12 \geq rc_A(Fl_6)$. Define a function $f: V(Fl_6) \to \{1, 2, 3, \dots, 12\}$ as a bijective function of all vertices in graph Fl_3 to a set of natural number $\{1, 2, 3, \dots, 12\}$ as provided below,

$$\begin{gathered} f(z) = 4 \\ f(x_1) = 1, \ f(x_2) = 5, \ f(x_3) = 2, \ f(x_4) = 7 \\ f(x_5) = 3, \ f(x_6) = 6, \ f(x_7) = 13, \ f(x_8) = 10 \\ f(x_9) = 12, \ f(x_{10}) = 8, \ f(x_{11}) = 11, \ f(x_{12}) = 9 \end{gathered}$$

Based on the function above, for every path $x_k x_l$ where $k \neq l$ through the points $(x_k - z - x_l)$, there two colors that must be passed $w(x_k z)$ and $w(z x_l)$. And for every path $y_k y_l$ where $k \neq l$ through the points $(y_k - z - y_l)$, there two colors that must be passed $w(y_k z)$ and $w(z y_l)$. We know that the weight of every spokes are different $w(x_k z) \neq w(z x_l)$ and $w(y_k z) \neq w(z y_l)$. It means for every two different vertices in graph Fl_6, at least there exist one rainbow path connecting them. So, $12 \geq rc_A(Fl_6)$. We have $12 \leq rc_A(Fl_6) \leq 12$, it means that $(Fl_6) = 12$

Case 5.

The rainbow antimagic connection number of graph Fl_7 is 14

Proof. Based on Proposition 1 and 3, we have $3 \leq rc_A(Fl_7)$. We prove that the lower bound of the rainbow antimagic connection number of Fl_7 is $14 \leq rc_A(Fl_7)$. Assume that $rc_A(Fl_7) < 14$, then we take $rc_A(Fl_7) = 13$. It means we should color all the edges of graph Fl_7 with 13 colors. As we know that graph Fl_7 has 14 spokes, so if it is colored with 13 colors, at least there are two spokes have the same color. For example,

$$\begin{aligned} & w(x_k z) = w(z x_l) \qquad ;1 \leq k \neq l \leq 2n \\ \leftrightarrow \quad & f(x_k) + f(z) = f(z) + f(x_l) \\ \leftrightarrow \quad & f(x_k) = f(x_l) \end{aligned}$$

There is a contradiction, based on antimagic labeling, every vertex has different label, so the assumption is wrong. It means $14 \leq rc_A(Fl_7)$.

Furthermore, we prove the upper bound of the rainbow antimagic connection number of Fl_7 is $14 \geq rc_A(Fl_7)$. Define a function $f: V(Fl_7) \to \{1, 2, 3, \dots, 15\}$ as a bijective function of all vertices in graph Fl_7 to a set of natural number $\{1, 2, 3, \dots, 15\}$ as provided below,

$$\begin{gathered} f(z) = 4, \quad f(x_1) = 1, \quad f(x_2) = 5 \\ f(x_3) = 7, \ f(x_4) = 2, \ f(x_5) = 8, \ f(x_6) = 3 \\ f(x_7) = 6, \ f(x_8) = 15, \ f(x_9) = 12, \ f(x_{10}) = 10 \\ f(x_{11}) = 14, \ f(x_{12}) = 9, \ f(x_{13}) = 13, \ f(x_{14}) = 11 \end{gathered}$$

Based on the function above, for every path $x_k x_l$ where $k \neq l$ through the points $(x_k - z - x_l)$, there two colors that must be passed $w(x_k z)$ and $w(z x_l)$. And for every path $y_k y_l$ where $k \neq l$ through the points $(y_k - z - y_l)$, there two colors that must be passed $w(y_k z)$ and $w(z y_l)$. We know that the weight of every spokes are different $w(x_k z) \neq w(z x_l)$ and $w(y_k z) \neq w(z y_l)$. It means for every two different vertices in graph Fl_7, at least there exist one rainbow path connecting them. So, $14 \geq rc_A(Fl_7)$. We have $14 \leq rc_A(Fl_7) \leq 14$, it means that $(Fl_7) = 14$

