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On The Packing $k-$Coloring Of Unicyclic Graph Family

Regina Ariningtyas
Department of Mathematics Education
University of Jember
Jember, Indonesia
reginaariningtyas97@gmail.com

Arika Indah Kristiana
Department of Mathematics Education
University of Jember
Jember, Indonesia
arika.fkip@unej.ac.id

Dafik
Department of Mathematics Education
University of Jember
Jember, Indonesia
d.dafik@unej.ac.id

Abstract: *All graph in this paper is connected graph. Let $d(u,v)$ be a distance between any vertex u and v in graph $G(V,E)$. A function $c:V(G)\rightarrow\{1,\ 2,\ \ldots,\ k\}$ is called a packing $k-$coloring if every two vertex of color i are at least distance $i+1$. $\chi_p(G)$ or packing chromatics number of graph G is the smallest integer of k which has packing coloring. Unicyclic graphs are graphs that only have one cycle. This graphs is denoted by C_l, which l is the length of the unicyclic graph. In this paper, we will study about packing $k-$coloring of graphs and determine their packing chromatic number. We have found the exact values of the packing coloring of unicyclic graph family.*

Keywords: Packing $k-$coloring, Packing chromatic number, Unicyclic graph

1. Introduction

A graph G is defined as a set of pairs (V,E) where V is a non-empty set of nodes whose elements are called vertex denoted by $V(G)=\{v_1,v_2,v_3,\ldots,v_n\}$. Vertex on a graph can be numbered with alphabet, natural numbers, or by using alphabet and numbers (natural numbers). E is the set and may be empty of unsorted pairs (u,v) of two points u and v on V, a edge with vertex (u) and (v) denoted by uv. This edge is depicted by lines connecting a pair of vertex whose elements are called edges which are denoted $E(G)=\{e_1,e_2,e_3,\ldots,e_n\}$[1]. Distance is the shortest path length from vertex u to vertex w on a graph G denoted by $d(u,w)$ [2]. Unicyclic graphs are graphs that only have one cycle. This graph can also be obtained from tree graphs added with new edges[3]. The unicyclic graph has n vertex and m edge, where the number of vertex and the number of edge in the unicyclic graph are the same. This graph is denoted by C_l, where l is the length of the unicyclic graph[4]. Coloring is the process of coloring elements in the graph, but each neighboring graph element must not have the same color and the number of colors produced must be as minimum as possible [5]. Given graph G is a connected graph, $\delta(u,v)$ is the distance between vertex u and v with $k-$packing integers, $k\geq1$ on graph G called chromatic numbers. A $k-$packing coloring of graph G is the mapping $\pi:V(G)\rightarrow\{1,2,\ldots,k\}$. So that graph G has the same color vertex as the distance $i+1$ from the previous vertex, where i is the color of that vertex[6]. Chromatic numbers in the coloring of a graph are denoted by χ_p [7]. According to Goddard et al. (2008) graph G in the form of cycles has a lower limit of broadcast coloring with the maximum number of chromatic numbers $\chi_b(C_n)=3$ for $n=3$ or multiples of 4 and $\chi_b(C_n)=4$ for other n. This is shown in the following proportions. Proportion for $n\geq3$, if n is 3 or a multiple of 4, then $\chi_b(C_n)=3$; if not $\chi_b(C_n)=4$[8].

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2. RESULT

Theorem 2.1. The packing chromatic numbers of $S_{n,n}$ graph for $n \geq 3$ are:

$$\chi_p(S_{n,n}) = \begin{cases} 4, & for\ n = 1\ and\ 2 \\ 5, & for\ n = 3{,}4\ and\ 6 \\ 4, & for\ n = 5\ and\ n \geq 7 \end{cases}$$

Proof. There are three cases in packing chromatic numbers of graph $S_{n,n}$, for $n = 1\ and\ 2$, for $n = 3{,}4\ and\ 6$, and for $n = 5\ and\ n \geq 7$. The explanation of the three cases is as follows.

