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The Properties of The Cevian Vector on Triangles in the Normed Space With Wilson's Angle.

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***Abstract:** This paper will discuss about expanding the understanding of the Ceva's line in Euclid's space into the normed space. The basic properties of the Ceva's line will be developed in the normed space. The definition of the Ceva vector in normed space consists of three, divide Ceva vector, heavy Ceva vector and high Ceva vector. Finally, the discussion will end by discussing the Stewart theorem.*

Keywords : Normed Space, Triangeles, divide Ceva vector, heavy Ceva vector , high Ceva vector , Stewart theorem.

1. Introduction

The angle between the two vectors in the Euclid $\mathbb{R}^2$ space is well known. In the Euclid space the angle between two vectors is defined using the product of the dot [8]. Furthermore, the angle between the two vectors in the inner product space has also been developed in [7, 11]. Likewise in normed space, angles between two vectors are also known, including angles P, I, g ([1], [2], [3], [4]), Thy angle [2] and Wilson's angle ([5],[6]).

The angle in the normed space discussed in this paper is the Wilson's angle introduced by Valentine and Wayment (1971). The study of Wilson's angle is discussed as follows :

Let $(V, \|.\|)$ be the normed space over the field $\mathbb{R}$, for any $x, y \in V$ is defined as a nonlinear functional :

$$2\langle x, y\rangle := \|x\|^2 + \|y\|^2 - \|x-y\|^2 \quad (1)$$

From the properties of the norm it belongs:

$$\|\|x\| - \|y\||^2 \le \|x-y\|^2$$
$$\Leftrightarrow \|x\|^2 - 2\|x\|\cdot\|y\| + \|y\|^2 \le \|x-y\|^2$$
$$\Leftrightarrow \langle x, y\rangle \le \|x\|\cdot\|y\| \quad (2)$$

Meanwhile :

$$\|x-y\|^2 \le (\|x\| + \|y\|)^2$$
$$\Leftrightarrow \|x-y\|^2 - \|x\|^2 - \|y\|^2 \le 2\|x\|\cdot\|y\|$$
$$\Leftrightarrow -\langle x, y\rangle \le \|x\|\cdot\|y\| \quad (3)$$

From equations (2) and (3) obtained :

$$|\langle x, y\rangle| \le \|x\|.\|y\|, \quad \forall\ x, y \in V \quad (4)$$

fulfill the inequality Cauchy-Schwarz [8]. Wilson's angle is defined as the angle between two vectors x and y that satisfy

$$\angle(x, y) := \arccos\left(\frac{\|x\|^2 + \|y\|^2 - \|x-y\|^2}{2\|x\|\cdot\|y\|}\right) \quad (5)$$

With Wilson's angle the cosine rule is obtained :

$$\|z\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\cdot\|y\|\cos\angle(x, y) \quad (6)$$

Furthermore, from equation (5) sine rules are obtained:

$$\|x\|.\|y\|\sin\angle(x, y) = K \quad (7)$$

With $K = 2\sqrt{s(s-\|x\|)(s-\|y\|)(s-\|z\|)}$ and

$$2s = \|x\| + \|y\| + \|z\|$$

2. MAIN RESULT

As you well know, the triangle in Euclid's space is known as the Ceva line. In triangles in normed space Ceva vectors will also be defined as follows.

Definition. 2.1. Let $(V, \|.\|, \angle_w)$ be, Normed space over the field $\mathbb{R}$. Vector $d \in V$ called the Ceva vector of triangles $\Delta[a_1, a_2, a_3]$ if there is $\alpha \in (0,1)$ such that it fulfills $\alpha a_i + d = a_j$ with $i \neq j$ for some $i, j = 1,2,3$.

