International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 4 Issue 2, February – 2020, Pages: 15-28

Convolution of an Extension of Al-Zughair Integral Transform for solving some LPDE'S

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Abstract: *The convolution plays an important role in many different physical applications such as the convolution to Laplace Transform.We offer in this paper the Convolution of an Extension of Al-Zughair Transform and how to be used in extraction of inverse of an Extension of Al-Zughair Transform and solving some type of Partial differential Equations with variable coefficients.*

1.Convolution of Laplace Transform

Definition(1.1)[1]:

The convolution of the two functions $f(x)$and $g(x)$ defined for $x > 0$ plays an important role in a number of different physical applications.

The convolution is given by the integral

$$(f * g)(x) = \int_0^x f(\tau)g(x - \tau)\, d\tau$$

Convolution theorem(1.2)[2]:

Let $f(x)$ and $g(x)$ are piecewise continuous functions on $[0, \infty)$ and of exponential order α , then

$$\mathcal{L}[(f * g)(x,t)] = \mathcal{L}[f(x,t)].\mathcal{L}[g(x,t)]$$

Definition(1.3)[3]:convolution of AL-Zughair Transform

The convolution of Al-Zughair Transform of the two functions $f(lnx)$ and $g(lnx)$ is defined for $lnx\epsilon\ [1, e]$ by

$$(f * x)(lnx) = \int_x^e f(ln\, u)\, g(\frac{lnx}{lnu})\frac{dlnu}{lnu} = \int_x^e f(lnu)\, g\left(\frac{lnx}{lnu}\right)\frac{du}{ulnu}$$

Where $lnu \neq 0 f$ and g are piecewise continuous function on $[1, e]$

2.Convolusion Of an extension of Al-Zughair Transform:

Definition(2.1):

The Convolution of an extension of Al-Zughair Transform of two functions f$(|lnt|)$ *and* $g(|lnt|)$

is defined for $t \in [e, e^{-1}]$ *by:*

International Journal of Engineering and Information Systems (IJEAIS)
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Vol. 4 Issue 2, February – 2020, Pages: 15-28

$(f * g)\ (|lnu|) = 2\int_t^e f(|lnu|)g\left(\left|\frac{lnt}{lnu}\right|\right)\frac{du}{ulnu}$

$$= 2\int_t^e f(|lnu|)g\left(\left|\frac{lnt}{lnu}\right|\right)\frac{du}{ulnu}$$

Where

$lnu \neq 0\ , f\ and\ g\ are\ piecewise\ continuous\ functions\ on\ [e, e^{-1}]$

Theorem(2.2) :

Let $f(|lnt|)$ *and* $g(|lnt|)$ *be two functions . An Extension of Al-Zughair convolution of* $f(|lnt|)$ *and* $g(|lnt|)$ *denoted by EZ[(f * g) (lnt)] is given by the relation*

$EZ[(f * g)\ (lnt)] = EZ[f(|lnt|)]\ .\ EZ[g(|lnt|)]$

Proof:

$EZ[f(|lnt|)]\ .\ EZ[g(|lnt|)] = [\int_{e^{-1}}^{e}\frac{(lnu)^{\mu}}{u}f(|lnu|)du\ ].[\int_{e^{-1}}^{e}\frac{(lnv)^{\mu}}{v}g(|lnv|)dv\ ]$

$$= [\int_{e^{-1}}^{e}(lnu)^{\mu}f(|lnu|)\frac{du}{u}].[\int_{e^{-1}}^{e}(lnv)^{\mu}g(|lnv|)\frac{dv}{v}]$$

$$= \int_{e^{-1}}^{e}[\int_{e^{-1}}^{e}(lnu\ .\ lnv)^{\mu}f(|lnu|)g(|lnv|)dlnv]dlnu$$

Let lnu.lnv=lnt

If $v=e^{-1} \quad \rightarrow \quad lnu\ .(-1) = lnt$

$$u^{-1} = t$$

And if $v=e \quad \rightarrow \quad lnu\ .(1) = lnt \qquad u=t$

Where lnu is fixed in the interior integral

$$\rightarrow \qquad lnu\ .dlnv = dlnt$$

$EZ[f(|lnu|)]\ .\ EZ[g(|lnv|)] = \int_{e^{-1}}^{e}[\int_{u^{-1}}^{u}(lnu\ )^{\mu}f(|lnu|)g(\left|\frac{lnt}{lnu}\right|)\frac{dlnt}{lnu}]dlnu$