Cases 1: for $n = 1\ and\ 2$. The equation $\chi_p(S_{n,n}) = 4$ is obtained so as to prove that the packing chromatic number of the graph Sn, n is equal to the equation, it needs to be proven using the lower and upper bound. First it is proved that the lower boundary of the chromatic packing number in the graph $S_{n,n}$ is $\chi_p(S_{n,n}) \geq 4$. Assume that $\chi_p(S_{n,n}) < 4$. Take $\chi_p(S_{n,n}) = 3$, coloring the graph $S_{n,n}$ with three colors, so that:

- The distance from vertex y_k to vertex x_{k+1} for $1 \leq k \leq n-1$ is two.
- Vertex y_i neightboring with vertex x_i and vice versa, so the colors at vertex y_i and vertex x_i cannot be the same.
- Vertex x_i neightboring with vertex x_{i+1} and vice versa, so the colors at vertex x_i and vertex x_{i+1} cannot be the same.
- The graph $S_{n,n}$ has three colors, where color 1 is the color found at vertex y_i with i odd, and color 1 also exists at vertex x_i with i even because it is spaced two.
- $c(y_i)$ when i is even and $c(x_i)$ when i is odd colorless 1, so $c(y_i)$ and $c(x_i)$ must be ≥ 2.
- Vertex y_k to vertex y_{k+2} is four, then the color may be $= 2$.
- Because from vertex x_i to vertex x_{i+1} with i odd spacing is two, the color $x_i \neq 2$ so $c(x_i)$ must be ≥ 3.
- If the graph $S_{n,n}$ is colored with three colors, then there are two points that are the same color as the distance at the previous vertex less than $i+1$, contradiction. So $\chi_p(S_{n,n}) \geq 4$.

Furthermore it is proved that the upper bound of the packing chromatic numbers in the graph $S_{n,n}$ is $\chi_p(S_{n,n}) \leq 4$. With the color function defined $c : V(G) \to \{1,2,3,\ldots\}$ as follows:

$$c(x_i) = \begin{cases} 1, & for\ i\ even \\ 3, & for\ i \equiv 1 (\mathrm{mod}\ 4) \\ 4, & for\ i \equiv 3 (\mathrm{mod}\ 4) \\ 5, & for\ i = 5\ dan\ 3 \leq n \leq 7 \\ 6, & for\ other\ i \end{cases}$$

$$c(y_i) = \begin{cases} 1, & for\ j\ odd \\ 2, & for\ j\ even \end{cases}$$

Based on the color function, it can be seen that:

- Every two vertice that are neighbors with color 1 must have a distance of at least two from the previous vertex.
- Every two neighboring colors of 2 have a minimum distance of three from the previous vertex.
- Every two vertice that are neighbors with color 3 must have a distance of at least four from the previous vertex.
- Every two vertice that are neighbors with color 4 have a minimum distance of five from the previous vertex.
- Every two vertice that are neighbors with color 5 must have a distance of at least six from the previous vertex.
- Each of the two neighbors of color 6 must have a distance of at least seven from the previous vertex.

So, every two points have the same color, for example i means having a minimum distance of $i+1$ from the previous vertex. So we get the upper bound of the coloring of the packing graph $S_{n,n}$ is $\chi_p(S_{n,n}) \leq 4$ for $n = 1\ and\ 2$. It is found that the lower and upper bound of the packing chromatic numbers on the graph $S_{n,n}$,for $n = 1\ and\ 2$ is $4 \leq \chi_p(S_{n,n}) \leq 4$. So it can be concluded that the packing chromatic number in the graph $S_{n,n}$ for $n = 1$ and 2 is $\chi_p(S_{n,n}) = 4$.

Case 2: for $n = 3{,}4\ and\ 6$. The equation $\chi_p(S_{n,n}) = 5$ is obtained, so as to prove that the chromatic packing number of graph $S_{n,n}$ is equal to the equation, it needs to be proven using the lower and upper bound. First it is proved that the lower boundary of the chromatic packing number in the graph $S_{n,n}$ is $\chi_p(S_{n,n}) \geq 5$. Assume that $\chi_p(S_{n,n}) < 5$. Take $\chi_p(S_{n,n}) = 4$, coloring graph $S_{n,n}$ with four colors, so that:

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- The distance from point yk to point y_{k+1} for $1 \le i \le n-1$ is two.
- Vertex y_i neightboring with vertex x_i and vice versa, so the colors at vertex y_i and vertex x_i cannot be the same.
- Vertex x_i neightboring with vertex x_{i+1} and vice versa, so the colors at vertex x_i and vertex x_{i+1} cannot be the same.
- The graph $S_{n,n}$ has four colors, where color 1 is the color found at vertex y_i with i odd because the distance of y_i to i odd is four, and color 1 is also found at vertex x_i with i even because the distance x_i to i is even two.
- $c(y_i)$ with i even and $c(x_i)$ with i odd colorless 1, so $c(y_i)$ and $c(x_i)$ colored ≥ 2.
- Vertex y_k to vertex y_{k+2} with k even spacing four, then the color may be $=2$.
- Because from vertex x_i to vertex x_{i+2} with i odd spacing two, then the color $x_i \ne 2$ so $c(x_i)$ must be ≥ 3.
- The distance x_i with i odd is two, then the color $x_i \ne 3$ so that $c(x_i)$ must be ≥ 4.
- If the graph $S_{n,n}$ is colored with four colors, then there are two points that are the same color as the distance at the previous vertex less than $i+1$, contradiction, so $\chi_p(S_{n,n}) \ge 5$.

