International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 4 Issue 1, January – 2020, Pages: 20-27

A Study of Double Sumudu Transform for Solving Differential Equations with Some Applications

Mohammed S. Mechee[1] and Abbas J. Naeemah [1]

Abstract

In this paper, we have studied the extension of Laplace and Sumudu transforms to functions of two variables. Speci cally, we studied these two transforms with their main properties as well as their applications in solving ordinary and partial differential equations. We provide a proof to the relationship between these two operators, i.e. double Laplace and double Sumudu transforms. We apply double Sumudu transform to nd the solution of Heat and Wave Equations.

Keywords: Sumudu Transform; Double; Differential Equations

1 Introduction

DEs are useful tools in mathematical models of life problems and applied mathematics. DEs have played very important role in different applications of mathematics for a long time and with the development of the computer their importance has increased. Thus, the investigation and the analysis of DEs had increase in applications leading to several mathe-matical problems; therefore,a number of methods (exact and approximated) can be used to nd the solution of DEs.The numerical methods can provide approximate solutions rather than the analytical solutions of the problems. In most times it may be complex to solve these equations analytically and thus are commonly solved by integral transforms such as Laplace and Fourier transforms and the advantage of these two methods lies in their ability to transform differential equations into algebraic equations, which allows a simple way to nd the solution. As we see the integral transform method is an effective way to solve some certain differential equations. Thus, in the literature there are a lot of works on the theory and applications of Laplace, Fourier, Mellin and other integral transforms (Debnath Bhatta, 2006) but a little on the power series transformation such as Sumudu transform, maybe because it is little known and not widely used yet. Sumudu transform was proposed originally by Watugala for solving ordinary and partial differential equations both in ordi-nary and in fractional sense as explained in the previous chapters.Watugala (2002) extended the transform to functions of two variables with emphasis on solutions to partial differen-tial equations. Kilicman & Gadain (2009) studied the relation between double Laplace and double Sumudu transforms and study most of double sumudu transform properties and applied it to the solution of ordinary differential equations and control engineering prob-lems. Then, K l çman & Eltayeb (2010) applied double Laplace and Sumudu transforms

[1]Department of Mathematics, Faculty of Education, Kufa university, Najaf, Iraq. Corresponding E-mail Address:mohammeds.abed@uokufa.edu.iq

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 4 Issue 1, January – 2020, Pages: 20-27

to solve the partial differential equations and show that there is a strong relationship be-tween them and there is also a relation between double Sumudu of convolution and double Laplace of convolution, Sumudu transform was extended to the distributions and some of their properties were studied. Eltayeb & Kilichman (2010a) produced a new equation by us-ing convolution and solved it by double Sumudu and double Laplace transforms, compared both the Laplace and Sumudu integral transforms and established a relationship between double Sumudu transform and double Laplace transforms. Eshag (2017) de ned Sumudu transform method and used it to solve the one dimensional heat equation and compare the results with the results of double Laplace transform. The double Sumudu transform also studied in (Eltayeb & Kilichman, 2010b; Eltayeb & Kilicman, 2010).

In this paper, we have studied double Sumudu transform with its main properties,studied double Laplace transform and its relation with double Sumudu transform and applied the double Sumudu transform for solving the heat and wave equations among other illustrative examples.

1.1 Definition

The double Sumudu transform can be de ned by (Eltayeb & K l çman, 2010):

$$F(u;v) = S_2[f(t;x);(u;v)] = \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-(\frac{t}{v}+\frac{x}{u})} f(t;x)\,dt\,dx$$

1.2 Properties of double Sumudu transform

1. The double Sumudu transform of the second partial derivative with respect to x has a form (Eltayeb & K l çman, 2010):

$$S_2\left[\frac{\partial^2 f(x;t)}{\partial x^2}\right] = \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-(\frac{t}{v}+\frac{x}{u})}\frac{\partial^2 f(x;t)}{\partial x^2}\,dt\,dx = \frac{1}{v}\int_0^\infty e^{-\frac{t}{v}}\left(\frac{1}{u}\int_0^\infty e^{-\frac{x}{u}}\frac{\partial^2 f(x;t)}{\partial x^2}\,dx\right)dt$$

