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Equivalence of Sine and Cosine Rules with Wilson's Angle in Normed Space

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Abstract: *In the Euclid space cosine rules have been proven to be equivalent to the sine rule. This paper will discuss the expansion of the properties of triangles in normed space. It will be examined about the sine rules, cosine rules and the triangle side rules in normed spaces. Furthermore this paper will prove that the rules of the sine and the rules of cosine in normed space are also equivalent.*

Keywords : *cosine rules, sine rules, normed space, Wilson angle.*

1. Introduction

Norm space is a vector space in which a normed function is defined.

Definition 1.

Let V be a vector space over a field $\mathbb{R}$, if defined association $\|\cdot\| := V \to \mathbb{R}$, who fulfills ;

a. $\|x\| \geq 0$ for each $x \in V$.
b. If $x \in V$ and $\|x\| = 0$ if and only if $x = 0$
c. $\|\alpha x\| = |\alpha|\|x\|$ for each $x \in V$ and $\alpha \in \mathbb{R}$.
d. $\|x + y\| \leq \|x\| + \|y\|$ for each $x, y \in V$.

Pair of vector spaces with norm functions is called normed space , symbolized by $(V, \|\,.\,\|)$.[4,5,6,7]

Let $(V, \|\,.\,\|)$ be a vector space over a field $\mathbb{R}$, for each $x, y \in V$ defined as a nonlinear function on V :

$$2\langle x, y\rangle_w := \|x\|^2 + \|y\|^2 - \|x - y\|^2$$

Of the norms possessed :

$$|\|x\| - \|y\||^2 \leq \|x - y\|^2$$
$$\Leftrightarrow \|x\|^2 - 2\|x\|\|y\| + \|y\|^2 \leq \|x - y\|^2$$
$$\Leftrightarrow \langle x, y\rangle_w \leq \|x\|.\|y\| \qquad 1.1$$
$$\|x - y\|^2 \leq (\|x\| + \|y\|)^2$$
$$\Leftrightarrow \|x - y\|^2 - \|x\|^2 - \|y\|^2 \leq 2\,\|x\|\|y\|$$
$$\Leftrightarrow -\langle x, y\rangle_w \leq \|x\|\|y\| \qquad 1.2$$

From the equation 1.1 and 1.2 this means $|\langle x, y\rangle_w| \leq \|x\|\|y\|$, for each $x, y \in V$.[1,2,3]

Theorem 1. Let $(V, \|\,.\,\|)$ be a normed space over a field $\mathbb{R}$, for each $x, y \in V$ defined as a nonlinear function on V [1,2,3] :

$$2\langle x, y\rangle_w := \|x\|^2 + \|y\|^2 - \|x - y\|^2$$

then the following statement is equivalent :

1. $|\langle x, y\rangle_w| \leq \|x\|\|y\|$
2. $\|x + y\| \leq \|x\| + \|y\|$
3. $|\|x\| - \|y\|| \leq \|x - y\|$

Proof :

$(1 \Rightarrow 2)$ $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2|\langle x, y\rangle_w|$
$$\leq \|x\|^2 + \|x\|^2 + 2\|x\|\|y\|$$
$$\leq (\|x\| + \|y\|)^2$$
$$\Leftrightarrow \|x + y\| \leq \|x\| + \|y\|$$

$(2 \Rightarrow 3)$ Note that $\|x\| = \|(x - y) + y\|$
$$\leq \|x - y\| + \|y\|$$
$$(\|x\| - \|y\|) \leq \|x - y\| \qquad 1.3$$
$$\|y\| = \|(y - x) + x\|$$
$$\leq \|x - y\| + \|x\|$$
$$-(\|x\| - \|y\|) \leq \|x - y\| \qquad 1.4$$

From the equation (1.3) dan (1.4) can be concluded that

$$|\|x\| - \|y\|| \leq \|x - y\|$$

$(3 \Rightarrow 1)$ Note that $|\|x\| - \|y\||^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|$
$$\leq \|x - y\|^2$$
$$\Leftrightarrow \|x\|^2 + \|y\|^2 - \|x - y\|^2 \leq 2\|x\|\|y\|$$
$$\Leftrightarrow 2\langle x, y\rangle_w \leq 2\|x\|\|y\|$$
$$\langle x, y\rangle_w \leq \|x\|\|y\| \qquad 1.5$$

