

# Fuzzy Laplace Transforms of the Fuzzy Riemann-Liouville Fractional Derivatives about Order $3 < \beta < 4$

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**Abstract:** The main aim of this paper is to find the formulas for fuzzy Riemann-Liouville fractional derivatives of the order  $0 < \beta < 4$  for fuzzy -valued function  $f$  and the formulas of fuzzy Laplace Transforms for fuzzy Riemann-Liouville fractional derivatives of the order  $3 < \beta < 4$  under Hukuhara difference(H- difference).

## 1. Introduction

Fractional calculus and fractional differential equations have undergone expanded study in recent years as a considerable interest both in mathematics and in applications. They were applied in modeling of many physical and chemical processes and in engineering [1].

There are many researchers have been worked on the field of fuzzy fractional differential equations (FFDEs) for example: Mohammad OH et al. [2] present an approximate analytical solution for fuzzy fractional initial value problems (FFIVs) using differential transform method, Salahshour S et al. [1] deal with the solution of FFDEs under Riemann-Liouville H-differentiability by fuzzy Laplace Transforms, Jafarian A et al. [3] used fractional fuzzy Laplace transformation to solve the fuzzy fractional eigenvalue differential equation .

This paper is arranged as follows: Basic concepts are given in Section 2. In Section 3, the formulas for fuzzy Riemann-Liouville fractional derivatives of the order  $0 < \beta < 4$  for fuzzy -valued function  $f$  and the formulas of fuzzy Laplace Transforms for fuzzy Riemann-Liouville fractional derivatives of the order  $3 < \beta < 4$ . In Section 4, example of FFIV of order  $3 < \beta < 4$  is given. In Section 5, conclusions are drawn.

## 2. Basic Concepts

Definition 2.1 [4] A fuzzy number  $u$  in parametric form is a pair  $(\underline{u}, \bar{u})$  of functions  $\underline{u}(r), \bar{u}(r)$ ,  $0 \leq r \leq 1$ , which satisfy the following requirements:

$\underline{u}(r)$  is a bounded non-decreasing left continuous function in  $(0, 1]$ , and right continuous at 0 .

$\bar{u}(r)$  is a bounded non-increasing left continuous function in  $(0, 1]$ , and right continuous at 0 .

$\underline{u}(r) \leq \bar{u}(r)$ ,  $0 \leq r \leq 1$ .

We denote the set of all real numbers by  $R$  and the set of all fuzzy numbers on  $E$  is indicated by  $E$  .

Definition 2.2 [5] Let  $x, y \in E$  . If there exists  $z \in E$  such that  $x + y = z$ , then  $z$  is called the H – difference of  $x$  and  $y$ , and it is denoted by  $x \ominus y$  . The sign "  $\ominus$  " always stands for H-difference and also note that  $x \ominus y \neq x + (-1)y$  .

Definition 2.3 [6] Let  $f(x)$  be continuous fuzzy-valued function, suppose that  $\int_0^\infty f(x) e^{-sx} dx$  is improper fuzzy Riemann-integrable

on  $[0, \infty)$  then  $\int_0^\infty f(x) e^{-sx} dx$  is called fuzzy Laplace transforms and is denoted as  $L[f(x)] = \int_0^\infty f(x) e^{-sx} dx$ , ( $s > 0$ ).

We have:

$$\int_0^\infty f(x) e^{-sx} dx = \left( \int_0^\infty \underline{f}(x; r) e^{-sx} dx, \int_0^\infty \bar{f}(x; r) e^{-sx} dx \right)$$

also by using the definition of classical Laplace transforms

$$\ell[\underline{f}(x; r)] = \int_0^\infty \underline{f}(x; r) e^{-sx} dx, \quad \ell[\bar{f}(x; r)] = \int_0^\infty \bar{f}(x; r) e^{-sx} dx$$

then, we follow:

$$L[f(x)] = (\ell[f(x; r)], \ell[\bar{f}(x; r)]).$$

Definition 2.4 [1] Let  $f \in C^F[a, b] \cap L^F[a, b]$ . The fuzzy Riemann–Liouville integral of fuzzy-valued function  $f$  is defined as following:

$$(I_{a+}^\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\beta}}, \quad x > a, \quad 0 < \beta \leq 1$$

Definition 2.5 [1] Let  $f \in C^F[a, b] \cap L^F[a, b]$  and  $x_0$  in  $(a, b)$  and  $\phi(x) = \frac{1}{\Gamma(1-\beta)} \int_a^x \frac{f(t)dt}{(x-t)^\beta}$ . We say that  $f(x)$  is Riemann–Liouville H-differentiable about order  $0 < \beta < 1$  at  $x_0$ , if there exists an element  $({}^{RL}D_{a+}^\beta f)(x_0) \in E$ , such that for all  $h > 0$ , sufficiently small, either:

$$({}^{RL}D_{a+}^\beta f)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi(x_0+h) \ominus \phi(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{\phi(x_0) \ominus \phi(x_0-h)}{h}$$

$$({}^{RL}D_{a+}^\beta f)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi(x_0) \ominus \phi(x_0+h)}{-h} = \lim_{h \rightarrow 0^+} \frac{\phi(x_0-h) \ominus \phi(x_0)}{-h}$$

For the sake of simplicity, we say that the fuzzy – valued function  $f$  is  ${}^{RL}\left[(i)-\beta\right]$ -differentiable if it is differentiable as in the definition 1.8 case (i), and is  ${}^{RL}\left[(ii)-\beta\right]$ -differentiable if it is differentiable as in the definition 1.8 case(ii).

### 3. Fuzzy Laplace Transforms of the Fuzzy Riemann-Liouville Fractional Derivatives of Order $3 < \beta < 4$ .

In this section, we define Riemann- Liouville fractional derivatives of order  $0 < \beta < 4$  for fuzzy –valued function  $f$  and also we find fuzzy Laplace transform for Riemann- Liouville fractional derivatives of order  $3 < \beta < 4$  under H-differentiability.

Definition 3.1. Let  $f(x) \in C^F[0, b] \cap L^F[0, b]$ ,  $\phi(x) = \frac{1}{\Gamma(\lceil \beta \rceil - \beta)} \int_0^x \frac{f(t)dt}{(x-t)^{1-\lceil \beta \rceil + \beta}}$  where  $\phi_1(x_0)$  and  $\phi_2(x_0)$  are the limits defined in a1 and a2 respectively and  $\phi_{1,1}(x_0), \phi_{1,2}(x_0), \phi_{2,1}(x_0), \phi_{2,2}(x_0)$  are the limits defined in b1, b2, b3 and b4 respectively and  $\phi_{1,1,1}(x_0), \phi_{1,1,2}(x_0), \phi_{1,2,1}(x_0), \phi_{1,2,2}(x_0), \phi_{2,1,1}(x_0), \phi_{2,1,2}(x_0), \phi_{2,2,1}(x_0)$  and  $\phi_{2,2,2}(x_0)$  are the limits defined in c1, c2, c3, c4, c5, c6, c7 and c8. respectively.  $f(x)$  is the Reimann-Liouville type fuzzy fractional differentiable function of order  $0 < \beta < 4$ ,  $\beta \neq 1, 2, 3$  at  $x_0 \in (0, b)$ , if there exists an element  $({}^{RL}D^\beta f)(x_0) \in C^F$  such that for all  $0 \leq r \leq 1$  and for  $h > 0$  sufficiently near zero either:

$$\text{a1. } ({}^{RL}D^\beta f)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi(x_0+h) \ominus \phi(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{\phi(x_0) \ominus \phi(x_0-h)}{h}$$

$$\text{a2. } ({}^{RL}D^\beta f)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi(x_0) \ominus \phi(x_0+h)}{-h} = \lim_{h \rightarrow 0^+} \frac{\phi(x_0-h) \ominus \phi(x_0)}{-h}$$

for  $0 < \beta < 1$  and either

$$\text{b1. } ({}^{RL}D^\beta f)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_1(x_0+h) \ominus \phi_1(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{\phi_1(x_0) \ominus \phi_1(x_0-h)}{h}$$

$$\text{b2. } ({}^{RL}D^\beta f)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_1(x_0) \ominus \phi_1(x_0+h)}{-h} = \lim_{h \rightarrow 0^+} \frac{\phi_1(x_0-h) \ominus \phi_1(x_0)}{-h}$$

$$\text{b3. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_2(x_0 + h) \ominus \phi_2(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{\phi_2(x_0) \ominus \phi_2(x_0 - h)}{h}$$

$$\text{b4. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_2(x_0) \ominus \phi_2(x_0 + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{\phi_2(x_0 - h) \ominus \phi_2(x_0)}{-h}$$

for  $1 < \beta < 2$  and either

$$\text{c1. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{1,1}(x_0 + h) \ominus \phi_{1,1}(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{\phi_{1,1}(x_0) \ominus \phi_{1,1}(x_0 - h)}{h}$$

$$\text{c2. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{1,1}(x_0) \ominus \phi_{1,1}(x_0 + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{\phi_{1,1}(x_0 - h) \ominus \phi_{1,1}(x_0)}{-h}$$

$$\text{c3. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{1,2}(x_0 + h) \ominus \phi_{1,2}(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{\phi_{1,2}(x_0) \ominus \phi_{1,2}(x_0 - h)}{h}$$

$$\text{c4. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{1,2}(x_0) \ominus \phi_{1,2}(x_0 + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{\phi_{1,2}(x_0 - h) \ominus \phi_{1,2}(x_0)}{-h}$$

$$\text{c5. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{2,1}(x_0 + h) \ominus \phi_{2,1}(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{\phi_{2,1}(x_0) \ominus \phi_{2,1}(x_0 - h)}{h}$$