The integral inside the brackets can be computed individually as follows:

$$\frac{1}{u}\int_0^\infty e^{-\frac{x}{u}}\frac{\partial^2 f(x;t)}{\partial x^2}\,dx = \frac{1}{u^2}F(u;t) - \frac{1}{u^2}f(t;0) - \frac{1}{u}\frac{\partial f(0;t)}{\partial x}$$

By taking Sumudu transform with respect to t , we get the double Sumudu transform as follows:

$$S_2\left[\frac{\partial^2 f(x;t)}{\partial x^2};(u;v)\right] = \frac{1}{u^2}F(u;v) - \frac{1}{u^2}f(0;v) - \frac{1}{u}\frac{\partial f(0;v)}{\partial x}$$

By the same way,

2. The double Sumudu transform of $\frac{\partial^2 f(t;x)}{\partial t^2}$ can be given by

$$S_2\left[\frac{\partial^2 f(t;x)}{\partial t^2};(u;v)\right] = \frac{1}{v^2}F(u;v) - \frac{1}{v^2}f(u;0) - \frac{1}{v}\frac{\partial f(u;0)}{\partial t}$$

3. The double Laplace transform was de ned by (Estrin and Higgins,1951)by the following form:

$$L_xL_t[f(x;t);(p;s)] = F(p;s) = \int_0^{\infty} e^{\;px} \int_0^{\infty} e^{\;st} f(x;t)dtdx$$

were $x; t > 0$ and $p; s$ are complex values. They de ned the rst-order partial derivative as :

$$L_xL_t[\frac{\partial f(x;t)}{\partial x};(p;s)] = pF(p;s) \quad f(0;s)$$

4. The double Laplace transform for the second-order partial derivative with respect to x can be given by:

$$L_xL_t[\frac{\partial^2 f(x;t)}{\partial x^2};(p;s)] = p^2F(p;s) \quad p\,f(0;s)\;\frac{\partial f(0;s)}{\partial x}$$

By the same way,

5. The double Laplace transform for the second order partial derivative with respect to x can be given by

$$L_xL_t[\frac{\partial^2 f(x;t)}{\partial t^2};(p;s)] = s^2F(p;s) \quad s\,f(0;s)\;\frac{\partial f(p;0)}{\partial t}$$

6. The double Laplace transform of a mixed partial derivatives can be deduced from the single Laplace transform as below:

$$L_xL_t[\frac{\partial f(x;t)}{\partial x\partial t};(p;s)] = psF(p;s) \quad p\,f(p;0) \quad s\,f(0;s) \quad f(0;0)$$

.

2 The Relation between double Laplace and double Sumudu transforms

The double Laplace and Sumudu transforms have a strong relation may be expressed either as:

1-

$$uvF(u;v) = L_2(f(x;y);(\frac{1}{u};\frac{1}{v}))$$

or

$$psF(p;s) = L_2(f(x;y);(\frac{1}{p};\frac{1}{s}))$$

whereL_2 represent the operation of double Laplace transform.In fact this relation is best illustrated by the fact that these two transforms interchange the images of $sin(x+t)$ and $cos(x+t)$

$$S_2[sin(ax+bt)] = L_2[cos(ax+bt)] = \frac{au+bv}{(1+v^2b^2)(1+u^2a^2)}$$

and

$$S_2[cos(ax+bt)] = L_2[sin(ax+bt)] = \frac{1 \quad vbau}{(1+v^2b^2)(1+u^2a^2)}$$

We note that the relationship between double Sumudu of convolution and double Laplace transform of convolution is:

$$S_2[(f\ \ g)(t;x);(u;v)] = \frac{1}{uv} L_2(fg)(t;x)$$

where

$$F_1(x;y)\ \ F_2(x;y) = \int_0^y \int_0^x F_1(x\ \ q_1; y\ \ q_2) F_2\, dq_1 dq_2$$

