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Expansion of Menelaus's Theorem in the Normed Space

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Abstract: *This paper will discuss the expansion of the Menelaus theorem. Menelaus's theorem has been proven in Euclid's space. Before discussing the Menelaus theorem, it is proven that the Ceva theorem in the normed space. Next will be defined Ceva vector.*

Keywords : Euclid Space, Normed Space, Triangeles, Ceva's theorem, Menelaus's theorem.

1. INTRODUCTION

The angle between the two vectors in the Euclid $\mathbb{R}^2$ space is well known. In the Euclid space the angle between two vectors is defined using the product of the dot [8]. Furthermore, the angle between the two vectors in the inner product space has also been developed in [7, 11]. Likewise in normed space, angles between two vectors are also known, including angles P, I, g ([1], [2], [3], [4]), Thy angle [2] and Wilson's angle [6].

The angle in the normed space discussed in this paper is the Wilson's angle introduced by Valentine and Wayment (1971). The study of Wilson's angle is discussed as follows :

Let $(V, \|.\|)$ be the normed space over the field $\mathbb{R}$, for any $x, y \in V$ is defined as a nonlinear functional :

$$2\langle x, y\rangle := \|x\|^2 + \|y\|^2 - \|x-y\|^2 \quad (1)$$

From the properties of the norm it belongs:

$$\left(\|x\| - \|y\|\right)^2 \leq \|x-y\|^2$$
$$\Leftrightarrow \|x\|^2 - 2\|x\|\cdot\|y\| + \|y\|^2 \leq \|x-y\|^2$$
$$\Leftrightarrow \langle x, y\rangle \leq \|x\|\cdot\|y\| \quad (2)$$

Meanwhile :

$$\|x-y\|^2 \leq \left(\|x\| + \|y\|\right)^2$$
$$\Leftrightarrow \|x-y\|^2 - \|x\|^2 - \|y\|^2 \leq 2\|x\|\cdot\|y\|$$
$$\Leftrightarrow -\langle x, y\rangle \leq \|x\|\cdot\|y\| \quad (3)$$

From equations (2) and (3) obtained :

$$\left|\langle x, y\rangle\right| \leq \|x\|.\|y\|, \quad \forall\ x, y \in V \quad (4)$$

fulfill the inequality Cauchy-Schwarz [8]. Wilson's angle is defined as the angle between two vectors x and y that satisfy

$$\angle(x, y) := \arccos\left(\frac{\|x\|^2 + \|y\|^2 - \|x-y\|^2}{2\|x\|\cdot\|y\|}\right) \quad (5)$$

With Wilson's angle the cosine rule is obtained :

$$\|z\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\cdot\|y\|\cos\angle(x, y) \quad (6)$$

Furthermore, from equation (5) sine rules are obtained:

$$\|x\|.\|y\|\sin\angle(x, y) = K \quad (7)$$

With $K = 2\sqrt{s(s-\|x\|)(s-\|y\|)(s-\|z\|)}$ and

$$2s = \|x\| + \|y\| + \|z\|$$

2. MAIN RESULT

Definition. 2.1. Let $(V, \|.\|)$ be Normed space for $a, b, c \in V \setminus \{0\}$, is defined $\Delta[a, b, c]$ as $\{a, b, c\}$ that fulfills $a + c = b$, which is completed by Wilson's angle $\angle(a, b)$, $\angle(-a, c)$, and $\angle(b, c)$.

Definition. 2.2. Let $(V, \|.\|)$ be Normed space for $d \in V \setminus \{0\}$, called the Ceva vector for $\Delta[a_1, a_2, a_3]$ if there is $\alpha \in (0,1)$ such that it fulfills $\alpha a_i + d = a_j$ with $i \neq j$.

Definition. 2.3. Let $(V, \|.\|)$ be Normed space for $d, e, f \in V \setminus \{0\}$, called allied ceva vector of $\Delta[a, b, c]$ if there is $\alpha_i \in (0,1)$, $i = 1,2,3,4,5,6$ such that it fulfills $(1-\alpha_6)f + a = (1-\alpha_5)e$, $(1-\alpha_5)e + c = (1-\alpha_4)d$, $(1-\alpha_6)f + b = (1-\alpha_4)d$

Theorem. 2.1. Let $(V, \|.\|)$ be Normed space. For triangle $\Delta[a, b, c]$, then the following statement is equivalent.