Example 2.1.
Let $\Delta[a, b, c]$, $\{a, b, c\}$ be, is the set of sequences contained in

$$\ell^1(\mathbb{R}) = \left\{ (a_n) \subseteq \mathbb{R} \;\middle|\; \sum_{n=1}^{\infty} |a_n| < \infty \right\}$$

and fulfills $a + c = b$ with $(a_n) = (1,0,0,\dots)$, $(b_n) = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, \dots\right)$ and $(c_n) = \left(-\frac{1}{2}, \frac{1}{2}, 0, 0, \dots\right)$. Next, if selected $\alpha = \frac{1}{2}$ will fulfill the equation $\frac{1}{2}b + d = a$ then, obtained $d = \left(\frac{3}{4}, -\frac{1}{4}, 0, \dots,\right)$, as the Ceva vector of triangles $\Delta[a, b, c]$.

Based on its properties, the Ceva vector is divided into three, namely the high Ceva vector, divide Ceva vector and heavy Ceva vector, more details are defined as follows.

Definition. 2.2. Let $(V, \|.\|, \angle_w)$ be, Normed space over the field $\mathbb{R}$. Vector $d \in V$ called the high Ceva vector of $\Delta[a_1, a_2, a_3]$ if there is $\alpha \in (0,1)$ such that it fulfills $\alpha a_i + d = a_j$ with $i \neq j$ and $\angle_W(\alpha a_i, d) = \frac{\pi}{2}$.

Example 2.2.
Suppose the set of continuous real functions :

$$\mathbf{C}([0,1]) = \{ f \mid f := [0,1] \rightarrow \mathbb{R}, \ f \ \text{continuous} \}$$

With norm :

$$\|f\| := \max_{x\in[0,1]}\{|f(x)|\}$$

Let $(\mathbf{C}([0,1]), \|.\|, \angle_w)$ be, norm space is defined $\Delta[a,b,c]$, $\{a,b,c\} \subseteq \mathbf{C}([0,1])$, with ;

$$a(t) := 1 + 4t,$$
$$b(t) := 6,$$
$$c(t) := 5 - 4t.$$
$$\|a\| := \max_{t\in[0,1]}\{|a(t)|\} = 5.$$
$$\|b\| := \max_{t\in[0,1]}\{|b(t)|\} = 6.$$
$$\|c\| := \max_{t\in[0,1]}\{|c(t)|\} = 5.$$

Select the Ceva vector $d(t) := 3 - 4t$ and $\alpha = \frac{2}{3}$, then obtained :

$$\angle_w\left(d, \frac{2}{3}b\right) = arccos\left(\frac{\|d\|^2 + \left\|\frac{2}{3}b\right\|^2 - \|a\|^2}{2\|d\|\left\|\frac{2}{3}b\right\|}\right)$$

$$= arccos\left(\frac{0}{24}\right) = \frac{\pi}{2}$$

Thus the ceva vector d is a high ceva vector.
Similarly the Ceva vector for is defined as follows :

Definition 2.3. Let $(V, \|.\|, \angle_w)$ be, Normed space over the field $\mathbb{R}$. Vector $d \in V$ called the divided Ceva vector $\Delta[a_1, a_2, a_3]$ if there is $\alpha \in (0,1)$ such that it fulfills $\alpha a_i + d = a_j$ with $i \neq j$ and $\angle_W(\alpha a_i, d) = \angle_W(\alpha a_k, d)$.

Example 2.3.
Suppose the set of continuous real functions :

$$\mathbf{C}([0,1]) = \{ f \mid f := [0,1] \rightarrow \mathbb{R}, \ f \ \text{continuous} \}$$

With norm :

$$\|f\| := \max_{x\in[0,1]}\{|f(x)|\}$$

Let $(\mathbf{C}([0,1]), \|.\|, \angle_w)$ be, norm space is defined $\Delta[a,b,c]$, $\{a,b,c\} \subseteq \mathbf{C}([0,1])$, with ;
;

$$a(t) := 1 + 4t,$$
$$b(t) := 6,$$
$$c(t) := 5 - 4t.$$
$$\|a\| := \max_{t\in[0,1]}\{|a(t)|\} = 5.$$
$$\|b\| := \max_{t\in[0,1]}\{|b(t)|\} = 6.$$
$$\|c\| := \max_{t\in[0,1]}\{|c(t)|\} = 5.$$