Table 1: Double Sumudu transform

f(x,t)	F(u,v)	F(p,s)
1	1	$\frac{1}{ps}$
xt	uv	$\frac{1}{(ps)^{1+1}}$
$(xt)^n$	$n!^2(uv)^n$	$\frac{n!^2}{(ps)^{n+1}}$
e^{ax+bt}	$\frac{1}{(1\ bv)(1\ au)}$	$\frac{1}{(p\ a)(s\ b)}$
$sin(ax+bt)$	$\frac{au+bv}{(1+v^2b^2)(1+u^2a^2)}$	$\frac{as+pb}{(s^2+b^2)(p^2+a^2)}$
$cos(ax+bt)$	$\frac{1\,vbau}{(1+v^2b^2)(1+u^2a^2)}$	$\frac{ps\ \ ab}{(s^2+b^2)(p^2+a^2)}$
$sinh(ax+bt)$	$\frac{1}{2}[\frac{1}{(1\ au)(1\ bv)}\ \ \frac{1}{(1+au)(1+bu)}]$	$\frac{1}{2}[\frac{1}{(s\ b)(p\ a)}\ \ \frac{1}{(s+b)(p+a)}]$
$cosh(ax+bt)$	$\frac{1}{2}[\frac{1}{(1\ au)(1\ bv)}+\frac{1}{(1+au)(1+bu)}]$	$\frac{1}{2}[\frac{1}{(s\ b)(p\ a)}+\frac{1}{(s+b)(p+a)}]$

3 Partial Defferential Equations

Example 3.1. *Solve the following linear rst order partial differential equation*

$$au_x + bu_y = 0; \tag{1}$$

With initial and boundary conditions:

$$u(x;0) = f(x); \quad x > 0; \quad u(0;y) = 0; \quad y > 0$$

Applying double Sumudu transform Equation (1) gives

$$aS_2[u_x] + S_2[u_y] = 0$$

or,

$$a[\frac{\bar{u}(u;v)\ \ S[u(0;y)]}{u}] + b[\frac{\bar{u}(u;v)\ \ S[u(x;0)]}{v}]$$

or,

$$\bar{u}(u;v)(\frac{av+bu}{uv}) = \frac{b\bar{f}(u)}{v}$$

or,

$$\bar{u}(u;v) = f(\bar{u})\ \frac{bu}{av+bu}$$

or,

$$\bar{\bar{u}}(u;v) = \bar{f}(u)\frac{1}{bu\frac{a}{v}+1} \tag{2}$$

where

$$\bar{f}(u) = S[u(x;0)] = S[f(x)]$$

The inverse sumudu transform to Equation (2) with respect to (v) gives

$$\bar{u}(u;y) = \bar{f}(u)e^{\frac{a}{bu}y}$$

The inverse transform to Equation (2) with respect to (u) yields the solution

$$u(u;v) = S^{1}[\bar{f}(u)e^{\frac{a}{bu}y}] = f(x)\quad d(x\quad \frac{ay}{b})$$

. by the convolution theorem,

$$u(u;v) = \int_0^x f(x\quad t)d(t\quad \frac{ab}{b})dt = f(x\quad \frac{ay}{b})$$

Example 3.2. *Solve the following rst-order partial differential equation*

$$u_x = u_y; \tag{3}$$

with initial and boundary conditions:

$$u(x;0) = f(x);\quad x > 0;\quad u(0;y) = g(y)\quad ; y > 0:$$

The application of double Sumudu transform to Equation (3) gives

$$S_2[u_x] = S_2[u_y]$$

or,

$$[\frac{\bar{u}(u;v)\quad S[u(0;y)]}{u}] = [\frac{\bar{u}(u;v)\quad S[u(x;0)]}{v}]$$

or,

$$v\bar{u}(u;v)\quad v\bar{f}_1(v) = u\bar{u}(u;v)\quad u\bar{f}_2(u)$$

or,

$$\bar{u}(u;v)(u\quad v) = (u\quad v)\quad \bar{f}_1(v)\bar{f}_2(u)$$

or,

$$\bar{\bar{u}}(u;v) = \bar{f}(v)\bar{f}(u)$$

where

$$\bar{f}_1(v) = S[u(0;y)] = S[g(y)] \tag{4}$$

$$\bar{f}_2(u) = S[u(x;0)] = S[f(x)]$$

Thus,by taking the inverse Equation (4), we have

$$u(x;y) = S^{1}[\bar{u}(u;v)] = S^{1}[\bar{f}_1(v)\bar{f}_2(u)]$$

In particular, if $u(x;0) = 1$ *and* $u(0;y) = 1$, *so that* $f_1(v) = 1$ *and* $f_2(u) = 1$,

then, $\bar{u}(u;v) = 1$

Thus,the inverse of the double Sumudu transform gives the

solution $u(x;y) = S^{1}[1] = 1$

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 4 Issue 1, January – 2020, Pages: 20-27