Furthermore it is proved that the upper bound of the packing chromatic numbers in the graph $S_{n,n}$ is $\chi_p(S_{n,n}) \le 5$. With the color function defined $c: V(G) \to \{1,2,3,\ldots\}$ as follows:

$$c(x_i) = \begin{cases} 1, & for\ i\ even \\ 3, & for\ i \equiv 1 (\text{mod } 4) \\ 4, & for\ i \equiv 3 (\text{mod } 4) \\ 5, & for\ i = 5\ dan\ 3 \le n \le 7 \\ 6, & for\ other\ i \end{cases}$$

$$c(y_i) = \begin{cases} 1, & for\ j\ odd \\ 2, & for\ j\ even \end{cases}$$

Based on the color function, it can be seen that:

- Every two vertice that are neighbors with color 1 must have a distance of at least two from the previous vertex.
- Every two neighboring colors of 2 have a minimum distance of three from the previous vertex.
- Every two vertice that are neighbors with color 3 must have a distance of at least four from the previous vertex.
- Every two vertice that are neighbors with color 4 have a minimum distance of five from the previous vertex.
- Every two vertice that are neighbors with color 5 must have a distance of at least six from the previous vertex.
- Each of the two neighbors of color 6 must have a distance of at least seven from the previous vertex.

So, every two points have the same color, for example i means having a minimum distance of $i+1$ from the previous vertex. So we get the upper bound of the coloring of the packing graph $S_{n,n}$ is $\chi_p(S_{n,n}) \le 5$ for $n = 3,4,\ and\ 6$. It is found that the lower and upper bound of the packing chromatic numbers on the graph $S_{n,n}$,for $n = 3,4,\ and\ 6$ is $5 \le \chi_p(S_{n,n}) \le 5$. So,it can be concluded that the packing chromatic number in the graph $S_{n,n}$ for $n = 3,4,\ and\ 6$ is $\chi_p(S_{n,n}) = 5$.

Case 3: for $n = 5\ and\ n \ge 7$. The equation $\chi_p(S_{n,n}) = 6$ is obtained, so to prove that the packing chromatic number of the graph $S_{n,n}$ is equal to the equation, it needs to be proven using the lower and upper bound. First it is proved that the lower boundary of the packing chromatic number in the graph $S_{n,n}$ is $\chi_p(S_{n,n}) \ge 6$. Assume that $\chi_p(S_{n,n}) < 6$. Take $\chi_p(S_{n,n}) = 5$, coloring graph $S_{n,n}$ with five colors, so that:

- The distance from vertex y_k to vertex x_{k+1} for $1 \le k \le n-1$ is two
- Vertex y_i neightboring with vertex x_i and vice versa, so the colors at vertex y_i and vertex x_i cannot be the same.
- Vertex x_i neightboring with vertex x_{i+1} and vice versa, so the colors at point x_i and vertex x_{i+1} cannot be the same.
- The graph $S_{n,n}$ has five colors, where color 1 is the color found at vertex y_i with i odd because the distance of y_i with i odd is four, and color 1 is also found at vertex x_i with i even because the distance x_i with i even is two.
- $c(y_i)$ with i even and $c(x_i)$ with i odd colorless 1, so $c(x_i)$ with i odd must be ≥ 2.

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- Vertex y_k to vertex y_{k+2} with k even is four, then the color may be $=2$.
- Because from vertex x_k to point y_{k+1} for k odd spacing two, the color $c(x_k)\neq 2$ so $c(x_k)$ must be ≥ 3.
- The distance x_i with i odd is two, then the color $c(x_i)\neq 3$ so $c(x_i)$ must be ≥ 4.
- Vertex x_k to vertex x_l is four, so the color $\neq 4$ so $c(x_l)$ must be ≥ 5.
- If the graph $S_{n,n}$ is colored with five colors, then there are two vertex that are the same color as the distance at the previous vertex less than $i+1$, contradiction, so $\chi_p(S_{n,n})\geq 6$.