$$\text{c6. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{2,1}(x_0) \ominus \phi_{2,1}(x_0 + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{\phi_{2,1}(x_0 - h) \ominus \phi_{2,1}(x_0)}{-h}$$

$$\text{c7. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{2,2}(x_0 + h) \ominus \phi_{2,2}(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{\phi_{2,2}(x_0) \ominus \phi_{2,2}(x_0 - h)}{h}$$

$$\text{c8. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{2,2}(x_0) \ominus \phi_{2,2}(x_0 + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{\phi_{2,2}(x_0 - h) \ominus \phi_{2,2}(x_0)}{-h}$$

for  $2 < \beta < 3$  and either

$$\text{d1. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{1,1,1}(x_0 + h) \ominus \phi_{1,1,1}(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{\phi_{1,1,1}(x_0) \ominus \phi_{1,1,1}(x_0 - h)}{h}$$

$$\text{d2. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{1,1,1}(x_0) \ominus \phi_{1,1,1}(x_0 + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{\phi_{1,1,1}(x_0 - h) \ominus \phi_{1,1,1}(x_0)}{-h}$$

$$\text{d3. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{1,1,2}(x_0 + h) \ominus \phi_{1,1,2}(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{\phi_{1,1,2}(x_0) \ominus \phi_{1,1,2}(x_0 - h)}{h}$$

$$\text{d4. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{1,1,2}(x_0) \ominus \phi_{1,1,2}(x_0 + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{\phi_{1,1,2}(x_0 - h) \ominus \phi_{1,1,2}(x_0)}{-h}$$

$$\text{d5. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{1,2,1}(x_0 + h) \ominus \phi_{1,2,1}(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{\phi_{1,2,1}(x_0) \ominus \phi_{1,2,1}(x_0 - h)}{h}$$

$$\text{d6. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{1,2,1}(x_0) \ominus \phi_{1,2,1}(x_0 + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{\phi_{1,2,1}(x_0 - h) \ominus \phi_{1,2,1}(x_0)}{-h}$$

$$\text{d7. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{1,2,2}(x_0 + h) \ominus \phi_{1,2,2}(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{\phi_{1,2,2}(x_0) \ominus \phi_{1,2,2}(x_0 - h)}{h}$$

$$\text{d8. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{1,2,2}(x_0) \ominus \phi_{1,2,2}(x_0 + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{\phi_{1,2,2}(x_0 - h) \ominus \phi_{1,2,2}(x_0)}{-h}$$

$$\text{d9. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{2,1,1}(x_0 + h) \ominus \phi_{2,1,1}(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{\phi_{2,1,1}(x_0) \ominus \phi_{2,1,1}(x_0 - h)}{h}$$

$$\text{d10. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{2,1,1}(x_0) \ominus \phi_{2,1,1}(x_0 + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{\phi_{2,1,1}(x_0 - h) \ominus \phi_{2,1,1}(x_0)}{-h}$$

$$\text{d11. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{2,1,2}(x_0 + h) \ominus \phi_{2,1,2}(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{\phi_{2,1,2}(x_0) \ominus \phi_{2,1,2}(x_0 - h)}{h}$$

$$\text{d12. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{2,1,2}(x_0) \ominus \phi_{2,1,2}(x_0 + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{\phi_{2,1,2}(x_0 - h) \ominus \phi_{2,1,2}(x_0)}{-h}$$

$$\text{d13. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{2,2,1}(x_0 + h) \ominus \phi_{2,2,1}(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{\phi_{2,2,1}(x_0) \ominus \phi_{2,2,1}(x_0 - h)}{h}$$

$$\text{d14. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{2,2,1}(x_0) \ominus \phi_{2,2,1}(x_0 + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{\phi_{2,2,1}(x_0 - h) \ominus \phi_{2,2,1}(x_0)}{-h}$$

$$\text{d15. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{2,2,2}(x_0 + h) \ominus \phi_{2,2,2}(x_0)}{h} = \lim_{h \rightarrow 0^+} \frac{\phi_{2,2,2}(x_0) \ominus \phi_{2,2,2}(x_0 - h)}{h}$$

$$\text{d16. } \left( {}^{RL}D^{\beta}f \right)(x_0) = \lim_{h \rightarrow 0^+} \frac{\phi_{2,2,2}(x_0) \ominus \phi_{2,2,2}(x_0 + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{\phi_{2,2,2}(x_0 - h) \ominus \phi_{2,2,2}(x_0)}{-h}$$

for  $3 < \beta < 4$ .

If the fuzzy valued function  $f(x)$  is differentiable as definition 3.1 cases (a1, b1, b3, c1, c3, c5, c7, d1, d3, d5, d7, d9, d11, d13,

d15) it is the Riemann-Liouville type differentiable in the first form and denoted by  $\left( {}^{RL}D_1^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{1,1}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{2,1}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{1,1,1}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{1,2,1}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{2,1,1}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{1,1,1,1}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{1,1,2,1}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{1,2,1,1}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{1,2,2,1}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{2,1,1,1}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{2,1,2,1}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{2,2,1,1}^{\beta}f \right)(x_0)$

and  $\left( {}^{RL}D_{2,2,2,1}^{\beta}f \right)(x_0)$  respectively. If the fuzzy valued function  $f(x)$  is differentiable as in definition 3.1 cases(a2, b2, b4, c2, c4, c6, c8, d2, d4, d6, d8, d10, d12, d14, d16) it is the Riemann-Liouville type differentiable in the second form and denoted by  $\left( {}^{RL}D_2^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{1,2}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{2,2}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{1,1,2}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{1,2,2}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{2,1,2}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{2,2,2}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{1,1,1,2}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{1,1,2,2}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{1,2,1,2}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{1,2,2,2}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{2,1,1,2}^{\beta}f \right)(x_0)$ ,  $\left( {}^{RL}D_{2,1,2,2}^{\beta}f \right)(x_0)$  and  $\left( {}^{RL}D_{2,2,2,2}^{\beta}f \right)(x_0)$  respectively.

Theorem 3.2. Let  $f(x) \in C^F[0, b] \cap L^F[0, b]$  be a fuzzy-valued function and  $f(x) = [f(x; r), \bar{f}(x; r)]$  for  $r \in [0, 1]$ ,  $0 < \beta < 4$  and  $x_0 \in (0, b)$ . Then

a1. If  $f(x)$  is  ${}^{RL}[i - \beta]$ -differentiable, then for  $0 < \beta < 1$

$$\left( {}^{RL}D_1^{\beta}f \right)(x_0) = [ \left( {}^{RL}D^{\beta}f_- \right)(x_0; r), \left( {}^{RL}D^{\beta}\bar{f}_- \right)(x_0; r) ]$$

a2. If  $f(x)$  is Riemann-Liouville type fuzzy fractional differentiable function in the second form, then for  $0 < \beta < 1$

$$\left( {}^{RL}D_2^{\beta}f \right)(x_0) = [ \left( {}^{RL}D^{\beta}\bar{f}_- \right)(x_0; r), \left( {}^{RL}D^{\beta}f_- \right)(x_0; r) ]$$

b1. If  $\left( {}^{RL}D_1^{\beta}f \right)(x)$  is Riemann-Liouville type fuzzy fractional differentiable function in the first form, then for  $1 < \beta < 2$

$$\left( {}^{RL}D_{1,1}^{\beta}f \right)(x_0) = [ \left( {}^{RL}D^{\beta}f_- \right)(x_0; r), \left( {}^{RL}D^{\beta}\bar{f}_- \right)(x_0; r) ]$$

b2. If  $({}^{RL}D_1^\beta f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable, then for  $1 < \beta < 2$

$$({}^{RL}D_{1,2}^\beta f)(x_0) = [({}^{RL}D^\beta \bar{f})(x_0; r), ({ }^{RL}D^\beta \underline{f})(x_0; r)]$$

b3. If  $({}^{RL}D_2^\beta f)(x)$  is  ${}^{RL}[i - \beta]$ -differentiable, then for  $1 < \beta < 2$

$$({}^{RL}D_{2,1}^\beta f)(x_0) = [({}^{RL}D^\beta \bar{f})(x_0; r), ({ }^{RL}D^\beta \underline{f})(x_0; r)]$$

b4. If  $({}^{RL}D_2^\beta f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable, then for  $1 < \beta < 2$

$$({}^{RL}D_{2,2}^\beta f)(x_0) = [({}^{RL}D^\beta \bar{f})(x_0; r), ({ }^{RL}D^\beta \bar{f})(x_0; r)]$$

c1. If  $({}^{RL}D_{1,1}^\beta f)(x)$  is  ${}^{RL}[i - \beta]$ -differentiable, then for  $2 < \beta < 3$

$$({}^{RL}D_{1,1,1}^\beta f)(x_0) = [({}^{RL}D^\beta \bar{f})(x_0; r), ({ }^{RL}D^\beta \bar{f})(x_0; r)]$$

c2. If  $({}^{RL}D_{1,1}^\beta f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable, then for  $2 < \beta < 3$

$$({}^{RL}D_{1,1,2}^\beta f)(x_0) = [({}^{RL}D^\beta \bar{f})(x_0; r), ({ }^{RL}D^\beta \underline{f})(x_0; r)]$$

c3. If  $({}^{RL}D_{1,2}^\beta f)(x)$  is  ${}^{RL}[i - \beta]$ -differentiable, then for  $2 < \beta < 3$

$$({}^{RL}D_{1,2,1}^\beta f)(x_0) = [({}^{RL}D^\beta \bar{f})(x_0; r), ({ }^{RL}D^\beta \underline{f})(x_0; r)]$$