Select the Ceva vector $d(t) := -3 + 4t$ and $\alpha = \frac{2}{3}$, then obtained :

$$\angle_w(a, d) = arccos\left(\frac{\|a\|^2 + \|d\|^2 - \left\|\frac{2}{3}b\right\|^2}{2\|a\|\|d\|}\right)$$

$$= arccos\left(\frac{3}{5}\right).$$

$$\angle_w(c, -d) = arccos\left(\frac{\|c\|^2 + \|-d\|^2 - \left\|\frac{1}{2}b\right\|^2}{2\|c\|\|-d\|}\right)$$

$$= arccos\left(\frac{3}{5}\right).$$

Because $\angle_w(a, -d) = \angle_w(c, -d)$ then $d(t) := -3 + 4t$, is the divide Ceva vector.

Example 2.4.
Let $\Delta[a,b,c]$, $\{a,b,c\}$ be, is the set of sequences contained in ;

$$\ell^1(\mathbb{R}) = \left\{ (a_n) \subseteq \mathbb{R} \;\middle|\; \sum_{n=1}^{\infty} |a_n| < \infty \right\}$$

such that it fulfills $a + c = b$ with $(a_n) = (1,0,0,\dots)$, $(b_n) = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, \dots\right)$ and $(c_n) = \left(-\frac{1}{2}, \frac{1}{2}, 0, 0, \dots\right)$. Next, if selected $\alpha = \frac{1}{2}$ then : was obtained $d = \left(\frac{3}{4}, -\frac{1}{4}, 0, \dots,\right)$, as a Ceva vector obtained of $\Delta[a,b,c]$.

$$\|a\| = 1.$$
$$\|b\| = 1.$$
$$\|c\| = 1.$$
$$\|d\| = 1.$$

$$\angle_w(a, d) = arccos\left(\frac{\|a\|^2 + \|d\|^2 - \left\|\frac{1}{2}b\right\|^2}{2\|a\|\|d\|}\right)$$

$$= arccos\left(\frac{7}{16}\right).$$

$$\angle_w(c, -d) = arccos\left(\frac{\|c\|^2 + \|-d\|^2 - \left\|\frac{1}{2}b\right\|^2}{2\|c\|\|-d\|}\right)$$

$$= arccos\left(\frac{7}{16}\right).$$

Because $\angle_w(a, d) = \angle_w(c, -d)$ then $(d_n) := \left(\frac{3}{4}, -\frac{1}{4}, 0, \dots,\right)$, is the vector **Ceva divide**.

Definition 2.4. Let $(V, \|.\|, \angle_w)$ be, Normed space over the field $\mathbb{R}$. Vector $d \in V$ called the heavy Ceva vector of

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$\Delta[a_1, a_2, a_3]$ if there is $\alpha \in (0,1)$ such that it fulfills $\alpha a_i + d = a_j$ with $i \neq j$ and $\alpha = \frac{1}{2}$.

Example 2.5

Let $\Delta[a,b,c]$ be, $\{a,b,c\} \subseteq \boldsymbol{L^3}([0,1])$ such that it fulfills $a + c = b$ with $a(t) = t^3$, $b(t) = t^2$ and $c(t) = t^2 - t^3$, then choose $\alpha = \frac{1}{2}$ dan vektor Ceva $d(t) = t^2 - \frac{1}{2}t^3$ so that each norm is obtained :

$$\|a\| = \left(\int_0^1 |t^3|^3\, dt\right)^{\frac{1}{3}} = \left(\int_0^1 t^9\, dt\right)^{\frac{1}{3}} = 0{,}464$$

$$\|b\| = \left(\int_0^1 |t^2|^3\, dt\right)^{\frac{1}{3}} = \left(\int_0^1 t^6\, dt\right)^{\frac{1}{3}} = 0{,}523$$

$$\|c\| = \left(\int_0^1 |t^2 - t^3|^3\, dt\right)^{\frac{1}{3}} = \left(\int_0^1 (t^6 - 3t^7 + 3t^8 - t^9)\, dt\right)^{\frac{1}{3}} = 0{,}106$$

$$d(t) = t^3 - \frac{1}{2}t^2 \quad , \qquad \text{heavy Ceva vector}$$

Theorem 2.1. (development of Stewart's Theorem).