Example 3.3. *Fourier's Heat Equation in a Quarter Plane The standard heat equation is*

$$
\begin{aligned}
u_t &= k u_{xx}; \quad x > 0; \quad t > o; \\
u(x; o) &= 0; \quad u(0; t) = 2T_0; \quad x > 0; \quad t > 0; \\
u_x(0; t) &= 0; \quad t > 0; \quad u(x; t) \to 0 \ \text{as} \ x \to \infty
\end{aligned} \tag{5}
$$

where T_0 is a constant
We apply the double Sumudu transform Equation (5)to

$$\text{obtain } S_2(u_t) = kS_2(u_{xx})$$

$$\frac{\bar{u}(u; v) - S[u(x; 0)]}{v} = k\left[\frac{\bar{u}(u; v) - Su(o; t)}{u^2} - \frac{S[u_x(0; t)]}{u}\right]$$

Hence,

$$
\begin{aligned}
\frac{\bar{u}(u; v)}{v} &= \frac{k\bar{u}(u; v) - 2kT_0}{u^2} \\
\bar{u}(u; v)\left(\frac{kv - u^2}{u^2 v}\right) &= \frac{2kT_0}{u^2} \\
\bar{u}(u; v) &= 2T_0 \frac{kv}{kv - u^2}
\end{aligned} \tag{6}
$$

The inverse of the double Sumudu transform to Equation (6) gives

$$
\begin{aligned}
u(x; t) &= 2T_0 S_2^{-1}\left[\frac{kv}{kv - u^2}\right] \\
&= 2T_0 S^{-1}[\cosh \sqrt{kv}\, x] \\
&= 2T_0 \left[\frac{e^{\sqrt{kv}\,x} + e^{-\sqrt{kv}\,x}}{2}\right] \\
&= T_0 S^{-1}[e^{\sqrt{kv}\,x} + e^{-\sqrt{kv}\,x}]
\end{aligned}
$$

The first term above vanishes because of $u(x; t) \to 0$ as $x \to \infty$. Hence,

$$u(x; t) = T_0 S^{-1}[e^{-\sqrt{kv}\,x}]$$

Now, inversion yields the solution

$$u(x; t) = T_0 \, erfc\left(\frac{x}{2\sqrt{kt}}\right) \tag{7}$$

Debnath (2016) solved Equation (5) using Laplace transform with the following solution

$$u(x; t) = T_0 \, erfc\left(\frac{x}{2\sqrt{kt}}\right) \tag{8}$$

Hence, the two solutions in equations (7) and (8) are identical

Example 3.4. *D'Alembert's Wave Equation in a Quarter Plane The standard wave equa-tion is*

$$c^2 u_{xx} = u_{tt}; \quad x > 0; \quad t > 0 \tag{9}$$

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 4 Issue 1, January – 2020, Pages: 20-27

with initial and boundary conditions

$$u(x;0) = f(x); \quad u_t(x;0) = g(x); \quad x > 0$$

$$u(0;t) = 0; \quad u_x(0;t) = 0$$

We apply the double Sumudu transform $\overline{u}(u;v) = S_2[u(x;t)]$ defined by

$$\overline{u}(u;v) = \frac{1}{uv}\int_0^{\infty}\int_0^{\infty} e^{-\left(\frac{x}{v}+\frac{t}{u}\right)} f(x;t)\,dx\,dt;$$

to the wave Equation (9)