c4. If  $({}^{RL}D_{1,2}^\beta f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable, then for  $2 < \beta < 3$

$$({}^{RL}D_{1,2,2}^\beta f)(x_0) = [({}^{RL}D^\beta \bar{f})(x_0; r), ({ }^{RL}D^\beta \bar{f})(x_0; r)]$$

c5. If  $({}^{RL}D_{2,1}^\beta f)(x)$  is  ${}^{RL}[i - \beta]$ -differentiable, then for  $2 < \beta < 3$

$$({}^{RL}D_{2,1,1}^\beta f)(x_0) = [({}^{RL}D^\beta \bar{f})(x_0; r), ({ }^{RL}D^\beta \underline{f})(x_0; r)]$$

c6. If  $({}^{RL}D_{2,1}^\beta f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable, then for  $2 < \beta < 3$

$$({}^{RL}D_{2,1,2}^\beta f)(x_0) = [({}^{RL}D^\beta \bar{f})(x_0; r), ({ }^{RL}D^\beta \bar{f})(x_0; r)]$$

c7. If  $({}^{RL}D_{2,2}^\beta f)(x)$  is  ${}^{RL}[i - \beta]$ -differentiable, then for  $2 < \beta < 3$

$$({}^{RL}D_{2,2,1}^\beta f)(x_0) = [({}^{RL}D^\beta \bar{f})(x_0; r), ({ }^{RL}D^\beta \bar{f})(x_0; r)]$$

c8. If  $({}^{RL}D_{2,2}^\beta f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable, then for  $2 < \beta < 3$

$$({}^{RL}D_{2,2,2}^\beta f)(x_0) = [({}^{RL}D^\beta \bar{f})(x_0; r), ({ }^{RL}D^\beta \underline{f})(x_0; r)]$$

d1. If  $({}^{RL}D_{1,1,1}^\beta f)(x)$  is  ${}^{RL}[i - \beta]$ -differentiable, then for  $3 < \beta < 4$

$$({}^{RL}D_{1,1,1,1}^\beta f)(x_0) = [({}^{RL}D^\beta \bar{f})(x_0; r), ({ }^{RL}D^\beta \bar{f})(x_0; r)]$$

d2. If  $({}^{RL}D_{1,1,1}^\beta f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable, then for  $3 < \beta < 4$

$$({}^{RL}D_{1,1,1,2}^\beta f)(x_0) = [({}^{RL}D^\beta \bar{f})(x_0; r), ({ }^{RL}D^\beta \underline{f})(x_0; r)]$$

d3. If  $({}^{RL}D_{1,1,2}^\beta f)(x)$  is  ${}^{RL}[i - \beta]$ -differentiable, then for  $3 < \beta < 4$

$$({}^{RL}D_{1,1,2,1}^\beta f)(x_0) = [({}^{RL}D^\beta \bar{f})(x_0; r), ({ }^{RL}D^\beta \underline{f})(x_0; r)]$$

d4. If  $({}^{RL}D_{1,1,2}^\beta f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable, then for  $3 < \beta < 4$

$$({}^{RL}D_{1,1,2,2}^\beta f)(x_0) = [({}^{RL}D^\beta \bar{f})(x_0; r), ({ }^{RL}D^\beta \bar{f})(x_0; r)]$$

d5. If  $({}^{RL}D_{1,2,1}^{\beta} f)(x)$  is  ${}^{RL}[i - \beta]$ -differentiable, then for  $3 < \beta < 4$

$$({}^{RL}D_{1,2,1,1}^{\beta} f)(x_0) = [({}^{RL}D^{\beta} f^-)(x_0; r), ({ }^{RL}D^{\beta} f_-)(x_0; r)]$$

d6. If  $({}^{RL}D_{1,2,1}^{\beta} f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable, then for  $3 < \beta < 4$

$$({}^{RL}D_{1,2,1,2}^{\beta} f)(x_0) = [({}^{RL}D^{\beta} f_-)(x_0; r), ({ }^{RL}D^{\beta} f^-)(x_0; r)]$$

d7. If  $({}^{RL}D_{1,2,2}^{\beta} f)(x)$  is  ${}^{RL}[i - \beta]$ -differentiable, then for  $3 < \beta < 4$

$$({}^{RL}D_{1,2,2,1}^{\beta} f)(x_0) = [({}^{RL}D^{\beta} f_-)(x_0; r), ({ }^{RL}D^{\beta} f^-)(x_0; r)]$$

d8. If  $({}^{RL}D_{1,2,2}^{\beta} f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable, then for  $3 < \beta < 4$

$$({}^{RL}D_{1,2,2,2}^{\beta} f)(x_0) = [({}^{RL}D^{\beta} f^-)(x_0; r), ({ }^{RL}D^{\beta} f_-)(x_0; r)]$$

d9. If  $({}^{RL}D_{2,1,1}^{\beta} f)(x)$  is  ${}^{RL}[i - \beta]$ -differentiable, then for  $3 < \beta < 4$

$$({}^{RL}D_{2,1,1,1}^{\beta} f)(x_0) = [({}^{RL}D^{\beta} f^-)(x_0; r), ({ }^{RL}D^{\beta} f_-)(x_0; r)]$$

d10. If  $({}^{RL}D_{2,1,1}^{\beta} f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable, then for  $3 < \beta < 4$

$$({}^{RL}D_{2,1,1,2}^{\beta} f)(x_0) = [({}^{RL}D^{\beta} f_-)(x_0; r), ({ }^{RL}D^{\beta} f^-)(x_0; r)]$$

d11. If  $({}^{RL}D_{2,1,2}^{\beta} f)(x)$  is  ${}^{RL}[i - \beta]$ -differentiable, then for  $3 < \beta < 4$

$$({}^{RL}D_{2,1,2,1}^{\beta} f)(x_0) = [({}^{RL}D^{\beta} f_-)(x_0; r), ({ }^{RL}D^{\beta} f^-)(x_0; r)]$$

d12. If  $({}^{RL}D_{2,1,2}^{\beta} f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable, then for  $3 < \beta < 4$

$$({}^{RL}D_{2,1,2,2}^{\beta} f)(x_0) = [({}^{RL}D^{\beta} f^-)(x_0; r), ({ }^{RL}D^{\beta} f_-)(x_0; r)]$$

d13. If  $({}^{RL}D_{2,2,1}^{\beta} f)(x)$  is  ${}^{RL}[i - \beta]$ -differentiable, then for  $3 < \beta < 4$

$$({}^{RL}D_{2,2,1,1}^{\beta} f)(x_0) = [({}^{RL}D^{\beta} f_-)(x_0; r), ({ }^{RL}D^{\beta} f^-)(x_0; r)]$$

d14. If  $({}^{RL}D_{2,2,1}^{\beta} f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable, then for  $3 < \beta < 4$

$$({}^{RL}D_{2,2,1,2}^{\beta} f)(x_0) = [({}^{RL}D^{\beta} f^-)(x_0; r), ({ }^{RL}D^{\beta} f_-)(x_0; r)]$$

d15. If  $({}^{RL}D_{2,2,2}^{\beta} f)(x)$  is  ${}^{RL}[i - \beta]$ -differentiable, then for  $3 < \beta < 4$

$$({}^{RL}D_{2,2,2,1}^{\beta} f)(x_0) = [({}^{RL}D^{\beta} f^-)(x_0; r), ({ }^{RL}D^{\beta} f_-)(x_0; r)]$$

d16. If  $({}^{RL}D_{2,2,2}^{\beta} f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable, then for  $3 < \beta < 4$

$$({}^{RL}D_{2,2,2,2}^{\beta} f)(x_0) = [({}^{RL}D^{\beta} f_-)(x_0; r), ({ }^{RL}D^{\beta} f^-)(x_0; r)]$$

where

$$({}^{RL}D^{\beta} f_-)(x_0; r) = \left[ \frac{1}{\Gamma(\lceil \beta \rceil - \beta)} \left( \frac{d}{dx} \right)^{\lceil \beta \rceil} \int_0^x \frac{f(t; r) dt}{(x-t)^{1-\lceil \beta \rceil+\beta}} \right]_{x=x_0},$$

$$({}^{RL}D^{\beta} f^-)(x_0; r) = \left[ \frac{1}{\Gamma(\lceil \beta \rceil - \beta)} \left( \frac{d}{dx} \right)^{\lceil \beta \rceil} \int_0^x \frac{f^-(t; r) dt}{(x-t)^{1-\lceil \beta \rceil+\beta}} \right]_{x=x_0}$$

proof We shall prove d11 as follows: Since  $({}^{RL}D_{2,1,2}^{\beta} f)(x)$ ,  $3 < \beta < 4$  is the Riemann-Liouville type fuzzy fractional differentiable function in the first form then from d11, of definition 3.1, we have:

$$\phi_{2,1,2}(x_0 + h) \ominus \phi_{2,1,2}(x_0) = [\underline{\phi}_{2,1,2}(x_0 + h; r) - \underline{\phi}_{2,1,2}(x_0; r), \bar{\phi}_{2,1,2}(x_0 + h; r) - \bar{\phi}_{2,1,2}(x_0; r)],$$

$$\phi_{2,1,2}(x_0) \ominus \phi_{2,1,2}(x_0 - h) = [\underline{\phi}_{2,1,2}(x_0; r) - \underline{\phi}_{2,1,2}(x_0 - h; r), \bar{\phi}_{2,1,2}(x_0; r) - \bar{\phi}_{2,1,2}(x_0 - h; r)].$$

$$\frac{1}{h}, h > 0$$

Multiplying both sides by  $\frac{1}{h}$ , we obtain:

$$\frac{\phi_{2,1,2}(x_0 + h) \ominus \phi_{2,1,2}(x_0)}{h} = \left[ \frac{\underline{\phi}_{2,1,2}(x_0 + h; r) - \underline{\phi}_{2,1,2}(x_0; r)}{h}, \frac{\bar{\phi}_{2,1,2}(x_0 + h; r) - \bar{\phi}_{2,1,2}(x_0; r)}{h} \right],$$

$$\frac{\phi_{2,1,2}(x_0) \ominus \phi_{2,1,2}(x_0 - h)}{h} = \left[ \frac{\underline{\phi}_{2,1,2}(x_0; r) - \underline{\phi}_{2,1,2}(x_0 - h; r)}{h}, \frac{\bar{\phi}_{2,1,2}(x_0; r) - \bar{\phi}_{2,1,2}(x_0 - h; r)}{h} \right].$$