Let d be, is the Ceva vector of $\Delta[a,b,c]$, then the rules apply :

$$\|d\|^2 = \frac{\|(1-\alpha)a\|\|b\|^2 + \|\alpha a\|\|c\|^2}{\|(1-\alpha)a\| + \|\alpha a\|} - \|\alpha a\|\|(1-\alpha)a\|$$

With $\alpha \in (0,1)$.

Proof .

Let d be, is the Ceva vector of $\Delta[a,b,c]$ means there are $\alpha \in (0,1)$ such that it fulfills $\alpha a + d = b$.

Note that equation :

$a + c = b \quad \Leftrightarrow \quad \alpha a + (1-\alpha)a + c = b$, with $0 < \alpha < 1$

$$\cos\angle_W(d, -\alpha a) = \frac{\|d\|^2 + \|\alpha a\|^2 - \|b\|^2}{2\|d\|\|xa\|}$$

$$\cos\angle_W(d, (1-\alpha)a) = \frac{\|d\|^2 + \|(1-\alpha)a\|^2 - \|c\|^2}{2\|d\|\|(1-\alpha)a\|}$$

Clear that $\angle_W(d, -\alpha a) + \angle_W(d, (1-\alpha)a) = \pi$. The result is obtained :

$$\Leftrightarrow \cos\angle_W(d, -\alpha a) + \cos\angle_W(d, (1-\alpha)a) = 0$$

$$\Leftrightarrow \frac{\|d\|^2 + \|\alpha a\|^2 - \|b\|^2}{2\|d\|\|\alpha a\|} + \frac{\|d\|^2 + \|(1-\alpha)a\|^2 - \|c\|^2}{2\|d\|\|(1-\alpha)a\|} = 0$$

$$\Leftrightarrow \|(1-\alpha)a\|(\|d\|^2 + \|\alpha a\|^2 - \|b\|^2) + \|\alpha a\|(\|d\|^2 + \|(1-\alpha)a\|^2 - \|c\|^2) = 0.$$

$$\begin{aligned}\Leftrightarrow \|(1-\alpha)a\|\|b\|^2 + \|\alpha a\|\|c\|^2 &= \|(1-\alpha)a\|\|d\|^2 + \|(1-\alpha)a\|\|\alpha a\|^2 + \|\alpha a\|\|d\|^2 + \|\alpha a\|\|(1\alpha)a\|^2 \\ &= \|d\|^2(\|(1-\alpha)a\| + \|\alpha a\|) + \|\alpha a\|\|(1-\alpha)a\|(\|\alpha a\| + \|(1-\alpha)a\|) \\ &= (\|(1-\alpha)a\| + \|\alpha a\|)(\|d\|^2 + \|\alpha a\|\|(1-\alpha)a\|)\end{aligned}$$

$$\|d\|^2 = \frac{|1-\alpha|\|a\|\|b\|^2 + |\alpha|\|a\|\|c\|^2}{|1-\alpha|\|a\| + |\alpha|\|a\|} - |\alpha|\|a\||1-\alpha|\|a\| .$$