Furthermore it is proved that the upper bound of the packing chromatic numbers in the graph $S_{n,n}$ is $\chi_p(S_{n,n})\leq 6$. With the color function defined $c:V(G)\rightarrow\{1,2,3,\ldots\}$ as follows:

$$c(x_i)=\begin{cases}1, & for\ i\ even\\ 3, & for\ i\equiv 1(\text{mod } 4)\\ 4, & for\ i\equiv 3(\text{mod } 4)\\ 5, & for\ i=5\ dan\ 3\leq n\leq 7\\ 6, & for\ other\ i\end{cases}$$

$$c(y_i)=\begin{cases}1, & for\ j\ odd\\ 2, & for\ j\ even\end{cases}$$

Based on the color function, it can be seen that:

- Every two vertice that are neighbors with color 1 must have a distance of at least two from the previous vertex.
- Every two neighboring colors of 2 have a minimum distance of three from the previous vertex.
- Every two vertice that are neighbors with color 3 must have a distance of at least four from the previous vertex.
- Every two vertice that are neighbors with color 4 have a minimum distance of five from the previous vertex.
- Every two vertice that are neighbors with color 5 must have a distance of at least six from the previous vertex.
- Each of the two neighbors of color 6 must have a distance of at least seven from the previous vertex.

So, every two points have the same color, for example i means having a minimum distance of $i+1$ from the previous vertex. So we get the upper bound of the coloring of the packing graph $S_{n,n}$ is $\chi_p(S_{n,n})\leq 6$ for $n=5\ and\ n\geq 7$. It is found that the lower and upper bound of the packing chromatic numbers on the graph $S_{n,n}$,for $n=5\ and\ n\geq 7$ is $6\leq\chi_p(S_{n,n})\leq 6$. So,it can be concluded that the packing chromatic number in the graph $S_{n,n}$ for $n=5\ and\ n\geq 7$ is $\chi_p(S_{n,n})=6$.

It can be concluded that **Theorem 2.1**, there are three cases, namely for $n=1\ and\ 2$, $n=3,4,\ and\ 6$, as well as for $n=5\ and\ n\geq 7$ which have been proven to be true from the evidence outlined above.

Theorem 2.2. The packing chromatic numbers on the graph $Cr_{n,m}$ for $n\geq 2$ and $m\geq 3$ are:

$$\chi_p(Cr_{n,m})=\begin{cases}3, & for\ n=1\ and\ n\equiv 2(\text{mod } 4)\\ 4, & for\ other\ n\end{cases}$$

Proof. There are two cases in the packing chromatic number of the $Cr_{n,m}$ graph for $n=1$ and $n\equiv 2(\text{mod } 4)$ as well as for other n. The explanation of the two cases is as follows.

Case 1: for $n=1$ and $n\equiv 2(\text{mod } 4)$. The equation $\chi_p(Cr_{n,m})=3$ is obtained so that to prove that the packing chromatic number of the graph $Cr_{n,m}$ is the same as the equation, it needs to be proven using the lower and upper bound. First it was proved that the lower boundary of the packing chromatic number in the graph $Cr_{n,m}$ is $\chi_p(Cr_{n,m})\geq 3$. Assume that $\chi_p(Cr_{n,m})<3$. Take $\chi_p(Cr_{n,m})=2$, coloring graph $Cr_{n,m}$ with two colors, so that:

- The distance from vertex y_k to vertex x_k for $1\leq k\leq n$ is two distances.
- The vertex y_j for $j=2$ neightboring with vertex x_i for $i=1$ and $i=n$ and vice versa, so the colors at vertex y_j and vertex x_i cannot be the same.
- The vertex x_i neightboring with vertex x_{i+1} and vice versa, so the colors at vertex x_i and vertex x_{i+1} cannot be the same.
- The vertex y_j neightboring with vertex y_{j+1} and vice versa, so the colors at vertex y_j and vertex y_{j+1} cannot be the same.
- The graph $Cr_{n,m}$ has two colors, where color 1 is the color found at vertex y_j with vertex j odd, and color 1

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also exists at vertex x_i with i odd due to distance between two.

- $c(y_j)$ when j is even and $c(x_i)$ when i is even colorless 1, so $c(y_j)$ and $c(x_i)$ must have color ≥ 2.
- Vertex y_k to vertex y_{k+4} for k is even the distance is four, then $c(y_k)$ may be colored $=2$
- The vertex y_k for $k=2$ to vertex x_l the distance is four, then $c(x_l)$ may be colored $=2$.
- The vertex x_k for $k \geq 4$ to vertex x_{k+4} the distance is four, so $c(x_{k+4})$ may be colored $=2$
- $c(y_j)$ when $j=2$ and $c(x_i)$ when i is even two apart, then $c(x_i) \neq 2$ and must be colored ≥ 3.
- If the graph $Cr_{n,m}$ is colored with two colors, then there are two points that are colored the same as the distance at the previous vertex less than $i+1$. So that the contradiction, so $\chi_p(Cr_{n,m}) \geq 3$.