By taking  $h \rightarrow 0^+$  on both sides of the above relation, we get:

$$({}^{RL}D^\beta f)(x_0) = \left[ \frac{d}{dx} \underline{\phi}_{2,1,2}(x_0; r), \frac{d}{dx} \bar{\phi}_{2,1,2}(x_0; r) \right]. \quad (3.1)$$

Now, since  $\phi_2(x_0)$  is equal to the limits defined in a2. of definition 3.1 then we have:

$$\phi(x_0) \ominus \phi(x_0 + h) = [\underline{\phi}(x_0; r) - \underline{\phi}(x_0 + h; r), \bar{\phi}(x_0; r) - \bar{\phi}(x_0 + h; r)],$$

$$\phi(x_0 - h) \ominus \phi(x_0) = [\underline{\phi}(x_0 - h; r) - \underline{\phi}(x_0; r), \bar{\phi}(x_0 - h; r) - \bar{\phi}(x_0; r)].$$

$$\frac{-1}{h}, h > 0$$

Multiplying both sides by  $\frac{-1}{h}$ , we obtain:

$$\frac{\phi(x_0) \ominus \phi(x_0 + h)}{-h} = \left[ \frac{\bar{\phi}(x_0 + h; r) - \bar{\phi}(x_0; r)}{h}, \frac{\underline{\phi}(x_0 + h; r) - \underline{\phi}(x_0; r)}{h} \right]$$

$$\frac{\phi(x_0 - h) \ominus \phi(x_0)}{-h} = \left[ \frac{\bar{\phi}(x_0; r) - \bar{\phi}(x_0 - h; r)}{h}, \frac{\underline{\phi}(x_0; r) - \underline{\phi}(x_0 - h; r)}{h} \right]$$

By taking  $h \rightarrow 0^+$  on both sides of the above relation , we get:

$$\phi_2(x_0) = [\bar{\phi}'(x_0; r), \underline{\phi}'(x_0; r)]$$

Then

$$\underline{\phi}_2(x_0; r) = \bar{\phi}'(x_0; r), \bar{\phi}_2(x_0; r) = \underline{\phi}'(x_0; r) \quad (3.2)$$

Now, since  $\phi_{2,1}(x_0)$  is equal to the limits defined in b3 of definition 3.1 then we have:

$$\phi_2(x_0 + h) \ominus \phi_2(x_0) = [\underline{\phi}_2(x_0 + h; r) - \underline{\phi}_2(x_0; r), \bar{\phi}_2(x_0 + h; r) - \bar{\phi}_2(x_0; r)],$$

$$\phi_2(x_0) \ominus \phi_2(x_0 - h) = [\underline{\phi}_2(x_0; r) - \underline{\phi}_2(x_0 - h; r), \bar{\phi}_2(x_0; r) - \bar{\phi}_2(x_0 - h; r)].$$

$$\frac{1}{h}, h > 0$$

Multiplying both sides by  $\frac{1}{h}$ , and using relation (3.2), we obtain

$$\frac{\phi_2(x_0 + h) \ominus \phi_2(x_0)}{h} = \left[ \frac{\bar{\phi}'(x_0 + h; r) - \bar{\phi}'(x_0; r)}{h}, \frac{\underline{\phi}'(x_0 + h; r) - \underline{\phi}'(x_0; r)}{h} \right]$$

$$\frac{\phi_2(x_0) \ominus \phi_2(x_0 - h)}{h} = \left[ \frac{\bar{\phi}'(x_0; r) - \bar{\phi}'(x_0 - h; r)}{h}, \frac{\underline{\phi}'(x_0; r) - \underline{\phi}'(x_0 - h; r)}{h} \right]$$

By taking  $h \rightarrow 0^+$  on both sides of the above relation, we get:

$$\phi_{2,1}(x_0) = [\bar{\phi}''(x_0; r), \underline{\phi}''(x_0; r)]$$

Then

$$\underline{\phi}_{2,1}(x_0; r) = \bar{\phi}''(x_0; r), \quad \bar{\phi}_{2,1}(x_0; r) = \underline{\phi}''(x_0; r) \quad (3.3)$$

Now, since  $\underline{\phi}_{2,1,2}(x_0)$  is equal to the limits defined in c6. of definition 3.1 then we have:

$$\underline{\phi}_{2,1}(x_0) \ominus \underline{\phi}_{2,1}(x_0 + h) = [\underline{\phi}_{2,1}(x_0; r) - \underline{\phi}_{2,1}(x_0 + h; r), \bar{\phi}_{2,1}(x_0; r) - \bar{\phi}_{2,1}(x_0 + h; r)],$$

$$\underline{\phi}_{2,1}(x_0 - h) \ominus \underline{\phi}_{2,1}(x_0) = [\underline{\phi}_{2,1}(x_0 - h; r) - \underline{\phi}_{2,1}(x_0; r), \bar{\phi}_{2,1}(x_0 - h; r) - \bar{\phi}_{2,1}(x_0; r)].$$

$$-\frac{1}{h}, \quad h > 0$$

Multiplying both sides by  $\frac{1}{h}$ , and using relation (3.3), we obtain

$$\frac{\underline{\phi}_{2,1}(x_0) \ominus \underline{\phi}_{2,1}(x_0 + h)}{-h} = \left[ \frac{\underline{\phi}''(x_0 + h; r) - \underline{\phi}''(x_0; r)}{h}, \frac{\bar{\phi}''(x_0 + h; r) - \bar{\phi}''(x_0; r)}{h} \right],$$

$$\frac{\underline{\phi}_{2,1}(x_0 - h) \ominus \underline{\phi}_{2,1}(x_0)}{-h} = \left[ \frac{\underline{\phi}''(x_0; r) - \underline{\phi}''(x_0 - h; r)}{h}, \frac{\bar{\phi}''(x_0; r) - \bar{\phi}''(x_0 - h; r)}{h} \right].$$

By taking  $h \rightarrow 0^+$  on both sides of the above relation, we get

$$\underline{\phi}_{2,1,2}(x_0) = [\underline{\phi}'''(x_0; r), \bar{\phi}'''(x_0; r)],$$

Then

$$\underline{\phi}_{2,1,2}(x_0; r) = \underline{\phi}'''(x_0; r), \quad \bar{\phi}_{2,1,2}(x_0; r) = \bar{\phi}'''(x_0; r) \quad (3.4)$$

Substituting (3.4) in (3.1) yields

$$\begin{aligned} (\text{RLD}^\beta f)(x_0) &= \left[ \frac{d^4}{dx^4} \underline{\phi}(x_0; r), \frac{d^4}{dx^4} \bar{\phi}(x_0; r) \right] \\ &= [(\text{RLD}^\beta f_-)(x_0; r), (\text{RLD}^\beta f_+)(x_0; r)]. \end{aligned}$$

Theorem 3.3. Suppose that  $f(x) \in C^F[0, \infty) \cap L^F[0, \infty)$  and  $3 < \beta < 4$  then :

1. If  $(\text{RLD}_{1,1,1}^\beta f)(x)$  is  ${}^{RL}[i - \beta]$ - differentiable fuzzy-valued function, then

$$\begin{aligned} L[(\text{RLD}_{1,1,1}^\beta f)(x)] &= s^\beta L[f(x)] \ominus s^3 (\text{RLD}^{\beta-4} f)(0) \ominus (\text{RLD}^{\beta-1} f)(0) \\ &\quad \ominus s(\text{RLD}^{\beta-2} f)(0) \ominus s^2 (\text{RLD}^{\beta-3} f)(0) \end{aligned}$$

2. If  $(\text{RLD}_{1,1,2}^\beta f)(x)$  is  ${}^{RL}[ii - \beta]$ - differentiable fuzzy-valued function, then

$$\begin{aligned} L[(\text{RLD}_{1,1,2}^\beta f)(x)] &= -s^3 (\text{RLD}^{\beta-4} f)(0) \ominus (-s^\beta) L[f(x)] - (\text{RLD}^{\beta-1} f)(0) \\ &\quad - s(\text{RLD}^{\beta-2} f)(0) - s^2 (\text{RLD}^{\beta-3} f)(0) \end{aligned}$$

3. If  $(\text{RLD}_{1,1,2}^\beta f)(x)$  is  ${}^{RL}[i - \beta]$ - differentiable fuzzy-valued function, then

$$\begin{aligned} L[(\text{RLD}_{1,1,2,1}^\beta f)(x)] &= -s^3 (\text{RLD}^{\beta-4} f)(0) \ominus (-s^\beta) L[f(x)] \ominus (\text{RLD}^{\beta-1} f)(0) \\ &\quad - s(\text{RLD}^{\beta-2} f)(0) - s^2 (\text{RLD}^{\beta-3} f)(0) \end{aligned}$$