Meanwhile $\|x - y\|^2 \leq (\|x\| + \|y\|)^2$
$$= \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|$$
$$\Leftrightarrow -\|x\|^2 - \|y\|^2 + \|x - y\|^2 \leq 2\|x\|\|y\|$$
$$\Leftrightarrow -2\langle x, y\rangle_w \leq 2\|x\|\|y\|$$
$$-\langle x, y\rangle_w \leq \|x\|\|y\| \qquad 1.6$$

From the equation (1.5) and (1.6) can be concluded that

$$|\langle x, y\rangle_w| \leq \|x\|\|y\| \qquad 1.7$$

From the equation (1.7) The defined angle in a normed space called Wilson's angle :

Foe each $x, y \in V\backslash\{0\}$, defined Wilson's angle [3,4,8]:

$$\angle_w(x, y) = \arccos\left(\frac{\|x\| + \|y\| - \|x - y\|}{2\|x\|\|y\|}\right)$$

2. RESULT

Let $(V, \|\,.\,\|)$ be normed space and for each $a, b, c \in V \setminus \{0\}$, defined $\Delta[a, b, c]$ as $\{a, b, c\}$ who fulfills $a + c = b$ which is equipped with an angle $\angle_w(a, b)$, $\angle_w(b, c)$, dan $\angle_w(c, -a)$. From this understanding, cosine rules are obtained for $\Delta[a, b, c]$

$$\|a\|^2 = \|b\|^2 + \|c\|^2 - 2\|b\|\|c\| \cos\angle_w(b, c)$$
$$\|b\|^2 = \|a\|^2 + \|c\|^2 - 2\|a\|\|c\| \cos\angle_w(-a, c)$$

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$$\|c\|^2 = \|a\|^2 + \|b\|^2 - 2\|b\|\|c\| \cos \angle_w(a,b)$$

With $K = 2\sqrt{s(s-\|a\|)(s-\|b\|)(s-\|c\|)}$ and $2s = \|a\| + \|b\| + \|c\|$ then $\|b\|\|c\| \sin \angle_w(b,c) = \|a\|\|c\| \sin \angle_w(-a,c) = \|a\|\|b\| \sin \angle_w(a,bc) = K$, called the sine rule of $\Delta[a,b,c]$.

$Theorem.2.$ Let $(V, \|.\|)$ be normed space and for each $a,b,c \in V \setminus \{0\}$, defined $\Delta[a,b,c]$ as $\{a,b,c\}$ who fulfills $a + c = b$ which is equipped with an angle $\angle_w(a,b)$, $\angle_w(b,c)$, dan $\angle_w(c,-a)$ then $\angle_w(b,c) + \angle_w(c,-a) + \angle_w(a,b) = \pi$

Proof .

$$\angle_w(b,c) + \angle_w(a,b) = \pi - \angle_w(c,-a)$$

$$\Leftrightarrow \cos\big(\angle_w(a,b) + \angle_w(b,c)\big) = \cos(\pi - \angle_w(c,-a))$$

$$\Leftrightarrow \cos\big(\angle_w(a,b) + \angle_w(b,c)\big) = -\cos \angle_w(c,-a)$$

$$\Leftrightarrow \cos \angle_w(a,b) \cos \angle_w(b,c) - \sin \angle_w(a,b) \sin \angle_w(b,c) = -\cos \angle_w(c,-a)$$

$$\Leftrightarrow \cos \angle_w(a,b) \cos \angle_w(b,c) - \sin \angle_w(a,b) \sin \angle_w(b,c) + \cos \angle_w(c,-a) = 0$$

$$= \frac{\|a\|^2 + \|b\|^2 - \|c\|^2}{2\|a\|\|b\|} \cdot \frac{\|b\|^2 + \|c\|^2 - \|a\|^2}{2\|b\|\|c\|}$$

$$+ \frac{\|a\|^2 + \|c\|^2 - \|b\|^2}{2\|a\|\|c\|} - \frac{K}{\|a\|\|b\|} \cdot \frac{K}{\|b\|\|c\|}$$