Case 6.
The rainbow antimagic connection number of graph Fl_n where $n \geq 8$ is $2n$

Proof. Based on Proposition 1 and 3, we have $3 \leq rc_A(Fl_n)$. We prove that the lower bound of the rainbow antimagic connection number of Fl_n is $2n \leq rc_A(Fl_n)$. Assume that $rc_A(Fl_n) < 2n$, then we take $rc_A(Fl_n) = 2n - 1$. It means we should color all the edges of graph Fl_n with $2n - 1$ colors. As we know that graph Fl_n has $2n$ spokes, so if it is colored with $2n - 1$ colors, at least there are two spokes have the same color. For example,

$$w(x_k z) = w(zx_l) \quad ;1 \leq k \neq l \leq 2n$$
$$\leftrightarrow \quad f(x_k) + f(z) = f(z) + f(x_l)$$
$$\leftrightarrow \quad f(x_k) = f(x_l)$$

There is a contradiction, based on antimagic labeling, every vertex has different label, so the assumption is wrong. It means $2n \leq rc_A(Fl_n)$.

Furthermore, we prove the upper bound of the rainbow antimagic connection number of Fl_n is $2n \geq rc_A(Fl_n)$. Define a function $f: V(Fl_n) \rightarrow \{1, 2, 3, \ldots, 2n + 1\}$ as a bijective function of all vertices in graph Fl_7 to a set of natural number $\{1, 2, 3, \ldots, 2n + 1\}$ as provided below,

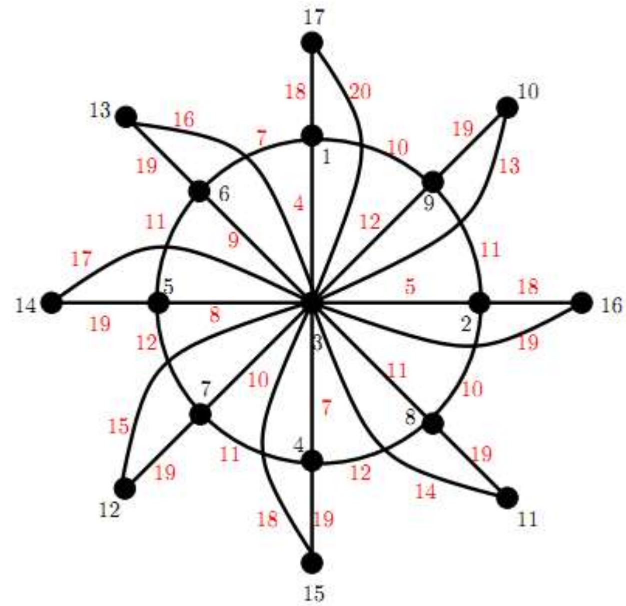
$$f(z) = 3$$

$$f(x_i) = \begin{cases} \frac{i+1}{2}, & for\ i \in [1,3] \\ \frac{i+3}{2}, & for\ i\ odd\ ; i \geq 5 \\ n - \frac{i}{2} + 2, & for\ i\ even \end{cases}$$

$$f(y_j) = \begin{cases} \frac{4n - j + 3}{2}, & \text{for } j \text{ odd} \\ n + \frac{j}{2} + 1, & \text{for } j \text{ even} \end{cases}$$

Thus, we have the set of the edge weight is $W = \left\{\frac{1+7}{2}, \frac{1+9}{2}, \frac{2n-i+10}{2}, n+2, n+3, n+4, \left\lfloor\frac{n+4}{2}\right\rfloor + 1, \frac{4n-i+9}{2}, \frac{2n+i+8}{2}, 2n+2, 2n+3\right\}$.