1. Let $a, b, c \in V \setminus \{0\}$ be, such that it fulfills

   $(1-\alpha_6)f + a = (1-\alpha_4)e$, $(1-\alpha_4)e + c = (1-\alpha_4)d$, $(1-\alpha_6)f + b = (1-\alpha_4)d$, with $\alpha_i \in (0,1)$, and i = 1,2,3,4,5,6

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2. $$\frac{\sin\angle(-b,(1-\alpha_6)f)}{\sin\angle(b,(1-\alpha_4)d)}\cdot\frac{\sin\angle(c,(1-\alpha_4)d)}{\sin\angle(-c,(1-\alpha_5)e)}\cdot\frac{\sin\angle(a,(1-\alpha_5)e)}{\sin\angle(-a,(1-\alpha_6)f)}=1$$

3. $$\frac{\|-\alpha_1 a\|}{\|(1-\alpha_1)a\|}\cdot\frac{\|-\alpha_3 c\|}{\|(1-\alpha_3)c\|}\cdot\frac{\|(1-\alpha_2)b\|}{\|-\alpha_2 b\|}=1$$

Proof.

$(1\Rightarrow 2)$

Pay attention $\Delta[(1-\alpha_6)f,(1-\alpha_4)d,b]$ with angles $\angle(-b,(1-\alpha_6)f)$, $\angle((1-\alpha_6)f,(1-\alpha_4)d)$ and $\angle(b,(1-\alpha_4)d)$. was obtained :

$$K_1=\|-b\|\cdot\|(1-\alpha_6)f\|\sin\angle(-b,(1-\alpha_6)f) \quad (8)$$

$$K_1=\|b\|\cdot\|(1-\alpha_4)d\|\sin\angle(b,(1-\alpha_4)d) \quad (9)$$

From equations (8) and (9) obtained :

$$\frac{\sin\angle(-b,(1-\alpha_6)f)}{\sin\angle(b,(1-\alpha_4)d)}=\frac{\|b\|\cdot\|(1-\alpha_4)d)\|}{\|-b\|\cdot\|(1-\alpha_6)f\|}=\frac{\|(1-\alpha_4)d\|}{\|(1-\alpha_6)f\|} \quad (10)$$

Pay attention $\Delta[(1-\alpha_5)e,(1-\alpha_4)d,c]$ with angles $\angle(-c,(1-\alpha_5)e)$, $\angle((1-\alpha_5)e,(1-\alpha_4)d)$ and $\angle(c,(1-\alpha_4)d)$. obtained :

$$K_2=\|c\|\cdot\|(1-\alpha_4)d\|\sin\angle(c,(1-\alpha_4)d) \quad (11)$$

$$K_2=\|-c\|\cdot\|(1-\alpha_5)e\|\sin\angle(-c,(1-\alpha_5)e) \quad (12)$$

From equations (11) and (12) obtained :

$$\frac{\sin\angle(c,(1-\alpha_4)d)}{\sin\angle(-c,(1-\alpha_5)e)}=\frac{\|-c\|\cdot\|(1-\alpha_5)e\|}{\|c\|\cdot\|(1-\alpha_4)d\|}=\frac{\|(1-\alpha_5)e\|}{\|(1-\alpha_4)d\|} \quad (13)$$

Pay attention $\Delta[(1-\alpha_6)f,(1-\alpha_5)e,a]$ with angles $\angle(-a,(1-\alpha_6)f)$, $\angle((1-\alpha_5)e,(1-\alpha_6)f)$ and $\angle(a,(1-\alpha_5)e)$. obtained :

$$K_3=\|-a\|\cdot\|(1-\alpha_6)f\|\sin\angle(-a,(1-\alpha_6)f) \quad (14)$$

$$K_3=\|a\|\cdot\|(1-\alpha_5)e\|\sin\angle(a,(1-\alpha_5)e) \quad (15)$$

From equations (14) and (15) obtained :

$$\frac{\sin\angle(a,(1-\alpha_5)e)}{\sin\angle(-a,(1-\alpha_6)f)}=\frac{\|-a\|\cdot\|(1-\alpha_6)f\|}{\|a\|\cdot\|(1-\alpha_5)e\|}=\frac{\|(1-\alpha_6)f\|}{\|(1-\alpha_5)e\|} \quad (16)$$