$$\frac{\|a\|^2 + \|c\|^2 - \|b\|^2}{2\|a\|\|c\|} - \frac{K^2}{\|a\|\|b\|^2\|c\|}$$

$$= \frac{(\|a\|^2 + \|b\|^2 - \|c\|^2)(\|b\|^2 + \|c\|^2 - \|a\|^2)}{4\|a\|\|b\|^2\|c\|} + \frac{\|a\|^2 + \|c\|^2 - \|b\|^2}{2\|a\|\|c\|} - \frac{K^2}{\|a\|\|b\|^2\|c\|}$$

$$= \frac{(-\|a\|^4 + \|b\|^4 - \|c\|^4)}{4\|a\|\|b\|^2\|c\|} + \frac{\|a\|^2 + \|c\|^2 - \|b\|^2}{2\|a\|\|c\|} - \frac{K^2}{\|a\|\|b\|^2\|c\|}$$

$$= \frac{(-\|a\|^4 - \|b\|^4 - \|c\|^4 + 2\|a\|^2\|b\|^2 + 2\|b\|^2\|c\|^2) - 4K^2}{4\|a\|\|b\|^2\|c\|}$$

$$= \frac{4\mathrm{K}^2 - 4\mathrm{K}^2}{4\|a\|\|b\|^2\|c\|} = 0$$

■

$Theorem.3.$ Let $(V, \|.\|)$ be normed space and for each $a,b,c \in V \setminus \{0\}$, defined $\Delta[a,b,c]$ as $\{a,b,c\}$ who fulfills $a + c = b$ which is equipped with an angle $\angle_w(a,b)$, $\angle_w(b,c)$, and $\angle_w(c,-a)$ then the cosine rule if and only if the rule is sine.

Proof.

$(\Longrightarrow)$ known cosine rules, will be proven to result in a sine rule :

Note that if :

$$\cos \angle(a,b) = \frac{\|a\|^2 + \|b\|^2 - \|c\|^2}{2\|a\|\|b\|}$$

than :

$$\sin^2 \angle(a,b) = 1 - \left(\frac{\|a\|^2 + \|b\|^2 - \|c\|^2}{2\|a\|\|b\|}\right)^2$$

$$= \frac{(2\|a\|\|b\|)^2}{4\|a\|^2\|b\|^2} - \left(\frac{\|a\|^2 + \|b\|^2 - \|c\|^2}{2\|a\|\|b\|}\right)^2$$

$$= \frac{(2\|a\|\|b\|)^2 - (\|a\|^2 + \|b\|^2 - \|c\|^2)^2}{4\|a\|^2\|b\|^2}$$

$$= \frac{(2\|a\|\|b\|) - (\|a\|^2 + \|b\|^2 - \|c\|^2)}{2\|a\|\|b\|} \cdot$$

$$\frac{\big((2\|a\|\|b\|) + (\|a\|^2 + \|b\|^2 - \|c\|^2)\big)}{2\|a\|\|b\|}$$

$$= \frac{(\|c\|^2) - (\|a\| - \|b\|)^2 \,\big((\|a\| + \|b\|)^2 - (\|c\|^2)\big)}{4\|a\|^2\|b\|^2}$$

$$= \frac{\big((\|c\|) - (\|a\| - \|b\|)\big(\|c\| + (\|a\| - \|b\|)\big)}{2\|a\|\|b\|}$$

$$\frac{(\|a\| + \|b\| - \|c\|)(\|a\| + \|b\| + \|c\|)}{2\|a\|\|b\|}$$

$$= \frac{(\|a\| + \|b\| + \|c\|)(\|c\| + \|b\| - \|a\|)}{2\|a\|\|b\|}$$

$$= \frac{(\|a\| + \|c\| - \|b\|)(\|a\| + \|b\| - \|c\|)}{2\|a\|\|b\|}$$

By specifying $2s = \|a\| + \|b\| + \|c\|$ then obtained :

$$\sin^2 \angle(a,b) = \frac{(2s)2(s - \|a\|)2(s - \|b\|)2(s - \|c\|)}{4\|a\|^2\|b\|^2}$$

$$= \frac{16s(s - \|a\|)(s - \|b\|)(s - \|c\|)}{4\|a\|^2\|b\|^2}$$

$$\|a\|\|b\| \sin \angle(a,b) = 2\sqrt{s(s - \|a\|)(s - \|b\|)(s - \|c\|)}$$

In the same way the other sine rules are obtained.