4. If  $(\text{RLD}_{1,1,2,2}^\beta f)(x)$  is  ${}^{RL}[ii - \beta]$ - differentiable fuzzy-valued function, then

$$\begin{aligned} L[(\text{RLD}_{1,1,2,2}^\beta f)(x)] &= s^\beta L[f(x)] \ominus s^3 (\text{RLD}^{\beta-4} f)(0) - (\text{RLD}^{\beta-1} f)(0) \\ &\quad \ominus s(\text{RLD}^{\beta-2} f)(0) \ominus s^2 (\text{RLD}^{\beta-3} f)(0) \end{aligned}$$

5.If  $({}^{RL}D_{1,2,1}^\beta f)(x)$  is  ${}^{RL}[i - \beta]$ - differentiable fuzzy- valued function, then

$$L\left[({}^{RL}D_{1,2,1,1}^\beta f)(x)\right] = -s^3({}^{RL}D^{\beta-4}f)(0) \ominus (-s^\beta)L[f(x)] \ominus ({}^{RL}D^{\beta-1}f)(0) \\ \ominus s({}^{RL}D^{\beta-2}f)(0) - s^2({}^{RL}D^{\beta-3}f)(0)$$

6.If  $({}^{RL}D_{1,2,1}^\beta f)(x)$  is  ${}^{RL}[ii - \beta]$ - differentiable fuzzy- valued function, then

$$L\left[({}^{RL}D_{1,2,1,2}^\beta f)(x)\right] = s^\beta L[f(x)] \ominus s^3({}^{RL}D^{\beta-4}f)(0) - ({}^{RL}D^{\beta-1}f)(0) \\ - s({}^{RL}D^{\beta-2}f)(0) \ominus s^2({}^{RL}D^{\beta-3}f)(0)$$

7. If  $({}^{RL}D_{1,2,2}^\beta f)(x)$  is  ${}^{RL}[i - \beta]$ - differentiable fuzzy – valued function, then

$$L\left[({}^{RL}D_{1,2,2,1}^\beta f)(x)\right] = s^\beta L[f(x)] \ominus s^3({}^{RL}D^{\beta-4}f)(0) \ominus ({}^{RL}D^{\beta-1}f)(0) \\ - s({}^{RL}D^{\beta-2}f)(0) \ominus s^2({}^{RL}D^{\beta-3}f)(0)$$

8.If  $({}^{RL}D_{1,2,2}^\beta f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable fuzzy- valued function, then

$$L\left[({}^{RL}D_{1,2,2,2}^\beta f)(x)\right] = -s^3({}^{RL}D^{\beta-4}f)(0) \ominus (-s^\beta)L[f(x)] - ({}^{RL}D^{\beta-1}f)(0) \\ \ominus s({}^{RL}D^{\beta-2}f)(0) - s^2({}^{RL}D^{\beta-3}f)(0)$$

9.If  $({}^{RL}D_{2,1,1}^\beta f)(x)$  is  ${}^{RL}[i - \beta]$  - differentiable fuzzy- valued function, then

$$L\left[({}^{RL}D_{2,1,1,1}^\beta f)(x)\right] = -s^3({}^{RL}D^{\beta-4}f)(0) \ominus (-s^\beta)L[f(x)] \ominus ({}^{RL}D^{\beta-1}f)(0) \\ \ominus s({}^{RL}D^{\beta-2}f)(0) \ominus s^2({}^{RL}D^{\beta-3}f)(0)$$

10.If  $({}^{RL}D_{2,1,1}^\beta f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable fuzzy- valued function, then

$$L\left[({}^{RL}D_{2,1,1,2}^\beta f)(x)\right] = s^\beta L[f(x)] \ominus s^3({}^{RL}D^{\beta-4}f)(0) - ({}^{RL}D^{\beta-1}f)(0) \\ - s({}^{RL}D^{\beta-2}f)(0) - s^2({}^{RL}D^{\beta-3}f)(0)$$

11.If  $({}^{RL}D_{2,1,2}^\beta f)(x)$  is  ${}^{RL}[i - \beta]$ -differentiable fuzzy- valued function, then

$$L\left[({}^{RL}D_{2,1,2,1}^\beta f)(x)\right] = s^\beta L[f(x)] \ominus s^3({}^{RL}D^{\beta-4}f)(0) \ominus ({}^{RL}D^{\beta-1}f)(0) \\ - s({}^{RL}D^{\beta-2}f)(0) - s^2({}^{RL}D^{\beta-3}f)(0)$$

12.If  $({}^{RL}D_{2,1,2}^\beta f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable fuzzy- valued function, then

$$L\left[({}^{RL}D_{2,1,2,2}^\beta f)(x)\right] = -s^3({}^{RL}D^{\beta-4}f)(0) \ominus (-s^\beta)L[f(x)] - ({}^{RL}D^{\beta-1}f)(0) \\ \ominus s({}^{RL}D^{\beta-2}f)(0) \ominus s^2({}^{RL}D^{\beta-3}f)(0)$$

13.If  $({}^{RL}D_{2,2,1}^\beta f)(x)$  is  ${}^{RL}[i - \beta]$ -differentiable fuzzy- valued function, then

$$L\left[\left({}^{RL}D_{2,2,1,1}^{\beta} f\right)(x)\right] = s^{\beta} L\left[f(x)\right] \ominus s^3 \left({}^{RL}D^{\beta-4} f\right)(0) \ominus \left({}^{RL}D^{\beta-1} f\right)(0) \\ \ominus s \left({}^{RL}D^{\beta-2} f\right)(0) - s^2 \left({}^{RL}D^{\beta-3} f\right)(0)$$

14. If  $\left({}^{RL}D_{2,2,1}^{\beta} f\right)(x)$  is  ${}^{RL}\left[i - \beta\right]$ -differentiable fuzzy-valued function, then

$$L\left[\left({}^{RL}D_{2,2,1,2}^{\beta} f\right)(x)\right] = -s^3 \left({}^{RL}D^{\beta-4} f\right)(0) \ominus (-s^{\beta}) L\left[f(x)\right] - \left({}^{RL}D^{\beta-1} f\right)(0) \\ - s \left({}^{RL}D^{\beta-2} f\right)(0) \ominus s^2 \left({}^{RL}D^{\beta-3} f\right)(0)$$

15. If  $\left({}^{RL}D_{2,2,2}^{\beta} f\right)(x)$  is  ${}^{RL}\left[i - \beta\right]$ -differentiable fuzzy-valued function, then

$$L\left[\left({}^{RL}D_{2,2,2,1}^{\beta} f\right)(x)\right] = -s^3 \left({}^{RL}D^{\beta-4} f\right)(0) \ominus (-s^{\beta}) L\left[f(x)\right] \ominus \left({}^{RL}D^{\beta-1} f\right)(0) \\ - s \left({}^{RL}D^{\beta-2} f\right)(0) \ominus s^2 \left({}^{RL}D^{\beta-3} f\right)(0)$$

16. If  $\left({}^{RL}D_{2,2,2,2}^{\beta} f\right)(x)$  is  ${}^{RL}\left[i - \beta\right]$ -differentiable fuzzy-valued function, then

$$L\left[\left({}^{RL}D_{2,2,2,2}^{\beta} f\right)(x)\right] = s^{\beta} L\left[f(x)\right] \ominus s^3 \left({}^{RL}D^{\beta-4} f\right)(0) - \left({}^{RL}D^{\beta-1} f\right)(0) \\ \ominus s \left({}^{RL}D^{\beta-2} f\right)(0) - s^2 \left({}^{RL}D^{\beta-3} f\right)(0)$$

Proof We prove 11 as follows: Since  $\left({}^{RL}D_{2,1,2}^{\beta} f\right)(x)$  is  ${}^{RL}\left[(i) - \beta\right]$ -differentiable fuzzy-valued function, then by theorem 3.2 we get :

$$\left({}^{RL}D_{2,1,2}^{\beta} f\right)(x) = \left[\left({}^{RL}D^{\beta} f\right)(x; r), \left({}^{RL}D^{\beta} \bar{f}\right)(x; r)\right]$$

Therefore, we get:

$$\left(\underline{{}^{RL}D^{\beta} f}\right)(x; r) = \left({}^{RL}D^{\beta} f\right)(x; r), \quad \left(\overline{{}^{RL}D^{\beta} f}\right)(x; r) = \left({}^{RL}D^{\beta} \bar{f}\right)(x; r) \quad (3.5)$$

Then from (3.5) we get:

$$L\left[\left({}^{RL}D_{2,1,2,1}^{\beta} f\right)(x)\right] = L\left[\left(\underline{{}^{RL}D^{\beta} f}\right)(x; r), \left(\overline{{}^{RL}D^{\beta} f}\right)(x; r)\right] \\ = \left[\ell\left[\left({}^{RL}D^{\beta} f\right)(x; r)\right], \ell\left[\left({}^{RL}D^{\beta} \bar{f}\right)(x; r)\right]\right] \quad (3.6)$$

By Laplace transform of ordinary Riemann-Liouville fractional derivative, equation (3.6) becomes:

$$L\left[\left({}^{RL}D_{2,1,2,1}^{\beta} f\right)(x)\right] = \left[s^{\beta} \ell\left[f(x; r)\right] - \left({}^{RL}D^{\beta-1} f\right)(0; r) - s \left({}^{RL}D^{\beta-2} f\right)(0; r) - s^2 \left({}^{RL}D^{\beta-3} f\right)(0; r)\right. \\ \left. - s^3 \left({}^{RL}D^{\beta-4} f\right)(0; r), s^{\beta} \ell\left[\bar{f}(x; r)\right] - \left({}^{RL}D^{\beta-1} \bar{f}\right)(0; r) - s \left({}^{RL}D^{\beta-2} \bar{f}\right)(0; r)\right. \\ \left. - s^2 \left({}^{RL}D^{\beta-3} \bar{f}\right)(0; r) - s^3 \left({}^{RL}D^{\beta-4} \bar{f}\right)(0; r)\right] \quad (3.7)$$