Multiply the equations (10), (13) and (16) then they are obtained :

$$\frac{\sin\angle(-b,(1-\alpha_6)f)}{\sin\angle(b,(1-\alpha_4)d)}\cdot\frac{\sin\angle(c,(1-\alpha_4)d)}{\sin\angle(-c,(1-\alpha_5)e)}\cdot\frac{\sin\angle(a,(1-\alpha_5)e)}{\sin\angle(-a,(1-\alpha_6)f)}=\frac{\|(1-\alpha_4)d\|}{\|(1-\alpha_6)f\|}\cdot\frac{\|(1-\alpha_5)e\|}{\|(1-\alpha_4)d\|}\cdot\frac{\|(1-\alpha_6)f\|}{\|(1-\alpha_5)e\|}=1$$

$(2\Rightarrow 3)$

$$\frac{\sin\angle(a,(1-\alpha_5)e)}{\sin\angle(-a,(1-\alpha_6)f)}\cdot\frac{\sin\angle(-b,(1-\alpha_6)f)}{\sin\angle(b,(1-\alpha_4)d)}\cdot\frac{\sin\angle(c,(1-\alpha_4)d)}{\sin\angle(-c,(1-\alpha_5)e)}=1$$

$$\Leftrightarrow\frac{\sin\angle(-\alpha_2 b,(1-\alpha_6)f)}{\sin\angle(-\alpha_1 a,(1-\alpha_6)f)}\cdot\frac{\sin\angle((1-\alpha_1)a,(1-\alpha_5)e)}{\sin\angle(-\alpha_3 c,(1-\alpha_5)e)}\cdot\frac{\sin\angle((1-\alpha_3)c,(1-\alpha_4)d)}{\sin\angle((1-\alpha_4)d,(1-\alpha_2)b)}=1$$

$$\Leftrightarrow \quad \frac{\dfrac{K_2}{\|-\alpha_2 b\|\cdot\|(1-\alpha_6)f\|}}{\dfrac{K_1}{\|-\alpha_1 a\|\cdot\|(1-\alpha_6)f\|}}\cdot$$

$$\frac{\dfrac{K_1}{\|(1-\alpha_1)a\|\cdot\|(1-\alpha_5)e\|}}{\dfrac{K_3}{\|-\alpha_3 c\|\cdot\|(1-\alpha_5)e\|}}$$

$$\frac{\dfrac{K_3}{\|(1-\alpha_3)c\|\cdot\|(1-\alpha_4)d\|}}{\dfrac{K_2}{\|(1-\alpha_4)d\|\cdot\|(1-\alpha_2)b\|}}=1$$

$$\Leftrightarrow \quad \frac{\|-\alpha_1 a\|}{\|(1-\alpha_1)a\|}\cdot\frac{\|-\alpha_3 c\|}{\|(1-\alpha_3)c\|}\cdot\frac{\|(1-\alpha_2)b\|}{\|-\alpha_2 b\|}=1$$

$(3 \Rightarrow 1)$

suppose $(1-\alpha_6)f+a \neq (1-\alpha_4)e$

let $(1-\alpha_6)f+a=\beta g$ be, then

$$\frac{\|-\alpha_1 a\|}{\|(1-\alpha_1)a\|}\cdot\frac{\|\beta c\|}{\|(1-\beta)c\|}\cdot\frac{\|(1-\alpha_2)b\|}{\|-\alpha_2 b\|}=1 \qquad (17)$$

Meanwhile known :

$$\frac{\|-\alpha_1 a\|}{\|(1-\alpha_1)a\|}\cdot\frac{\|-\alpha_3 c\|}{\|(1-\alpha_3)c\|}\cdot\frac{\|(1-\alpha_2)b\|}{\|-\alpha_2 b\|}=1 \qquad (18)$$

From equations (17) and (18) obtained :

$$\frac{\|\beta c\|}{\|(1-\beta)c\|}=\frac{\|-\alpha_3 c\|}{\|(1-\alpha_3)c\|} \qquad (19)$$

$\alpha_3=\beta$ or $f=g$ ■

Example 2.1.