$(\Longleftarrow)$ Known sine rules will be proven to result in a cosine rule.

Note that :

$$K^2(\|a\|^2 + \|b\|^2 - \|c\|^2)$$

$$= \|a\|^2\|b\|^2\|c\|^2(\sin^2 \angle(b,c) + \sin^2 \angle(c,-a) - \sin^2 \angle(a,b)).$$

$$= \|a\|^2\|b\|^2\|c\|^2(\sin^2 \angle(b,c) + \sin^2 \angle(c,-a) - \sin^2(\angle(b,c) + \angle(c,-a))).$$

$$= \|a\|^2\|b\|^2\|c\|^2(\sin^2 \angle(b,c) + \sin^2 \angle(c,-a)$$

$$-(\sin \angle(b,c) \cos \angle(c,-a) + \cos \angle(b,c) \sin \angle(c,-a))^2)$$

$$= \|a\|^2\|b\|^2\|c\|^2(\sin^2 \angle(b,c) + \sin^2 \angle(c,-a)$$

$$
\begin{aligned}
&-\sin^2\angle(b,c)\cos^2\angle(c,-a)-\cos^2\angle(\mathrm{b,c})\sin^2\angle(\mathrm{c,-a}) \\
&\quad -2\sin\angle(\mathrm{b,c})\cos\angle(\mathrm{c,-a})\cos\angle(\mathrm{b,c})\sin\angle(\mathrm{c,-a})) \\
&= \|a\|^2\|b\|^2\|c\|^2(\sin^2\angle_w(b,c)\,(1-\cos^2\angle_w(c,-a)) \\
&\quad +\sin^2\angle_w(c,-a)(1-\cos^2\angle_w(b,c)) \\
&\quad -2\sin\angle_w(b,c)\cos\angle_w(c,-a)\cos\angle_w(b,c)\sin\angle_w(c,-a)) \\
&= \|a\|^2\|b\|^2\|c\|^2(\sin^2\angle_w(b,c)\,(\sin^2\angle_w(c,-a)) \\
&\quad +\sin^2\angle_w(c,-a)(\sin^2\angle_w(b,c)) \\
&\quad -2\sin\angle_w(b,c)\cos\angle_w(c,-a)\cos\angle_w(b,c)\sin\angle_w(c,-a) \\
&= 2\|a\|^2\|b\|^2\|c\|^2(\sin^2\angle_w(b,c)\,(\sin^2\angle_w(c,-a)) \\
&\quad -\sin\angle(b,c)\cos\angle_w(c,-a)\cos\angle_w(b,c)\sin\angle_w(c,-a) \\
&= 2\|a\|^2\|b\|^2\|c\|^2(\sin\angle_w(b,c)\sin\angle_w(c,-a) \\
&\quad (\sin\angle_w(b,c)\sin\angle_w(c,-a) \\
&\quad -\cos\angle_w(b,c)\cos\angle_w(c,-a))) \\
&= 2\|a\|^2\|b\|^2\|c\|^2(\sin\angle_w(b,c)\sin\angle_w(c,-a)\cos\angle_w(a,b)) \\
&= 2\,\|\mathrm{a}\|K\|b\|K(\cos\angle_w(a,b)) \\
&= 2\,K^2\|\mathrm{a}\|\|b\|(\cos\angle(a,b))
\end{aligned}
$$

Then obtained :

$\|a\|^2+\|b\|^2-\|c\|^2 = 2\|a\|\|b\|\cos\angle(a,b)$ or

$\|c\|^2 = \|a\|^2+\|b\|^2-2\|a\|\|b\|\cos\angle(a,b)$

In the same way another cosine rule can be obtained .