Furthermore, it is proven that the upper bound of the packing chromatic numbers on the graph $Cr_{n,m}$ is $\chi_p(Cr_{n,m}) \leq 3$. With the color function defined $c: V(G) \to \{1,2,3,\ldots\}$ as follows:

$$c(x_i) = \begin{cases} 1, & for\ i\ odd \\ 2, & for\ i \equiv 0 (\text{mod } 4) \\ 3, & for\ i \equiv 2 (\text{mod } 4) \\ 4, & for\ other\ i \end{cases}$$

$$c(y_j) = \begin{cases} 1, & for\ j\ odd \\ 2, & for\ j \equiv 2 (\text{mod } 4) \\ 3, & for\ j \equiv 0 (\text{mod } 4) \end{cases}$$

Based on the color function above, it can be seen that:

- Every two vertice that are neighbors with color 1 must have a distance of at least two from the previous vertex.
- Every two vertice neighboring colors of 2 have a minimum distance of three from the previous vertex.
- Every two vertice that are neighbors with color 3 must have a distance of at least four from the previous vertex.
- Every two vertice that are neighbors with color 4 have a minimum distance of five from the previous vertex.

So, every two vertex have the same color, for example i means having a minimum distance of $i+1$ from the previous vertex. So we get the upper bound of the coloring of the packing on the graph $Cr_{n,m}$ is $\chi_p(Cr_{n,m}) \leq 3$ for $n=1$ and $n \equiv 2 (\text{mod } 4)$. It is found that the lower and upper bound of the packing chromatic numbers in the graph $Cr_{n,m}$, for $n=1$ and $n \equiv 2 (\text{mod } 4)$ are $3 \leq \chi_p(Cr_{n,m}) \leq 3$. So, it can be concluded that the chromatic number packing at graph $Cr_{n,m}$, for $n=1$ and $n \equiv 2 (\text{mod } 4)$ is $\chi_p(Cr_{n,m}) = 3$.

Case 2: for other than $n=1$ and $n \equiv 2 (\text{mod } 4)$. The equation $\chi_p(Cr_{n,m}) = 4$ is obtained so that to prove that the packing chromatic number of the graph $Cr_{n,m}$ is the same as the equation, it needs to be proven using the lower and upper bound. First it is proved that the lower boundary of the packing chromatic number in the graph $Cr_{n,m}$ is $\chi_p(Cr_{n,m}) \geq 4$. Assume that $\chi_p(Cr_{n,m}) < 4$. Take $\chi_p(Cr_{n,m}) = 3$, coloring graph $Cr_{n,m}$ with three colors, so that:

- The distance from vertex y_k to vertex x_k for $1 \leq k \leq n$ is two distances.
- The vertex y_j for $j=2$ neightboring with vertex x_i for $i=1$ and $i=n$ and vice versa, so the colors at vertex y_j and vertex x_i cannot be the same.
- The vertex x_i neightboring with vertex x_{i+1} and vice versa, so the colors at vertex x_i and vertex x_{i+1} cannot be the same.
- The vertex y_j is neightboring with vertex y_{j+1} and vice versa, so the colors at the vertex y_j and the vertex y_{j+1} cannot be the same.
- The graph $Cr_{n,m}$ has three colors, where color 1 is the color found at vertex y_j with j is odd, and color 1 is also found at vertex x_i with i is odd due to distance between two.
- $c(y_j)$ when j is even and $c(x_i)$ when i is even colorless 1, so $c(y_j)$ and $c(x_i)$ must have color ≥ 2.
- Vertex y_k to vertex y_{k+4} for k is even four distance, then $c(y_k)$ may be colored $=2$.
- The vertex y_k for $k=2$ to vertex x_l is four distance, then $c(x_l)$ may be colored $=2$.

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- The vertex x_k for $k \geq 4$ to vertex x_{k+4} is four distance, so $c(x_{k+4})$ may be colored $= 2$.
- $c(y_j)$ when $j = 2$ and $c(x_i)$ when i is even two apart, then $c(x_i) \neq 2$ and must be colored ≥ 3.
- If the graph $Cr_{n,m}$ is colored with three colors, then there are two vertex that are the same color as the distance at the previous vertex less than $i+1$. So it's a contradiction, so $\chi_p(Cr_{n,m}) \geq 4$.