Let $\Delta[a,b,c]$ be, $\{a,b,c\}$ is the set of rows contained in :

$$\ell^1(\mathbb{R})=\left\{(a_n)\subseteq\mathbb{R}\;\middle|\;\sum_{n=1}^{\infty}|a_n|<\infty\right\}$$

and fulfill $a+c=b$ with $(a_n)=(5,0,0,\dots)$, $(b_n)=(11,0,0,0,\dots)$ and $(c_n)=(6,0,0,\dots)$. Next, if selected $\alpha_1=\frac{1}{3}, \alpha_2=\frac{1}{4}, \alpha_3=\frac{6}{7}$ then :

$$d=\left(\frac{24}{7},0,\dots,\right), \qquad e=\left(\frac{28}{3},0,\dots,\right),$$

$$f=\left(\frac{14}{4},0,\dots,\right),$$

$$\|a\|=5$$

$$\|b\|=11$$

$$\|c\|=6$$

$$\|d\|=\frac{24}{7}$$

$$\|e\|=\frac{28}{3}$$

$$\|f\|=\frac{14}{4}$$

From here obtained ;

$$\frac{\|(1-\alpha_2)b\|}{\|\alpha_2 b\|}\cdot\frac{\|(1-\alpha_1)a\|}{\|\alpha_1 a\|}\cdot\frac{\|(1-\alpha_3)c\|}{\|-\alpha_3 c\|}$$

$$\frac{\left\|\frac{1}{7}b\right\|}{\left\|\frac{6}{7}b\right\|}\cdot\frac{\left\|\frac{2}{3}a\right\|}{\left\|\frac{1}{3}a\right\|}\cdot\frac{\left\|\frac{3}{4}c\right\|}{\left\|\frac{1}{4}c\right\|}=1\cdot$$

Theorem 2.2. Let $(V,\|.\|)$ be Normed space for $a,b,c\in V\setminus\{0\}$, is defined $\Delta[a,b,c]$ as $\{a,b,c\}$ that fulfills $a+c=b$, which is completed by Wilson's angle $\angle(a,b)$, $\angle(-a,c)$, and $\angle(b,c)$.

$(\alpha_2-1)a+d=c+e$, $\alpha_3 b+e=\alpha_1 d$, $\alpha_2 a+(\alpha_3-1)b=(\alpha_1-1)d$, with $0<\alpha_i<1$, $i=1,2,3$, If and only if

$$\frac{\|\alpha_3 b\|}{\|(\alpha_3-1)b\|}\cdot\frac{\|\alpha_2 a\|}{\|(\alpha_2-1)a\|}\cdot\frac{\|c+e\|}{\|e\|}=1$$

Proof.

$(\Longrightarrow)$

Pay attention $\Delta[(\alpha_2-1)a,d,c+e]$

$$\sin\angle(d,c+e)=\frac{K_1}{\|d\|\|c+e\|}$$

$$\sin\angle\big((\alpha_2-1)a,d\big)=\frac{K_1}{\|(\alpha_2-1)a\|\|d\|}$$

$$\frac{\sin\angle((\alpha_2-1)a,d)}{\sin\angle(d,c+e)}=\frac{\frac{K_1}{\|(\alpha_2-1)a\|\|d\|}}{\frac{K_1}{\|d\|\|c+e\|}}=\frac{\|c+e\|}{\|(\alpha_2-1)a\|}, \quad (20)$$