■

Theorem. 4. Let $(V,\|\,.\,\|)$ be normed space and for each $a,b,c\in V\setminus\{0\}$, defined $\Delta[a,b,c]$ as $\{a,b,c\}$ who fulfills $a+c=b$ which is equipped with an angle $\angle_w(a,b)$, $\angle_w(b,c)$, dan $\angle_w(c,-a)$ then the cosine rule if and only if the side length rule.

Proof.

($\Longrightarrow$)The known cosine rule will be proven to result in the rule of the side length.

From the rules of cosine obtained:

$$\|a\|\cos\angle_w(a,b) = \|a\|\frac{\|a\|^2+\|b\|^2-\|c\|^2}{2\|a\|\|b\|}$$

$$\|c\|\cos\angle_w(b,c) = \|c\|\frac{\|b\|^2+\|c\|^2-\|a\|^2}{2\|b\|\|c\|}$$

$$\|a\|\cos\angle_w(a,b)+\|c\|\cos\angle(b,c) = \frac{\|a\|^2+\|b\|^2-\|c\|^2}{2\|b\|}+\frac{\|b\|^2+\|c\|^2-\|a\|^2}{2\|b\|} = \|b\|$$

In the same way the long rule of the other side is obtained :

($\Longleftarrow$) Knowing the rules of the side length will be proven to result in a cosine rule.

From the rules of cosine obtained :

$$\|a\|\|\mathrm{b}\|\cos\angle_w(a,b)+\|a\|\,\|c\|\cos\angle_w(c,-a) = \|a\|^2 \quad 2.1$$

$$\|a\|\|\mathrm{b}\|\cos\angle_w(a,b)+\|b\|\,\|c\|\cos\angle_w(b,c) = \|b\|^2 \quad 2.2$$

$$\|b\|\|\mathrm{c}\|\cos\angle_w(b,c)+\|a\|\|\mathrm{c}\|\cos\angle_w(c,-a) = \|c\|^2 \quad 2.3$$

By eliminating the equation the cosine rule is obtained :$\|a\|^2+\|b\|^2-\|c\|^2 = 2\,\|a\|\|\mathrm{b}\|\cos\angle_w(a,b)$ or $\|a\|^2+\|b\|^2-2\,\|a\|\|\mathrm{b}\|\cos\angle_w(a,b) = \|c\|^2$, then in the same way another person can get it.

■

Theorem. 5. Let $(V,\|\,.\,\|)$ be normed space and for each $a,b,c\in V\setminus\{0\}$, defined $\Delta[a,b,c]$ as $\{a,b,c\}$ who fulfills $a+c=b$ which is equipped with an angle $\angle_w(a,b)$, $\angle_w(b,c)$, and $\angle_w(c,-a)$, If $\|a\| = \|c\|$ than $\angle_w(a,b) = \angle_w(b,c)$.

Proof.

Let $\|a\| = \|c\|$ be then obtained :

$\|b\|\cos\angle_w(a,b)+\|\mathrm{c}\|\cos\angle_w(c,-a)$

$= \|b\|\cos\angle_w(b,c)+\|\mathrm{a}\|\cos\angle_w(c,-a)$ so that it is obtained

$\cos\angle_w(a,b) = \cos\angle_w(b,c)$ because $\angle_w(a,b),\angle_w(b,c)\in[0,\pi]$ than $\angle_w(a,b)-\angle_w(b,c)$

■

Example 1.

Let $\Delta[a,b,c]$ be, $\{a,b,c\}$ is a set of sequences contained in the following sequence space,

$$\ell^1(\mathbb{R}) = \left\{(a_n)\subseteq\mathbb{R}\;\middle|\;\sum_{n=1}^{\infty}|a_n| < \infty\right\}$$

And the vectors fulfill $a+c=b$ with the sequence as follows,

$(a_n) = (1,0,0,\dots)$, $(b_n) = \left(\frac{1}{2},\frac{1}{2},0,0,\dots\right)$ and $(c_n) = \left(-\frac{1}{2},\frac{1}{2},0,0,\dots\right)$.