Since  $\left({}^{RL}D_{2,1,2}^{\beta} f\right)(x)$  is  ${}^{RL}\left[(i) - \beta\right]$ -differentiable fuzzy-valued function and

$2 < \beta - 1 < 3$ ,  $1 < \beta - 2 < 2$  and  $0 < \beta - 3 < 1$ , then by theorem 3.2, we get :

$$\left(\underline{{}^{RL}D^{\beta-1} f}\right)(0; r) = \left({}^{RL}D^{\beta-1} f\right)(0; r), \quad \left(\overline{{}^{RL}D^{\beta-1} f}\right)(0; r) = \left(\overline{{}^{RL}D^{\beta-1} f}\right)(0; r),$$

$$\left(\underline{{}^{RL}D^{\beta-2} f}\right)(0; r) = \left(\overline{{}^{RL}D^{\beta-2} f}\right)(0; r), \quad \left(\overline{{}^{RL}D^{\beta-2} f}\right)(0; r) = \left(\underline{{}^{RL}D^{\beta-2} f}\right)(0; r),$$

$$\left(\underline{{}^{RL}D^{\beta-3} f}\right)(0; r) = \left(\overline{{}^{RL}D^{\beta-3} f}\right)(0; r), \quad \left(\overline{{}^{RL}D^{\beta-3} f}\right)(0; r) = \left(\underline{{}^{RL}D^{\beta-3} f}\right)(0; r).$$

Then, equation (3.7) becomes:

$$\begin{aligned}
 L\left[\left({}^{RL}D_{2,1,2,1}^{\beta} f\right)(x)\right] &= \left[s^{\beta} \ell\left[f(x;r)\right] - \left({}^{RL}D^{\beta-1}f\right)(0;r) - s\left(\overline{{}^{RL}D^{\beta-2}f}\right)(0;r) - s^2\left(\overline{{}^{RL}D^{\beta-3}f}\right)(0;r)\right. \\
 &\quad - s^3\left(\overline{{}^{RL}D^{\beta-4}f}\right)(0;r), s^{\beta} \ell\left[\bar{f}(x;r)\right] - \left(\overline{{}^{RL}D^{\beta-1}f}\right)(0;r) - s\left(\overline{{}^{RL}D^{\beta-2}f}\right)(0;r) \\
 &\quad \left. - s^2\left(\overline{{}^{RL}D^{\beta-3}f}\right)(0;r) - s^3\left(\overline{{}^{RL}D^{\beta-4}f}\right)(0;r)\right] \\
 &= s^{\beta} L\left[f(x)\right] \ominus s^3\left({}^{RL}D^{\beta-4}f\right)(0) \ominus \left({}^{RL}D^{\beta-1}f\right)(0) - s\left({}^{RL}D^{\beta-2}f\right)(0) \\
 &\quad - s^2\left({}^{RL}D^{\beta-3}f\right)(0)
 \end{aligned}$$

#### 4. Application

Example 4.1. Consider the following FFIVP:

$$\begin{aligned}
 \left({}^{RL}D^{\beta} y\right)(x) &= \sigma \quad ; \quad \sigma = (r-1, 1-r), \quad 3 < \beta < 4, \\
 \left({}^{RL}D^{\beta-1} y\right)(0) &= \left({}^{RL}D^{\beta-2} y\right)(0) = \left({}^{RL}D^{\beta-3} y\right)(0) = \left({}^{RL}D^{\beta-4} y\right)(0) = {}^{RL}y_0^{(\beta-1)} \in E.
 \end{aligned} \tag{4.1}$$

We note that

$$\begin{aligned}
 \left(\overline{{}^{RL}D^{\beta-1}y}\right)(0;r) &= \left(\overline{{}^{RL}D^{\beta-2}y}\right)(0;r) = \left(\overline{{}^{RL}D^{\beta-3}y}\right)(0;r) = \left(\overline{{}^{RL}D^{\beta-4}y}\right)(0;r) = {}^{RL}\underline{y}_0^{(\beta-1)}(r), \\
 \left(\overline{{}^{RL}D^{\beta-1}y}\right)(0;r) &= \left(\overline{{}^{RL}D^{\beta-2}y}\right)(0;r) = \left(\overline{{}^{RL}D^{\beta-3}y}\right)(0;r) = \left(\overline{{}^{RL}D^{\beta-4}y}\right)(0;r) = {}^{RL}\bar{y}_0^{(\beta-1)}(r).
 \end{aligned}$$

By taking fuzzy Laplace transform for both sides of equation (2.46) we get

$$L\left[\left({}^{RL}D^{\beta} y\right)(x)\right] = L[\sigma] \tag{4.2}$$

Now we have  $2^4 = 16$  cases as follows:

Case 1 If  $\left({}^{RL}D_{1,1,1}^{\beta} f\right)(x)$  is  ${}^{RL}[i-\beta]$ -differentiable, then equation (4.2) becomes:

$$s^{\beta} L\left[y(x)\right] \ominus s^3\left({}^{RL}D^{\beta-4}y\right)(0) \ominus \left({}^{RL}D^{\beta-1}y\right)(0) \ominus s\left({}^{RL}D^{\beta-2}y\right)(0) \ominus s^2\left({}^{RL}D^{\beta-3}y\right)(0) = L[\sigma]$$

Therefore we have :

$$\begin{aligned}
 s^{\beta} \ell\left[\underline{y}(x;r)\right] &= \frac{r-1}{s} + {}^{RL}\underline{y}_0^{(\beta-1)}(r)(1+s+s^2+s^3), \\
 s^{\beta} \ell\left[\bar{y}(x;r)\right] &= \frac{1-r}{s} + {}^{RL}\bar{y}_0^{(\beta-1)}(r)(1+s+s^2+s^3).
 \end{aligned}$$

Finally, we determine the solution of FFIVP (4.1) as follows:

$$\begin{aligned}
 \underline{y}(x;r) &= (r-1) \frac{x^{\beta}}{\Gamma(\beta+1)} + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) \\
 \bar{y}(x;r) &= (1-r) \frac{x^{\beta}}{\Gamma(\beta+1)} + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right)
 \end{aligned}$$

Case 2 If  $\left({}^{RL}D_{1,1,1}^{\beta} f\right)(x)$  is  ${}^{RL}[ii-\beta]$ -differentiable, then equation (4.2) becomes:

$$-s^3\left({}^{RL}D^{\beta-4}y\right)(0) \ominus (-s^{\beta}) L\left[y(x)\right] - \left({}^{RL}D^{\beta-1}y\right)(0) - s\left({}^{RL}D^{\beta-2}y\right)(0) - s^2\left({}^{RL}D^{\beta-3}y\right)(0) = L[\sigma] \quad \text{Therefore we have :}$$

$$s^{\beta} \ell\left[\underline{y}(x;r)\right] = \frac{1-r}{s} + {}^{RL}\underline{y}_0^{(\beta-1)}(r)(1+s+s^2+s^3)$$

$$s^{\beta} \ell\left[\bar{y}(x;r)\right] = \frac{r-1}{s} + {}^{RL}\bar{y}_0^{(\beta-1)}(r)(1+s+s^2+s^3)$$

Finally, we determine the solution of FFIVP (4.1) as follows:

$$\underline{y}(x;r) = (1-r) \frac{x^{\beta}}{\Gamma(\beta+1)} + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right)$$

$$\bar{y}(x; r) = (r-1) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right)$$

Case 3 If  $({}^{RL}D_{1,1,2}^\beta f)(x)$  is  ${}^{RL}[i-\beta]$ -differentiable, then equation (4.2) becomes:

$$-s^3 ({}^{RL}D^{\beta-4}y)(0) \ominus (-s^\beta) L[y(x)] \ominus ({}^{RL}D^{\beta-1}y)(0) - s ({}^{RL}D^{\beta-2}y)(0) - s^2 ({}^{RL}D^{\beta-3}y)(0) = L[\sigma]$$

Therefore we have :

$$s^\beta \ell[y(x; r)] = \frac{1-r}{s} + {}^{RL}\underline{y}_0^{(\beta-1)}(r) (s+s^2+s^3) + {}^{RL}\bar{y}_0^{(\beta-1)}(r)$$

$$s^\beta \ell[\bar{y}(x; r)] = \frac{r-1}{s} + {}^{RL}\bar{y}_0^{(\beta-1)}(r) (s+s^2+s^3) + {}^{RL}\underline{y}_0^{(\beta-1)}(r)$$

Finally, we determine the solution of FFIVP (4.1) as follows :

$$\underline{y}(x; r) = (1-r) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \frac{x^{\beta-1}}{\Gamma(\beta)}$$

$$\bar{y}(x; r) = (r-1) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \frac{x^{\beta-1}}{\Gamma(\beta)}$$

Case 4 If  $({}^{RL}D_{1,1,2}^\beta f)(x)$  is  ${}^{RL}[ii-\beta]$ -differentiable, then equation (4.2) becomes:

$$s^\beta L[y(x)] \ominus s^3 ({}^{RL}D^{\beta-4}y)(0) - ({}^{RL}D^{\beta-1}y)(0) \ominus s ({}^{RL}D^{\beta-2}y)(0) \ominus s^2 ({}^{RL}D^{\beta-3}y)(0) = L[\sigma]$$