Special shape of Cevian vector for $\alpha = \frac{1}{2}$ then :

$$\begin{aligned}\|d\|^2 &= \frac{\frac{1}{2}\|a\|\|b\|^2 + \frac{1}{2}\|a\|\|c\|^2}{\frac{1}{2}\|a\| + \frac{1}{2}\|a\|} - \frac{1}{2}\|a\|\frac{1}{2}\|a\| \\ &= \frac{\frac{1}{2}\|a\|(\|b\|^2 + \|c\|^2)}{\|a\|} - \frac{1}{4}\|a\|^2 \\ &= \frac{2\|b\|^2 + 2\|c\|^2 - \|a\|^2}{4}\end{aligned}$$

$$4\|d\|^2 = 2\|b\| + 2\|c\| - \|a\|$$

■

The following is given an example in the space $\mathbf{C}([0,1])$, which explains that for each triangle $\Delta[a,b,c]$. Select Ceva vectors d , such that the expansion of Stewart's theorem applies.

Example 2.6.

Suppose the set of continuous real functions :

$$\mathbf{C}([0,1]) = \{\ f \mid f := [0,1] \to \mathbb{R},\ f \text{ continuous } \}$$

With norm :

$$\|f\| := \max_{x\in[0,1]}\{|f(x)|\}$$

Let $(\mathbf{C}([0,1]), \|\,.\,\|, \angle_w)$ be, norm space is defined $\Delta[a,b,c]$, $\{a,b,c\} \subseteq \mathbf{C}([0,1])$, with;

$$a(t) := 1,$$

$$b(t) := 4,$$
$$c(t) := 3.$$
$$\|a\| := \max_{t\in[0,1]}\{|a(t)|\} = 1.$$
$$\|b\| := \max_{t\in[0,1]}\{|b(t)|\} = 4.$$
$$\|c\| := \max_{t\in[0,1]}\{|c(t)|\} = 3.$$

Select the high Ceva vector $d(t) := -\frac{1}{3}$ and $\alpha = \frac{1}{3}$, then obtained :

$$\|d\| := \max_{t\in[0,1]}\{|d(t)|\} = \frac{1}{3}.$$

from the Theorem **2.1.** obtained :

$$\|d\|^2 = \frac{|1-\alpha|\|a\|\|b\|^2 + |\alpha|\|a\|\|c\|^2}{|1-\alpha|\|a\| + |\alpha|\|a\|} - |\alpha|\|a\||1-\alpha|\|a\|$$

For $\alpha = \frac{1}{3}$ was obtained :

$$= \frac{\frac{4}{3}\cdot 3^2 + \frac{8}{3}\cdot 1^2}{4} - \frac{4}{3}\cdot\frac{8}{3}$$
$$= \frac{1}{9}$$

So obtained $\|d\| = \frac{1}{3}$

Example 2.7.
Let $\Delta[a,b,c]$, $\{a,b,c\}$ be, is the set of sequences contained in ;

$$\ell^1(\mathbb{R}) = \left\{ (a_n) \subseteq \mathbb{R} \;\middle|\; \sum_{n=1}^{\infty} |a_n| < \infty \right\}$$

such that it fulfills $a + c = b$ with $(a_n) = (1,0,0,\dots)$, $(b_n) = (4,0,0,0,\dots)$ and $(c_n) = (3,0,0,\dots)$. Next, if selected $\alpha = \frac{1}{3}$ then obtained $(d_n) = \left(-\frac{1}{3}, 0, \dots,\right)$, as the Ceva vector of $\Delta[a,b,c]$.

$$\|a\| = 1$$
$$\|b\| = 4$$
$$\|c\| = 3$$
$$\|d\| = \frac{1}{3}$$

Using Stewart's theorem :

$$\|d\|^2 = \frac{|1-\alpha|\|a\|\|b\|^2 + |\alpha|\|a\|\|c\|^2}{|1-\alpha|\|a\| + |\alpha|\|a\|} - |\alpha|\|a\||1-\alpha|\|a\|$$

$$= \frac{\frac{4}{3}\cdot 3^2 + \frac{8}{3}\cdot 1^2}{4} - \frac{4}{3}\cdot\frac{8}{3}$$
$$= \frac{1}{9}$$

So obtained : $\|d\| = \frac{1}{3}$

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