Then the norms of each are obtained :

$$\|(a_n)\| = \sum_{n=1}^{\infty}|a_n|$$

$$\|(a_n)\| = |1|+|0|+|0|+\dots = 1$$

$$\|(b_n)\| = \left|\frac{1}{2}\right|+\left|\frac{1}{2}\right|+|0|+\dots = 1$$

$$\|(c_n)\| = \left|-\frac{1}{2}\right|+\left|\frac{1}{2}\right|+|0|+\dots = 1$$

So that it is obtained :

$$\angle_w(a,b) = arccos\left(\frac{\|a\|^2+\|b\|^2-\|c\|^2}{2\|a\|\|b\|}\right) = arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\angle_w(-a,c) = arccos\left(\frac{\|a\|^2+\|c\|^2-\|b\|^2}{2\|a\|\|c\|}\right) = arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\angle_w(b,c) = arccos\left(\frac{\|b\|^2+\|c\|^2-\|a\|^2}{2\|a\|\|c\|}\right) = arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Thus the number of the three angles $\Delta[a,b,c]$ that is :

$$\angle_W(a,b)+\angle_W(b,c)+\angle_W(-a,c) = \frac{\pi}{3}+\frac{\pi}{3}+\frac{\pi}{3} = \pi$$

Example 2.

Suppose the set of functions is continuous real :

$$\mathbf{C}([0,1]) = \{ f \mid f : [0,1] \rightarrow \mathbb{R},\ f \quad \text{continuous} \}$$

With the norm :

$$\|f\| : \max_{x\in[0,1]}\{|f(x)|\}$$

Let $\Delta[a,b,c]$ be, $\{a,b,c\} \subseteq \mathbf{C}([0,1])$, with ;

$$a(t) = t,$$
$$b(t) = 1,$$
$$c(t) = 1 - t,$$

Obtained each norm is :

$$\|a\| : \max_{t\in[0,1]}\{|a(t)|\} = 1$$
$$\|b\| : \max_{t\in[0,1]}\{|b(t)|\} = 1$$
$$\|c\| : \max_{t\in[0,1]}\{|c(t)|\} = 1$$

obtained :

$$\angle_w(a,b) = arccos\left(\frac{\|a\|^2 + \|b\|^2 - \|c\|^2}{2\|a\|\|b\|}\right)$$
$$= arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$
$$\angle_w(-a,c) = arccos\left(\frac{\|a\|^2 + \|c\|^2 - \|b\|^2}{2\|a\|\|c\|}\right)$$
$$= arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$
$$\angle_w(b,c) = arccos\left(\frac{\|b\|^2 + \|c\|^2 - \|a\|^2}{2\|a\|\|c\|}\right) = arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Thus the number of the three angles $\Delta[a,b,c]$ that is :

$$\angle_W(a,b) + \angle_W(b,c) + \angle_W(-a,c) = \frac{\pi}{3} + \frac{\pi}{3} + \frac{\pi}{3} = \pi$$

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REFERENCES

[1] Zakir M, Eridani, dan Fatmawati, 2018. The New Triangle in Normed Space, Global Journal of Pure and Applied Mathematics.ISSN 0973-1768 Volume 14, Number 3 (2018), pp. 369–375.

[2] Zakir M, Eridani, dan Fatmawati, 2018a. Expansion of Ceva Theorem in the Normed Space with the Angle of Wilson, International Journal of Science and Research, Volume 7, Number 1, Januari 2018, ISSN (Online): 2319-7064.

[3] Milicic, P.M, 2011. The Thy-Angle and g-Angle in a Quasi-Inner Product Space, *Mathematica Moravica*, Vol. 15-2; 41 – 46.

[4] Anton H dan Rorres C, 2010, *Elementary Linear Algebra*, Tenth Edition, John Wiley & Sons, Inc

[5] Brown, A.L., dan Page, A. 1970. Elements of Functional Analysis. Van Nostrand Reinhold Company, London.

[6] Euclid. 2008. *Euclid's Elemens of Geometry*, (edisi Revisi, Richard Fitzpatrick),

[7] Gibson, C,G, 2004, *Elementary Euclidean Geometry*, Cambridge University Prees, New York.

[8] Kreyszig. E 1978. *Introductory Functional Analysis With Applications*, John Wiley & Sons. Inc. New York.