Based on the set of the edge weight above, for every path $x_k x_l$ where $k \neq l$ through the points $(x_k - z - x_l)$, there two colors that must be passed $w(x_k z)$ and $w(zx_l)$. And for every path $y_k y_l$ where $k \neq l$ through the points $(y_k - z - y_l)$, there two colors that must be passed $w(y_k z)$ and $w(zy_l)$. We know that the weight of every spokes are different $w(x_k z) \neq w(zx_l)$ and $w(y_k z) \neq w(zy_l)$. It means for every two different vertices in graph Fl_n, at least there exist one rainbow path connecting them. So, $2n \geq rc_A(Fl_n)$. We have $2n \leq rc_A(Fl_n) \leq 2n$, it means that $(Fl_n) = 2n$. The rainbow antimagic connection for Fl_8 is provided below



Theorem 2.2 The rainbow connection number of gear graph is $rc(G_n) \leq rc_A(G_n) \leq n + 2$, for $n \geq 3$

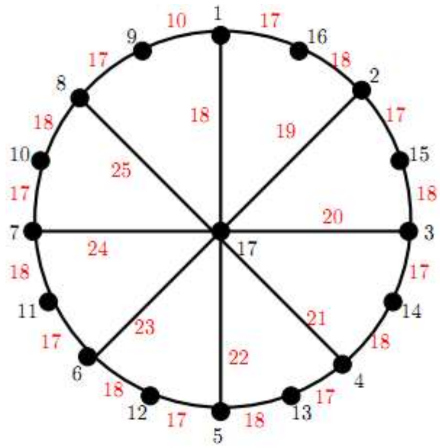
Let (G_n) be a gear graph with order n . The vertex set of (G_n) is $V(G_n) = \{z\} \cup \{x_i\ ; 1 \leq i \leq n\} \cup \{y_j\ ; 1 \leq j \leq n\}$ and the edge set is $E(G_n) = \{zx_i\ ; 1 \leq i \leq n\} \cup \{x_i y_j\ ; 1 \leq i = j \leq n\} \cup \{x_i y_j\ ; 2 \leq i \leq n\ ; j = i - 1\} \cup \{x_1 y_n\}$. The cardinality of the vertex set is $|V(G_n)| = 2n + 1$ and the cardinality of the edge set is $|E(G_n)| = 3n$.

Proof. Based on Proposition 1 and 2, we have $3 = rc(G_n) \leq rc_A(G_n)$ for $n = 3$, and $4 = rc(G_n) \leq rc_A(G_n)$ for $n \geq 4$. Furthermore, we prove that $rc_A(G_n) \leq n + 2$. Define a bijective function of all vertices in graph G_n as follows

$$f(z) = 2n + 1\ , \quad f(x_i) = i\ , \quad f(y_j) = 2n + 1 - j$$

Thus, we have the set of the edge weight is $W = \{2n + 1\ , 2n + 2\ , 2n + 3\ , \ldots, 2n + i, 2n + i + 1\ , n + 2\}$.

Based on the set of the edge weight above, for every path $x_k x_l$ where $k \neq l$ through the points $(x_k - z - x_l)$, there two colors that must be passed $w(x_k z)$ and $w(zx_l)$. And for every path $y_k y_l$ where $k \neq l$ through the points $(y_k - z - y_l)$, there two colors that must be passed $w(y_k z)$ and $w(zy_l)$. We know that the weight of every spokes are different $w(x_k z) \neq w(zx_l)$ and $w(y_k z) \neq w(zy_l)$. It means for every two different vertices in graph G_n, at least there exist one rainbow path connecting them. So, $n + 2 \geq rc_A(G_n)$, and we have $rc(G_n) \leq rc_A(G_n) \leq n + 2$ .The rainbow antimagic connection for G_8 is provided below



3. CONCLUSION

In this paper, we have studied the rainbow antimagic connection number of gear graph and flower graph. We can find the exact value of flower graph, but for gear graph we can only find the upper bound of its rainbow antimagic connection number. The rainbow antimagic connection number of gear graph is considered to be NP-problem, thus finding the rainbow antimagic connection number is still being open for the researchers to study.

4. ACKNOWLEDGEMENT

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