Furthermore, it is proved that the upper bound of the packing chromatic numbers on the graph $Cr_{n,m}$ is $\chi_p(Cr_{n,m}) \leq 4$. With the color function defined $c: V(G) \to \{1,2,3,\ldots\}$ as follows:

$$c(x_i) = \begin{cases} 1, & for\ i\ odd \\ 2, & for\ i \equiv 0 (\bmod\ 4) \\ 3, & for\ i \equiv 2 (\bmod\ 4) \\ 4, & for\ other\ i \end{cases}$$

$$c(y_j) = \begin{cases} 1, & for\ \ j\ odd \\ 2, & for\ \ j \equiv 2 (\bmod\ 4) \\ 3, & for\ \ j \equiv 0 (\bmod\ 4) \end{cases}$$

Based on the color function above, it can be seen that:

- Every two vertice that are neighbors with color 1 must have a distance of at least two from the previous vertex.
- Every two vertice neighboring colors of 2 have a minimum distance of three from the previous vertex.
- Every two vertice that are neighbors with color 3 must have a distance of at least four from the previous vertex.
- Every two vertice that are neighbors with color 4 have a minimum distance of five from the previous vertex.

So, every two vertex have the same color, for example i means having a minimum distance of $i+1$ from the previous vertex. So we get the upper bound of the coloring of the packing on the graph $Cr_{n,m}$ is $\chi_p(Cr_{n,m}) \leq 4$ for other $n = 1$ and $n \equiv 2 (\bmod\ 4)$. It is found that the lower and upper bound of the packing chromatic numbers in the graph $Cr_{n,m}$, for other $n = 1$ and $n \equiv 2 (\bmod\ 4)$ are $4 \leq \chi_p(Cr_{n,m}) \leq 4$. So, it can be concluded that the chromatic number packing at graph $Cr_{n,m}$, for other $n = 1$ and $n \equiv 2 (\bmod\ 4)$ is $\chi_p(Cr_{n,m}) = 4$. It can be concluded that **Theorem 2.2** there are two cases, for $n = 1$ and $n \equiv 2 (\bmod\ 4)$ as well as for n other than $n = 1$ and $n \equiv 2 (\bmod\ 4)$ which have been proven true from the evidence outlined above.

Theorem 2.3. The packing chromatic numbers in the graph C_n^n for $n \geq 2$ are:

$$\chi(C_n^n) = \begin{cases} 2, & for\ \ n = 1 \\ 3, & for\ \ n = 2\ \ and\ \ n \equiv 3 (\bmod\ 4) \\ 4, & for\ \ other\ \ n \end{cases}$$

Proof. There are three cases in packing chromatic numbers of graph C_n^n, for $n = 1$, $n = 2\ \ and\ \ n \equiv 3 (\bmod\ 4)$, as well as for other n. The explanation of the three cases is as follows.

Case 1: for $n = 1$. The equation $\chi_p(C_n^n) = 2$ is obtained so that to prove that the packing chromatic number of the graph C_n^n is equal to the equation, it needs to be proven using the lower and upper bound. First it was proved that the lower boundary of the packing chromatic number in the graph C_n^n is $\chi_p(C_n^n) \geq 2$. Assume that $\chi_p(C_n^n) < 2$. Take $\chi_p(C_n^n) = 1$, coloring the graph C_n^n with one color, so :

- The distance from vertex y_k to vertex x_l is two.
- The vertex y_j neightboring with vertex x_i for $i = 1$ and vice versa, so the colors at the vertex y_j and vertex x_i cannot be the same.
- The vertex x_i neighboring with vertex x_{i+1} and vice versa, so the colors at vertex x_i and vertex x_{i+1} cannot be the same.
- The graph C_n^n has one color, where color 1 is the color found at vertex y_j and color 1 is also found at vertex x_i with i even because it is spaced two apart.
- Because from vertex x_i to vertex x_{i+1} and vertex x_i to vertex y_j for $i = 1$ is one distance, the color $c(x_i) \neq 1$ so $c(x_i)$ must be ≥ 2.
- If the graph C_n^n is colored with one color, then there are two vertex that are colored the same as the distance at the previous vertex less than $i+1$, contradiction. So $\chi_p(C_n^n) \geq 2$.