Therefore we have :

$$s^\beta \ell[y(x; r)] = \frac{r-1}{s} + {}^{RL}\underline{y}_0^{(\beta-1)}(r) (s+s^2+s^3) + {}^{RL}\bar{y}_0^{(\beta-1)}(r)$$

$$s^\beta \ell[\bar{y}(x; r)] = \frac{1-r}{s} + {}^{RL}\bar{y}_0^{(\beta-1)}(r) (s+s^2+s^3) + {}^{RL}\underline{y}_0^{(\beta-1)}(r)$$

Finally, we determine the solution of FFIVP (4.1) as follows :

$$\underline{y}(x; r) = (r-1) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \frac{x^{\beta-1}}{\Gamma(\beta)}$$

$$\bar{y}(x; r) = (1-r) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \frac{x^{\beta-1}}{\Gamma(\beta)}$$

Case 5 If  $({}^{RL}D_{1,2,1}^\beta f)(x)$  is  ${}^{RL}[i-\beta]$ -differentiable, then equation (4.2) becomes:

$$-s^3 ({}^{RL}D^{\beta-4}y)(0) \ominus (-s^\beta) L[y(x)] \ominus ({}^{RL}D^{\beta-1}y)(0) \ominus s ({}^{RL}D^{\beta-2}y)(0) - s^2 ({}^{RL}D^{\beta-3}y)(0) = L[\sigma]. \text{ Therefore we have :}$$

$$s^\beta \ell[y(x; r)] = \frac{1-r}{s} + {}^{RL}\underline{y}_0^{(\beta-1)}(r) (s^2+s^3) + {}^{RL}\bar{y}_0^{(\beta-1)}(r) (1+s),$$

$$s^\beta \ell[\bar{y}(x; r)] = \frac{r-1}{s} + {}^{RL}\bar{y}_0^{(\beta-1)}(r) (s^2+s^3) + {}^{RL}\underline{y}_0^{(\beta-1)}(r) (1+s)$$

Finally, we determine the solution of FFIVP (2.46) as follows :

$$\underline{y}(x; r) = (1-r) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-3}}{\Gamma(\beta-2)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-2}}{\Gamma(\beta-1)} \right)$$

$$\bar{y}(x; r) = (r-1) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-3}}{\Gamma(\beta-2)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-2}}{\Gamma(\beta-1)} \right)$$

Case 6 If  $({}^{RL}D_{1,2,1}^\beta f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable, then equation (4.2) becomes:

$$s^\beta L[y(x)] \ominus s^3 ({}^{RL}D^{\beta-4}y)(0) \ominus ({}^{RL}D^{\beta-1}y)(0) \ominus s ({}^{RL}D^{\beta-2}y)(0) \ominus s^2 ({}^{RL}D^{\beta-3}y)(0) = L[\sigma]$$

Therefore we have :

$$s^\beta \ell[y(x; r)] = \frac{r-1}{s} + {}^{RL}\underline{y}_0^{(\beta-1)}(r)(s^2 + s^3) + {}^{RL}\bar{y}_0^{(\beta-1)}(r)(1+s),$$

$$s^\beta \ell[\bar{y}(x; r)] = \frac{1-r}{s} + {}^{RL}\bar{y}_0^{(\beta-1)}(r)(s^2 + s^3) + {}^{RL}\underline{y}_0^{(\beta-1)}(r)(1+s).$$

Finally, we determine the solution of FFIIVP (2.46) as follows :

$$\underline{y}(x; r) = (r-1) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-3}}{\Gamma(\beta-2)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-2}}{\Gamma(\beta-1)} \right)$$

$$\bar{y}(x; r) = (1-r) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-3}}{\Gamma(\beta-2)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-2}}{\Gamma(\beta-1)} \right)$$

Case 7 If  $({}^{RL}D_{1,2,2}^\beta f)(x)$  is  ${}^{RL}[i - \beta]$ -differentiable, then equation (4.2) becomes:

$$s^\beta L[y(x)] \ominus s^3 ({}^{RL}D^{\beta-4}y)(0) \ominus ({}^{RL}D^{\beta-1}y)(0) \ominus s ({}^{RL}D^{\beta-2}y)(0) \ominus s^2 ({}^{RL}D^{\beta-3}y)(0) = L[\sigma]$$

Therefore we have :

$$s^\beta \ell[y(x; r)] = \frac{r-1}{s} + {}^{RL}\underline{y}_0^{(\beta-1)}(r)(1+s^2 + s^3) + {}^{RL}\bar{y}_0^{(\beta-1)}(r)s,$$

$$s^\beta \ell[\bar{y}(x; r)] = \frac{1-r}{s} + {}^{RL}\bar{y}_0^{(\beta-1)}(r)(1+s^2 + s^3) + {}^{RL}\underline{y}_0^{(\beta-1)}(r)s.$$

Finally, we determine the solution of FFIIVP (2.46) as follows :

$$\underline{y}(x; r) = (r-1) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \frac{x^{\beta-2}}{\Gamma(\beta-1)}$$

$$\bar{y}(x; r) = (1-r) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \frac{x^{\beta-2}}{\Gamma(\beta-1)}$$

Case 8 If  $({}^{RL}D_{1,2,2}^\beta f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable, then equation (4.2) becomes:

$$-s^3 ({}^{RL}D^{\beta-4}y)(0) \ominus (-s^\beta)L[y(x)] \ominus ({}^{RL}D^{\beta-1}y)(0) \ominus s ({}^{RL}D^{\beta-2}y)(0) \ominus s^2 ({}^{RL}D^{\beta-3}y)(0) = L[\sigma].$$

Therefore we have :

$$s^\beta \ell[y(x; r)] = \frac{1-r}{s} + {}^{RL}\underline{y}_0^{(\beta-1)}(r)(1+s^2 + s^3) + {}^{RL}\bar{y}_0^{(\beta-1)}(r)s,$$

$$s^\beta \ell[\bar{y}(x; r)] = \frac{r-1}{s} + {}^{RL}\bar{y}_0^{(\beta-1)}(r)(1+s^2 + s^3) + {}^{RL}\underline{y}_0^{(\beta-1)}(r)s$$

Finally, we determine the solution of FFIIVP (2.46) as follows:

$$\underline{y}(x; r) = (1-r) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \frac{x^{\beta-2}}{\Gamma(\beta-1)},$$

$$\bar{y}(x; r) = (r-1) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \frac{x^{\beta-2}}{\Gamma(\beta-1)}.$$

Case 9 If  $({}^{RL}D_{2,1,1}^\beta f)(x)$  is  ${}^{RL}[i - \beta]$ -differentiable ,then equation (4.2) becomes:

$$-s^3 ({}^{RL}D^{\beta-4}y)(0) \ominus (-s^\beta)L[y(x)] \ominus ({}^{RL}D^{\beta-1}y)(0) \ominus s ({}^{RL}D^{\beta-2}y)(0) \ominus s^2 ({}^{RL}D^{\beta-3}y)(0) = L[\sigma].$$

Therefore we have :

$$s^\beta \ell \left[ \underline{y}(x; r) \right] = \frac{1-r}{s} + {}^{RL} \underline{y}_0^{(\beta-1)}(r) s^3 + {}^{RL} \bar{y}_0^{(\beta-1)}(r) (1+s+s^2),$$

$$s^\beta \ell \left[ \bar{y}(x; r) \right] = \frac{r-1}{s} + {}^{RL} \bar{y}_0^{(\beta-1)}(r) s^3 + {}^{RL} \underline{y}_0^{(\beta-1)}(r) (1+s+s^2).$$

Finally, we determine the solution of FFIIVP (2.46) as follows:

$$\underline{y}(x; r) = (1-r) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL} \bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} \right) + {}^{RL} \underline{y}_0^{(\beta-1)}(r) \frac{x^{\beta-4}}{\Gamma(\beta-3)},$$

$$\bar{y}(x; r) = (r-1) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL} \underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} \right) + {}^{RL} \bar{y}_0^{(\beta-1)}(r) \frac{x^{\beta-4}}{\Gamma(\beta-3)}.$$

Case 10 If  $({}^{RL} D_{2,1,1}^\beta f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable, then equation (4.2) becomes:

$$s^\beta L \left[ y(x) \right] \ominus s^3 ({}^{RL} D^{\beta-4} y)(0) - ({}^{RL} D^{\beta-1} y)(0) - s ({}^{RL} D^{\beta-2} y)(0) - s^2 ({}^{RL} D^{\beta-3} y)(0) = L[\sigma].$$

Therefore we have :

$$s^\beta \ell \left[ \underline{y}(x; r) \right] = \frac{r-1}{s} + {}^{RL} \underline{y}_0^{(\beta-1)}(r) s^3 + {}^{RL} \bar{y}_0^{(\beta-1)}(r) (1+s+s^2),$$

$$s^\beta \ell \left[ \bar{y}(x; r) \right] = \frac{1-r}{s} + {}^{RL} \bar{y}_0^{(\beta-1)}(r) s^3 + {}^{RL} \underline{y}_0^{(\beta-1)}(r) (1+s+s^2).$$

Finally, we determine the solution of FFIIVP (2.46) as follows:

$$\underline{y}(x; r) = (r-1) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL} \bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} \right) + {}^{RL} \underline{y}_0^{(\beta-1)}(r) \frac{x^{\beta-4}}{\Gamma(\beta-3)},$$

$$\bar{y}(x; r) = (1-r) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL} \underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} \right) + {}^{RL} \bar{y}_0^{(\beta-1)}(r) \frac{x^{\beta-4}}{\Gamma(\beta-3)}.$$