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$If\ g(|lnt|) = 0\ \ for\ \ lnt\ <\ lnu^{-1}\quad \rightarrow\quad g\left(\left|\frac{lnt}{lnu}\right|\right) = 0\ \ for\ (lnt < lnu)\ \rightarrow t < u$

$\rightarrow\quad EZ[f(|lnu|)]\ .\ EZ[g(|lnv|)] = \int_{e^{-1}}^{e}[\ \int_{e^{-1}}^{e}(lnu\ )^{\mu}\ f(|lnu|)g(\left|\frac{lnt}{lnu}\right|)dlnt]\ \frac{dlnu}{lnu}$

$$= \int_{e^{-1}}^{e}[\ \int_{e^{-1}}^{e}(lnu\ )^{\mu}\ f(|lnu|)g(\left|\frac{lnt}{lnu}\right|)\ \frac{dlnu}{lnu}]dlnt$$

$$= \int_{e^{-1}}^{e}(lnt)^{\mu}\ [-\int_{e^{-1}}^{t} f(|lnu|)g\left(\left|\frac{lnt}{lnu}\right|\right)\frac{dlnu}{lnu} + \int_{t}^{e} f(|lnu|)g\left(\left|\frac{lnt}{lnu}\right|\right)\frac{dlnu}{lnu}]\ dlnt$$

$$= \int_{e^{-1}}^{e}\frac{(lnt)^{\mu}}{t}[-\int_{e^{-1}}^{t} f(|lnu|)g\left(\left|\frac{lnt}{lnu}\right|\right)\frac{dlnu}{lnu} + \int_{t}^{e} f(|lnu|)g\left(\left|\frac{lnt}{lnu}\right|\right)\frac{dlnu}{lnu}]\ dlnt$$

$= EZ[\ (f * g)\ (|lnu|)]$

3. Properties of the convolution are given as follows:

*1- f * g = g * f , the convolution is commutative*

*2- c (f * g) = cf * g =f * cg , c is constant*

*3- f * (g * h) = (f * g) * h , associative property*

*4- f * (g + h) = (f * g) + (f * h) , distributive property .*

Proof (1) :

$(f * g)\ (|lnu|) = 2\int_{t}^{e} f(|lnu|)g\left(\left|\frac{lnt}{lnu}\right|\right)\frac{dlnu}{lnu}$

$$f\ and\ g\ are\ piecewise\ continuous\ on\ [e^{-1}, e]$$

$Now\ ,\ let\ lnv = \frac{lnt}{lnu}\ \ \rightarrow lnu = \frac{lnt}{lnv}\ \ ,\ lnv \neq 0$

$$lnt = lnu\ .\ lnv\quad \rightarrow lnu\ .\ dlnv + lnv\ .\ dlnu = 0$$

$$\frac{dlnu}{lnu} = -\frac{dlnv}{lnv}\ \ ,\qquad lnu \neq 0\ \ , lnv \neq 0$$

$$if\ u = t\ \ \rightarrow\ \ lnv = \frac{lnt}{lnt} = 1\qquad \rightarrow\ \ v = e$$

$$u = e\ \ \rightarrow\ \ lnv = lnt\quad \rightarrow\ \ v = t$$

$(f * g)\ (|lnu|) = 2\int_{e}^{t} f\left(\left|\frac{lnt}{lnu}\right|\right)g(|lnv|)\frac{-dlnv}{lnv}$

$$= 2\int_{t}^{e} g(|lnv|)\, f\left(\left|\frac{lnt}{lnu}\right|\right)\frac{dlnv}{lnv} = (\,g * f)\,(|lnu|)$$