Furthermore, it is proven that the upper bound of the packing chromatic number in the graph C_n^n is $\chi_p(C_n^n) \leq 2$. With

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the color function defined $c:V(G)\rightarrow\{1,2,3,\ldots\}$ as follows:

$$c(x_i)=\begin{cases}1, & for\ i\ even\\ 2, & for\ i\equiv 1(\text{mod}\ \ 4)\\ 3, & for\ i=2\ when\ n=2\ and\ i=3\ when\ n\geq 3\\ 4, & for\ i=n-1\ when\ n\equiv 1(\text{mod}\ \ 4)\end{cases}$$

$$c(y_j)=1,\ \ for\ each\ j$$

Based on the color function above, it can be seen that:

- Every two vertice that are neighbors with color 1 must have a distance of at least two from the previous vertex.
- Every two neighboring colors of 2 have a minimum distance of three from the previous vertex.
- Every two vertice that are neighbors with color 3 must have a distance of at least four from the previous vertex.
- Every two vertice that are neighbors with color 4 have a minimum distance of five from the previous vertex.

So, every two vertex have the same color, for example i means having a minimum distance of $i+1$ from the previous vertex. So that the upper bound of the coloration of the packing on the graph C_n^n is $\chi_p(C_n^n)\leq 2$ for $n=1$. It is found that the lower bound and the upper bound of the packing chromatic numbers on the graph C_n^n, for $n=1$ is $2\leq\chi_p(C_n^n)\leq 2$. So, it can be concluded that the chromatic number packing in the graph C_n^n, for $n=1$ is $\chi_p(C_n^n)=2$.

Case 2: for $n=2$ and $n\equiv 3(\text{mod}\ \ 4)$. The equation $\chi_p(C_n^n)=3$ is obtained so that to prove that the packing chromatic number of the graph C_n^n is equal to the equation, it needs to be proven using the lower and upper bound. First it is proved that the lower boundary of the packing chromatic number in the graph C_n^n is $\chi_p(C_n^n)\geq 3$. Assume that $\chi_p(C_n^n)<3$. Take $\chi_p(C_n^n)=2$, coloring the graph C_n^n with two colors, so :

- The distance from vertex y_k to vertex x_l is two.
- The vertex y_j neightboring with vertex x_i for $i=1$ and vice versa, so the colors at vertex y_j and vertex x_i cannot be the same.
- The vertex x_i neightboring with vertex x_{i+1} and vice versa, so the colors at vertex x_i and vertex x_{i+1} cannot be the same.
- The graph C_n^n has two colors, where color 1 is the color found at vertex y_j and color 1 is also found at vertex x_i with i is even because it is spaced two apart.
- $c(x_i)$ with i odd colorless 1, so $c(x_i)\geq 2$.
- Vertex x_k to vertex x_{k+4} for k is odd, distance is four, so the color may be $=2$.
- Because from vertex x_i with i is odd spaced two, the color $c(x_i)\neq 2$ so $c(x_i)\geq 3$.
- If the graph C_n^n is colored with two colors, then there are two vertex that are the same color as the distance at the previous vertex less than $i+1$. So that it is a contradiction, so $\chi_p(C_n^n)\geq 3$.

Furthermore, it is proved that the upper limit of the chromatic packing number in the graph Cnn is $\chi_p(C_n^n)\leq 3$.

With the color function defined $c:V(G)\rightarrow\{1,2,3,\ldots\}$ as follows:

$$c(x_i)=\begin{cases}1, & for\ i\ even\\ 2, & for\ i\equiv 1(\text{mod}\ \ 4)\\ 3, & for\ i=2\ when\ n=2\ and\ i=3\ when\ n\geq 3\\ 4, & for\ i=n-1\ when\ n\equiv 1(\text{mod}\ \ 4)\end{cases}$$

$$c(y_j)=1,\ \ for\ each\ j$$

Based on the color function above, it can be seen that:

- Every two vertice that are neighbors with color 1 must have a distance of at least two from the previous vertex.
- Every two neighboring colors of 2 have a minimum distance of three from the previous vertex.
- Every two vertice that are neighbors with color 3 must have a distance of at least four from the previous vertex.
- Every two vertice that are neighbors with color 4 have a minimum distance of five from the previous vertex.

So, every two vertex have the same color, for example i means having a minimum distance of $i+1$ from the previous vertex. So that the upper bound of the coloration of the packing on the graph C_n^n is $\chi_p(C_n^n)\leq 3$ for $n=2$ and $n\equiv 3(\text{mod}\ \ 4)$.

It is found that the lower bound and the upper bound of the packing chromatic numbers on the graph C_n^n, for $n=2$ and $n\equiv 3(\text{mod}\ \ 4)$ is $3\leq\chi_p(C_n^n)\leq 3$. So, it can be concluded that the chromatic number packing in the graph C_n^n, for $n=2$ and $n\equiv 3(\text{mod}\ \ 4)$ is $\chi_p(C_n^n)=3$.