Case 11 If  $({}^{RL} D_{2,1,2}^\beta f)(x)$  is  ${}^{RL}[i - \beta]$ -differentiable, then equation (4.2) becomes:

$$s^\beta L \left[ y(x) \right] \ominus s^3 ({}^{RL} D^{\beta-4} y)(0) \ominus ({}^{RL} D^{\beta-1} y)(0) - s ({}^{RL} D^{\beta-2} y)(0) - s^2 ({}^{RL} D^{\beta-3} y)(0) = L[\sigma].$$

Therefore we have :

$$s^\beta \ell \left[ \underline{y}(x; r) \right] = \frac{r-1}{s} + {}^{RL} \underline{y}_0^{(\beta-1)}(r) (1+s^3) + {}^{RL} \bar{y}_0^{(\beta-1)}(r) (s+s^2)$$

$$s^\beta \ell \left[ \bar{y}(x; r) \right] = \frac{1-r}{s} + {}^{RL} \bar{y}_0^{(\beta-1)}(r) (1+s^3) + {}^{RL} \underline{y}_0^{(\beta-1)}(r) (s+s^2)$$

Finally, we determine the solution of FFIIVP (2.46) as follows:

$$\underline{y}(x; r) = (r-1) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL} \underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL} \bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} \right)$$

$$\bar{y}(x; r) = (1-r) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL} \bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL} \underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} \right)$$

Case 12 If  $({}^{RL} D_{2,1,2}^\beta f)(x)$  is  ${}^{RL}[ii - \beta]$ -differentiable, then equation (4.2) becomes:

$$-s^3 ({}^{RL} D^{\beta-4} y)(0) \ominus (-s^\beta) L[y(x)] - ({}^{RL} D^{\beta-1} y)(0) \ominus s ({}^{RL} D^{\beta-2} y)(0) \ominus s^2 ({}^{RL} D^{\beta-3} y)(0) = L[\sigma] \quad \text{Therefore we have :}$$

$$s^\beta \ell[\underline{y}(x; r)] = \frac{1-r}{s} + {}^{RL}\underline{y}_0^{(\beta-1)}(r)(1+s^3) + {}^{RL}\bar{y}_0^{(\beta-1)}(r)(s+s^2)$$

$$s^\beta \ell[\bar{y}(x; r)] = \frac{r-1}{s} + {}^{RL}\bar{y}_0^{(\beta-1)}(r)(1+s^3) + {}^{RL}\underline{y}_0^{(\beta-1)}(r)(s+s^2)$$

Finally, we determine the solution of FFIIVP (2.46) as follows:

$$\underline{y}(x; r) = (1-r) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} \right),$$

$$\bar{y}(x; r) = (r-1) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} \right)$$

Case 13 If  $({}^{RL}D_{2,2,1}^\beta f)(x)$  is  ${}^{RL}[i-\beta]$ -differentiable, then equation (4.2) becomes:

$$s^\beta L[\underline{y}(x)] \ominus s^3({}^{RL}D^{\beta-4}y)(0) \ominus ({}^{RL}D^{\beta-1}y)(0) \ominus s({}^{RL}D^{\beta-2}y)(0) - s^2({}^{RL}D^{\beta-3}y)(0) = L[\sigma].$$

Therefore we have :

$$s^\beta \ell[\underline{y}(x; r)] = \frac{r-1}{s} + {}^{RL}\underline{y}_0^{(\beta-1)}(r)(1+s+s^3) + {}^{RL}\bar{y}_0^{(\beta-1)}(r)s^2$$

$$s^\beta \ell[\bar{y}(x; r)] = \frac{1-r}{s} + {}^{RL}\bar{y}_0^{(\beta-1)}(r)(1+s+s^3) + {}^{RL}\underline{y}_0^{(\beta-1)}(r)s^2$$

Finally, we determine the solution of FFIIVP (2.46) as follows :

$$\underline{y}(x; r) = (r-1) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \frac{x^{\beta-3}}{\Gamma(\beta-2)}$$

$$\bar{y}(x; r) = (1-r) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \frac{x^{\beta-3}}{\Gamma(\beta-2)}$$

Case 14 If  $({}^{RL}D_{2,2,1}^\beta f)(x)$  is  ${}^{RL}[(ii)-\beta]$ -differentiable, then equation (4.2) becomes:

$$-s^3({}^{RL}D^{\beta-4}y)(0) \ominus (-s^\beta)L[\underline{y}(x)] - ({}^{RL}D^{\beta-1}y)(0) - s({}^{RL}D^{\beta-2}y)(0) \ominus s^2({}^{RL}D^{\beta-3}y)(0) = L[\sigma].$$

Therefore we have :

$$s^\beta \ell[\underline{y}(x; r)] = \frac{1-r}{s} + {}^{RL}\underline{y}_0^{(\beta-1)}(r)(1+s+s^3) + {}^{RL}\bar{y}_0^{(\beta-1)}(r)s^2$$

$$s^\beta \ell[\bar{y}(x; r)] = \frac{r-1}{s} + {}^{RL}\bar{y}_0^{(\beta-1)}(r)(1+s+s^3) + {}^{RL}\underline{y}_0^{(\beta-1)}(r)s^2$$

Finally, we determine the solution of FFIIVP (2.46) as follows:

$$\underline{y}(x; r) = (1-r) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \frac{x^{\beta-3}}{\Gamma(\beta-2)}$$

$$\bar{y}(x; r) = (r-1) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \frac{x^{\beta-3}}{\Gamma(\beta-2)}$$

Case 15 If  $({}^{RL}D_{2,2,2}^\beta f)(x)$  is  ${}^{RL}[i-\beta]$ -differentiable, then equation (4.2) becomes:

$$-s^3({}^{RL}D^{\beta-4}y)(0) \ominus (-s^\beta)L[\underline{y}(x)] \ominus ({}^{RL}D^{\beta-1}y)(0) - s({}^{RL}D^{\beta-2}y)(0) \ominus s^2({}^{RL}D^{\beta-3}y)(0) = L[\sigma] \quad \text{Therefore we have :}$$

$$s^\beta \ell[\underline{y}(x; r)] = \frac{1-r}{s} + {}^{RL}\underline{y}_0^{(\beta-1)}(r)(s+s^3) + {}^{RL}\bar{y}_0^{(\beta-1)}(r)(1+s^2),$$

$$s^\beta \ell[\bar{y}(x; r)] = \frac{r-1}{s} + {}^{RL}\bar{y}_0^{(\beta-1)}(r)(s+s^3) + {}^{RL}\underline{y}_0^{(\beta-1)}(r)(1+s^2).$$

Finally, we determine the solution of FFIIVP (2.46) as follows:

$$\underline{y}(x; r) = (1-r) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} \right)$$

$$\bar{y}(x; r) = (r-1) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} \right)$$

Case 16 If  $({}^{RL}D_{2,2,2}^\beta f)(x)$  is  $^{RL}[ii-\beta]$ -differentiable, then equation (4.2) becomes:

$$s^\beta L[\underline{y}(x)] \ominus s^3({}^{RL}D^{\beta-4}y)(0) - ({}^{RL}D^{\beta-1}y)(0) \ominus s({}^{RL}D^{\beta-2}y)(0) - s^2({}^{RL}D^{\beta-3}y)(0) = L[\sigma].$$

Therefore we have :

$$s^\beta \ell[\underline{y}(x; r)] = \frac{r-1}{s} + {}^{RL}\underline{y}_0^{(\beta-1)}(r)(s+s^3) + {}^{RL}\bar{y}_0^{(\beta-1)}(r)(1+s^2)$$

$$s^\beta \ell[\bar{y}(x; r)] = \frac{1-r}{s} + {}^{RL}\bar{y}_0^{(\beta-1)}(r)(s+s^3) + {}^{RL}\underline{y}_0^{(\beta-1)}(r)(1+s^2)$$

Finally , we determine the solution of FFIIVP (2.46) as follows :

$$\underline{y}(x; r) = (r-1) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} \right)$$

$$\bar{y}(x; r) = (1-r) \frac{x^\beta}{\Gamma(\beta+1)} + {}^{RL}\bar{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-2}}{\Gamma(\beta-1)} + \frac{x^{\beta-4}}{\Gamma(\beta-3)} \right) + {}^{RL}\underline{y}_0^{(\beta-1)}(r) \left( \frac{x^{\beta-1}}{\Gamma(\beta)} + \frac{x^{\beta-3}}{\Gamma(\beta-2)} \right)$$

## 5. Conclusions

In this paper, definition of fuzzy Riemann-Liouville fractional derivatives about the order  $0 < \beta < 4$  for fuzzy-valued function  $f$  is introduced and, fuzzy Laplace transforms for fuzzy Riemann-Liouville fractional derivatives of the order  $3 < \beta < 4$  are found under H-differentiability. FFIIVP of the order  $3 < \beta < 4$  is solved .

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## تحويلات لابلاس الضبابية لمشتقات ريمان- ليوفيل الكسورية الضبابية

من الرتبة  $4 < \beta < 3$

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### المستخلص

الهدف الرئيسي من هذا البحث هو إيجاد صيغ المشتقات الكسورية الضبابية من النوع ريمان- ليوفيل من الرتبة  $4 < \beta < 0$  لدالة القيمة الضبابية  $f$ , وكذلك إيجاد صيغ تحويلات لابلاس الضبابية للمشتقات الكسورية الضبابية من النوع ريمان- ليوفيل من الرتبة  $4 < \beta < 3$  من باستخدام فرق  $H$ .