Proof(2):

$$c(f\ *\ g)\,(|lnt|) = 2c\int_{t}^{e} f(|lnu|)g\left(\left|\frac{lnt}{lnu}\right|\right)\frac{dlnu}{lnu}$$

$$= 2\int_{t}^{e} cf(|lnu|)g\left(\left|\frac{lnt}{lnu}\right|\right)\frac{dlnu}{lnu} = (cf\ *\ g)\,(|lnu|)$$

$by\ the\ same\ method\ we\ can\ prove\ \ c(f\ *\ g)(|lnu|) = (f\ *\ cg)\,(|lnu|)$

Proof (3):

$$f\ *\ (g\ *\ h)(|lnt|) =\ 2\int_{t}^{e} f(|lnu|)(g * h)\left(\left|\frac{lnt}{lnu}\right|\right)\frac{dlnu}{lnu}\ ,\qquad lnu \neq 0$$

$$= 2\int_{t}^{e} f(|lnu|)\left(2\int_{t/u}^{e} g(|lnu|)h\left(\frac{lnt/lnu}{lnv}\right)\frac{dlnv}{lnv}\right)\frac{dlnu}{lnu}\ ,$$

$$lnu \neq o\ ,\qquad lnv \neq 0$$

$$let\ lnv = \frac{ln\tau}{lnu}\ \rightarrow\ dlnv = \frac{dln\tau}{lnu}\ \ ,lnu \neq 0$$

$$= 2\int_{t}^{e} f(|lnt|)\left(2\int_{t}^{u} g(\left|\frac{ln\tau}{lnu}\right|)\,h\left(\left|\frac{lnt}{ln\tau}\right|\right)\frac{dln\tau}{lnu.\,lnv}\right)\frac{dlnu}{lnu}$$

$$= 2\int_{t}^{e}\left(2\int_{\tau}^{e} f(|lnu|)g\left(\left|\frac{ln\tau}{lnu}\right|\right)\frac{dlnu}{lnu}\right)h\left(\left|\frac{lnt}{ln\tau}\right|\right)\frac{dlnu}{lnu}$$

$$= 2\int_{t}^{e} (f * g)(|ln\tau|)\,h\left(\left|\frac{lnt}{ln\tau}\right|\right)\frac{dln\tau}{ln\tau}$$

$$= \big((f * g) * h\big)(|lnt|)$$

Proof(4):

$$f\ *\ (g +\ h)(|lnt|) = 2\int_{t}^{e} f(|lnu|)(g + h)\left(\left|\frac{lnt}{lnu}\right|\right)\frac{dlnu}{lnu}$$

$$= 2\int_{t}^{e} f(|lnu|)\left(g\left(\frac{lnt}{lnu}\right)\frac{dlnu}{lnu} + h\left(\frac{lnt}{lnu}\right)\frac{dlnu}{lnu}\right)$$

$$= 2\int_{t}^{e} f(|lnu|)g\left(\frac{lnt}{lnu}\right)\frac{dlnu}{lnu} + 2\int_{t}^{e} f(|lnu|)h\left(\frac{lnt}{lnu}\right)\frac{dlnu}{lnu}$$