Pay attention $\Delta[\alpha_3 b, e, \alpha_1 d]$

$$\sin\angle(\alpha_1 d,e)=\frac{K_2}{\|\alpha_1 d\|\|e\|}$$

$$\sin\angle(\alpha_3 b,\alpha_1 d)=\frac{K_2}{\|\alpha_3 b\|\|\alpha_1 d\|}$$

$$\frac{\sin\angle(\alpha_1 d,e)}{\sin\angle(\alpha_3 b,\alpha_1 d)}=\frac{\frac{K_2}{\|\alpha_1 d\|\|e\|}}{\frac{K_2}{\|\alpha_3 b\|\|\alpha_1 d\|}}=\frac{\|\alpha_3 b\|}{\|e\|}, \quad (21)$$

Pay attention $\Delta[\alpha_2 a, (\alpha_3-1)b, (\alpha_1-1)d]$

$$\sin\angle(\alpha_2 a,(\alpha_1-1)d)=\frac{K_3}{\|\alpha_2 a\|\|(\alpha_1-1)d\|}$$

$$\sin\angle((\alpha_3-1)b,(\alpha_1-1)d)=\frac{K_3}{\|(\alpha_3-1)b\|\|(\alpha_1-1)d\|}$$

$$\frac{\sin\angle(\alpha_3-1)b,(\alpha_1-1)d)}{\sin\angle(\alpha_2 a,(\alpha_1-1)d)}=\frac{\frac{K_3}{\|(\alpha_3-1)b\|\|(\alpha_1-1)d\|}}{\frac{K_3}{\|\alpha_2 a\|\|(\alpha_1-1)d\|}}=\frac{\|\alpha_2 a\|}{\|(\alpha_3-1)b\|}, \quad (22)$$

Multiply the equations (20), (21), (22) then get :

$$\frac{\|c+e\|}{\|(\alpha_2-1)a\|}\cdot\frac{\|\alpha_3 b\|}{\|e\|}\cdot\frac{\|\alpha_2 a\|}{\|(\alpha_3-1)b\|}=\frac{\sin\angle((\alpha_2-1)a,d)}{\sin\angle(d,c+e)}\cdot\frac{\sin\angle(\alpha_1 d,e)}{\sin\angle(\alpha_3 b,\alpha_1 d)}\cdot\frac{\sin\angle(\alpha_3-1)b,(\alpha_1-1)d)}{\sin\angle(\alpha_2 a,(\alpha_1-1)d)}$$

$$\frac{\|c+e\|}{\|e\|}\cdot\frac{\|\alpha_3 b\|}{\|(\alpha_3-1)b\|}\cdot\frac{\|\alpha_2 a\|}{\|(\alpha_2-1)a\|}=\frac{\sin\angle((\alpha_2-1)a,d)}{\sin\angle(\alpha_2 a,(\alpha_1-1)d)}\cdot\frac{\sin\angle(\alpha_1 d,e)}{\sin\angle(d,c+e)}\cdot\frac{\sin\angle(\alpha_3-1)b,(\alpha_1-1)d)}{\sin\angle(\alpha_3 b,\alpha_1 d)}=1$$

$$\frac{\|c+e\|}{\|e\|}\cdot\frac{\|\alpha_3 b\|}{\|(\alpha_3-1)b\|}\cdot\frac{\|\alpha_2 a\|}{\|(\alpha_2-1)a\|}=1$$

($\Leftarrow$)

Suppose $(\alpha_2-1)a+d\neq c+e$ or $(\alpha_2-1)a+d=c+t$, this matter :

$$\frac{\|c+t\|}{\|e\|}\cdot\frac{\|\alpha_3 b\|}{\|(\alpha_3-1)b\|}\cdot\frac{\|\alpha_2 a\|}{\|(\alpha_2-1)a\|}=1 \quad (23)$$

$$\frac{\|c+e\|}{\|e\|}\cdot\frac{\|\alpha_3 b\|}{\|(\alpha_3-1)b\|}\cdot\frac{\|\alpha_2 a\|}{\|(\alpha_2-1)a\|}=1 \quad (24)$$

From equation (23), (24) is obtained :

$$\frac{\|c+t\|}{\|e\|}=\frac{\|c+e\|}{\|e\|},$$

means it $t=e$, or $(\alpha_2-1)a+d=c+e$

■

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