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Case 3: for n other than $n=1$, $n=2$ and $n\equiv 3(\bmod\ 4)$. The equation $\chi_p(C_n^n)=4$ is obtained so that to prove that the chromatic packing number of the graph C_n^n is equal to the equation, it needs to be proven using the lower and upper limits. First it was proved that the lower boundary of the chromatic packing number in graph C_n^n is $\chi_p(C_n^n)\geq 4$. Assume that $\chi_p(C_n^n)<4$. Take $\chi_p(C_n^n)=3$, coloring the graph C_n^n with three colors, so :

- The distance from vertex y_k to vertex x_l is two.
- The vertex y_j neightboring with vertex x_i for $i=1$ and vice versa, so the colors at the vertex y_j and vertex x_i cannot be the same.
- The vertex x_i neightboring with vertex x_{i+1} and vice versa, so the colors at vertex x_i and vertex x_{i+1} cannot be the same.
- The graph C_n^n has three colors, where color 1 is the color found at vertex y_j and color 1 is also found at vertex x_i with i is even because it is spaced two apart.
- $c(x_i)$ with i an odd colorless 1, so $c(x_i)$ must be ≥ 2.
- Vertex x_k to vertex x_{k+4} for k odd, distance is four, so the color may be $=2$.
- Since the distance x_i with i an odd is two, the color $x_i\neq 2$ so $c(x_i)$ must be ≥ 3.
- If the graph C_n^n is colored with three colors, then there are two vertex that are the same color as the distance at the previous vertex less than $i+1$, contradiction. So $\chi_p(C_n^n)\geq 4$.

Furthermore, it is proven that the upper limit of the chromatic packing number in the graph Cnn is χp (Cnn) ≤ 4. With the color function defined c: V (G) → {1,2,3, ...} as follows:

$$c(x_i)=\begin{cases}1, & for\ i\ even\\ 2, & for\ i\equiv 1(\bmod\ 4)\\ 3, & for\ i=2\ when\ n=2\ and\ i=3\ when\ n\geq 3\\ 4, & for\ i=n-1\ when\ n\equiv 1(\bmod\ 4)\end{cases}$$

$$c(y_j)=1,\quad for\ each\ j$$

Based on the color function above, it can be seen that:

- Every two vertice that are neighbors with color 1 must have a distance of at least two from the previous vertex.
- Every two neighboring colors of 2 have a minimum distance of three from the previous vertex.
- Every two vertice that are neighbors with color 3 must have a distance of at least four from the previous vertex.
- Every two vertice that are neighbors with color 4 have a minimum distance of five from the previous vertex.

So, every two vertex have the same color, for example i means having a minimum distance of $i+1$ from the previous vertex. So that the upper bound of the coloration of the packing on the graph C_n^n is $\chi_p(C_n^n)\leq 4$ for other $n=1$, $n=2$ and $n\equiv 3(\bmod\ 4)$.

It is found that the lower bound and the upper bound of the packing chromatic numbers on the graph C_n^n, for other $n=1$, $n=2$ and $n\equiv 3(\bmod\ 4)$ is $4\leq\chi_p(C_n^n)\leq 4$. So, it can be concluded that the chromatic number packing in the graph C_n^n, for other $n=1$, $n=2$ and $n\equiv 3(\bmod\ 4)$ is $\chi_p(C_n^n)=4$.

It can be concluded that **Theorem 2.3** there are three cases for $n=1$, $n=2$ and $n\equiv 3(\bmod\ 4)$, and for n other than $n=1, n=2$ and $n\equiv 3(\bmod\ 4)$ which have been proven the truth of the evidence outlined above.

3. CONCLUSION

In this paper, we have learned about packing coloring of unicyclic graph families. It can be concluded that the exact value of the chromatic number packing on graph $S_{n,n}$ is

$$\chi_p(S_{n,n})=\begin{cases}4, & for\ n=1\ and\ 2\\ 5, & for\ n=3,4\ and\ 6\\ 4, & for\ n=5\ and\ n\geq 7\end{cases},$$

the chromatic number packing on graph $Cr_{n,m}$ is

$$\chi_p(Cr_{n,m})=\begin{cases}3, & for\ n=1\ and\ n\equiv 2(\bmod\ 4)\\ 4, & for\ other\ n\end{cases},$$

and the chromatic packing number on graph C_n^n is

$$\chi(C_n^n)=\begin{cases}2, & for\ n=1\\ 3, & for\ n=2\ and\ n\equiv 3(\bmod\ 4)\\ 4, & for\ other\ n\end{cases}.$$

Hence the following problem arises naturally.

4. ACKNOWLEDGEMENT

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