$$= (f * g)(|lnt|) + (f * h)(|lnt|)$$

1. Table of an extension of Al-Zughair integral transform for fundamental functions.

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Function, f(x)	$EZ[f(x)] = \int_{e^{-1}}^{e} \frac{(lnx)^{\mu}}{x} f(\lvert lnt \rvert)dx = F(\mu)$		*Regional of convergence*
K *k is constant*	$= \frac{2k}{\mu + 1}$ $if\, \mu$ *is an even number*	0 $if\, \mu$ *is an odd number*	$\mu > -1$
$(\lvert lnx \rvert)^{n}$ $, n \in R$	$= \frac{2}{\mu + (n+1)}$ $if\, (\mu + n)$ *is an even number*	0 $if\ (\mu + n)$ *is an odd number*	$\mu > -(n+1)$
$(ln(\lvert lnx \rvert))^{n}$	$= \frac{(-1)^{n} 2 * n!}{(\mu + 1)^{n+1}}$ $if\, \mu$ *is an even number*	0 $if\, \mu$ *is an odd number*	$\mu > -1$
$sin(aln(\lvert lnx \rvert))$	$= \frac{-2a}{(\mu + 1)^{2} + a^{2}}$ $if\, \mu \pm a$ *is an even Number*	0 $if\, \mu \pm a$ *is an odd number*	$\mu > -1$
$cos(aln(\lvert lnx \rvert))$	$= \frac{2(\mu + 1)}{(\mu + 1)^{2} + a^{2}}$ $if\, \mu \pm a$ *is an even number*	0 $if\, \mu \pm a$ *is an odd number*	$\mu > -1$
$sinh(aln(\lvert lnx \rvert))$	$= \frac{-2a}{(\mu + 1)^{2} - a^{2}}$ $if\, \mu \pm a$ *is an even Number*	0 $if\, \mu \pm a$ *is an odd number*	$\lvert \mu + 1 \rvert > a$ $a \in \mathbb{R}$
$cosh(aln(\lvert lnx \rvert))$	$= \frac{2(\mu + 1)}{(\mu + 1)^{2} - a^{2}}$ $if\, \mu \pm a$ *is an even number*	0 $if\, \mu \pm a$ *is an odd number*	$\lvert \mu + 1 \rvert > a$ $a \in \mathbb{R}$

2. The table here shows some the derivatives:

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Vol. 4 Issue 2, February – 2020, Pages: 15-28

if μ is an even number	
n=1	$2u(x,1) - (\mu+1)\,EZ[u(x,\lvert lnt\rvert)]$
n=2	$-2(\mu+2)u(x,1) + (\mu+2)(\mu+1)\,EZ[u(x,\lvert lnt\rvert)]$
n=3	$2u_{tt}(x,1) + 2(\mu+3)(\mu+2)u(x,1) - (\mu+3)(\mu+2)(\mu+1)\,EZ[u(x,\lvert lnt\rvert)]$
if μ is an odd number	
n=1	$-(\mu+1)\,EZ[u(x,lnt)]$
n=2	$2u_t(x,1) + (\mu+2)(\mu+1)\,EZ[u(x,\lvert lnt\rvert)]$
n=3	$-2(\mu+3)u_t(x,1) - (\mu+3)(\mu+2)(\mu+1)EZ[u(x,\lvert lnt\rvert)]$

Example (3.1):

To find $EZ^{-1}\left[\frac{1}{(\mu+3)(\mu+5)}\right]$

We have used the usual method

$$\frac{1}{(\mu+3)(\mu+5)} = \frac{A}{(\mu+3)} + \frac{B}{(\mu+5)} = \frac{A\mu+5A+B\mu+3B}{(\mu+3)(\mu+5)}$$

A+B=0

5A+3B=1

$$A = \frac{-1}{2} \qquad , \qquad B = \frac{1}{2}$$

$$EZ^{-1}\left[\frac{1}{(\mu+3)(\mu+5)}\right] = EZ^{-1}\left[\frac{1/2}{(\mu+3)} + \frac{-1/2}{(\mu+5)}\right]$$

$$= \frac{1}{4}(lnt)^2 - \frac{1}{4}(lnt)^4$$

Now , we will use the convolution method we get:

$$EZ^{-1}\left[\frac{1}{(\mu+3)(\mu+5)}\right]=EZ^{-1}\left[\frac{1}{(\mu+3)}*\frac{1}{(\mu+5)}\right]$$

$$=\frac{1}{2}(lnt)^2*\frac{1}{2}(lnt)^4$$

If $f(|lnu|)=\frac{1}{2}(lnt)^2$ *and* $g(|lnt|)=\frac{1}{2}(lnt)^4$

$$(f*g)(|lnu|)=2\int_t^e\frac{1}{2}(lnu)^2\frac{1}{2}\left(\left|\frac{lnt}{lnu}\right|\right)^4\frac{dlnu}{lnu}$$

$$=\frac{1}{2}(lnt)^4\int_t^e(lnu)^{-3}dlnu$$

$$=\frac{1}{2}(lnt)^4\left[\frac{(lnu)^{-2}}{-2}\Big|_t^e\right]$$

$$=\frac{1}{4}(lnt)^2-\frac{1}{4}(lnt)^4$$

Example (3.2):