$$\begin{aligned}
c^2 S_2[u_{xx}] &= S_2[u_{tt}] \\
c^2\left[\frac{\overline{u}(u;v) - S[u(o;t)]}{u^2} - \frac{S[u_x(0;t)]}{u}\right] &= \frac{\overline{u}(u;v) - S[u(x;0)]}{v^2} - \frac{S[u_t(x;0)]}{v} \\
\frac{c^2\overline{u}(u;v)}{u^2} &= \frac{\overline{u}(u;v) - \overline{f}(u)}{v^2} - \frac{\overline{g}(u)}{v} \\
\overline{u}(u;v)\left(\frac{u^2 - c^2v^2}{u^2v^2}\right) &= \frac{\overline{f}(u) + v\overline{g}(u)}{v^2} \\
\overline{u}(u;v) &= \frac{u^2\overline{f}(u) + vu^2\overline{g}(u)}{u^2 - c^2v^2} \\
\overline{u}(u;v) &= \overline{f}(u)\frac{1}{1 - \frac{c^2v^2}{u^2}} + \frac{u\overline{g}(u)}{c}\,\frac{\frac{cv}{u}}{1 - \frac{c^2v^2}{u^2}}
\end{aligned} \tag{10}$$

where,

$$\overline{g}(v) = S[u_t(0;t)] = S[g(t)]$$

$$\overline{f}(u) = S[u(x;0)] = S[f(x)]$$

Taking the inverse of double Sumudu transform for Equation (10)

$$\begin{aligned}
u(x;t) &= S_2^{-1}\left[\overline{f}(u)\frac{1}{1 - \frac{c^2v^2}{u^2}} + \frac{u\overline{g}(u)}{c}\,\frac{\frac{cv}{u}}{1 - \frac{c^2v^2}{u^2}}\right]; \\
&= S^{-1}\left[\overline{f}(u)\cosh\frac{c}{u}t\right] + S^{-1}\left[\frac{u\overline{g}(u)}{c}\sinh\frac{c}{u}t\right]; \\
&= S^{-1}\left[\overline{f}(u)\left(\frac{e^{\frac{ct}{u}} + e^{-\frac{ct}{u}}}{2}\right)\right] + \frac{1}{c}S^{-1}\left[u\overline{g}(u)\left(\frac{e^{\frac{ct}{u}} - e^{-\frac{ct}{u}}}{2}\right)\right]; \\
&= \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] \\
&+ \frac{1}{2c}\left[\int_0^{x+ct} g(t)\,dt - \int_0^{x-ct} g(t)\,dt\right]:
\end{aligned} \tag{11}$$

Hence,

$$u(x;t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(t)\,dt: \tag{12}$$

This is the celebrated D'Alembert solution of the wave equation,where we have used

$$S^{-1}[\bar{f}(u)e^{-\frac{a}{u}}] = f(x+a) \text{ and } S^{-1}[u\bar{g}(u)] = \int_0^t g(t)dt$$

Debnath (2016) solved Equation (9) using Laplace transform with the following solution

$$u(x;t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(t)dt: \quad (13)$$

Hence, the two solutions in equations (12) and (13) are identical

Conflict of Interests

The authors declare that there is no con ict of interests regarding the publication of this paper.

Acknowledgements

The authors would like to thank university of Kufa for supporting this research project.

References

Debnath, L. 2016, The double Laplace transforms and their properties with applications to functional, integral and partial differential equations, International Journal of Applied and Computational Mathematics, 2, 223

Debnath, L., & Bhatta, D. 2006, Integral transforms and their applications (Chapman and Hall/CRC)

Eltayeb, H., & Kilichman, A. 2010a, On double Sumudu transform and double Laplace transform, Malaysian journal of mathematical sciences, 4, 17

. 2010b, On some applications of a new integral transform, International Journal of Math-ematical Analysis, 4, 123

Eltayeb, H., & Kilicman, A. 2010, A note on the Sumudu transforms and differential equa-tions, Applied Mathematical Sciences, 4, 1089

Eltayeb, H., & K l çman, A. 2010, On double Sumudu transform and double Laplace trans-form, Malaysian journal of mathematical sciences, 4, 17

Eshag, M. O. 2017, On Double Laplace Transform and Double Sumudu Transform, Amer-ican Journal Of Engineering Research, 6, 312

K l çman, A., & Eltayeb, H. 2010, A note on integral transforms and partial differential equations, Applied Mathematical Sciences, 4, 109

Kilicman, A., & Gadain, H. 2009, An application of double Laplace transform and double Sumudu transform, Lobachevskii Journal of Mathematics, 30, 214

Watugala, G. 2002, The Sumudu transform for functions of two variables, Mathematical Engineering in Industry, 8, 293