To find $EZ^{-1}\left[\frac{1}{(\mu+5)^2(\mu+4)}\right]$

We have used the usual method

$$\frac{1}{(\mu+5)^2(\mu+4)}=\frac{A}{(\mu+5)}+\frac{B}{(\mu+5)^2}+\frac{C}{(\mu+4)}$$
$$=\frac{A(\mu+5)(\mu+4)+B(\mu+4)+C(\mu+5)^2}{(\mu+5)^2(\mu+4)}$$

A+C=0

9A+B+10C=0

20A+4B+25C=1

A=-1 , B=-1 , C=1

$$EZ^{-1}\left[\frac{1}{(\mu+5)^2(\mu+4)}\right]=\frac{-1}{(\mu+5)}+\frac{-1}{(\mu+5)^2}+\frac{1}{(\mu+4)}$$

$$=\frac{-1}{2}(lnt)^4+\frac{1}{2}(lnt)^4(ln(lnt))+\frac{1}{2}(lnt)^3$$

Now , we will use the convolution method we get

$$EZ^{-1}\left[\frac{1}{(\mu+5)^2(\mu+4)}\right]=EZ^{-1}\left[\frac{1}{(\mu+5)^2}*\frac{1}{(\mu+4)}\right]$$

$$=\frac{-1}{2}(lnt)^4(ln(lnt))*\frac{1}{2}(lnt)^3$$

If $f(|lnt|)=\frac{-1}{2}(lnt)^4(ln(lnt))$ *and* $g(|lnt|)=\frac{1}{2}(lnt)^3$

$$(f*g)(|lnt|)=2\int_t^e\frac{-1}{2}(lnu)^4\,ln\big((|lnu|)\big)\frac{1}{2}\left(\left|\frac{lnt}{lnu}\right|\right)^3\frac{dlnu}{lnu}$$

$$=-\frac{(lnt)^3}{2}\int_t^e ln|lnu|\,dlnu$$

$$u=ln|lnu| \qquad dv=dlnu$$

$$du=\frac{1}{lnu}dlnu \qquad v=lnu$$

$$=-\frac{(lnt)^3}{2}\left[lnu\,(ln(|lnu|))\Big|_t^e-\int_t^e dlnu\right]$$

$$=-\frac{(lnt)^3}{4}[-lnt\,(ln(|lnt|))-1+lnt]$$

$$=\frac{(lnt)^4}{2}(ln(|lnt|))+\frac{(lnt)^3}{2}-\frac{(lnt)^4}{2}$$

Example(3.3):

To solve the PDE by an Extension of Al-Zughair transform

$$(lnt)u_t(x,|lnt|)+4\,u(x,|lnt|)=\begin{cases}(lnt)^{-3}(ln(|lnt|)) & if\,\mu\ is\ an\ even\ no.\\ (lnt)^{-5} & if\ \mu\ is\ an\ odd\ no.\end{cases}$$

$$u(x,1)=0$$

If μ is an even number

Take EZ to both sides

$$2u(x,1)-(\mu+1)EZ[u(x,|lnt|)]+4EZ[u(x,|lnt|)]=\frac{-2}{(\mu-2)^2}$$

$$-(\mu+1)EZ[u(x,|lnt|)]+4EZ[u(x,|lnt|)]=\frac{-2}{(\mu-2)^2}$$

$$-(\mu-3)EZ[u(x,|lnt|)]\;=\frac{-2}{(\mu-2)^2}$$

$$EZ[u(x,|lnt|)]=\frac{2}{(\mu-2)^2(\mu-3)}$$

$Take\ EZ^{-1}\ to\ both\ sides$

$$u(x,|lnt|)=EZ^{-1}\left[\frac{2}{(\mu-2)^2(\mu-3)}\right]$$

Now we will get the solution by the convolution method

$$u(x,|lnt|)=EZ^{-1}\left[\frac{2}{(\mu-2)^2}*\frac{1}{(\mu-3)}\right]$$

$$=\frac{-1}{2}(lnt)^{-3}(ln(|lnt|))*(lnt)^{-4}$$

$$u(x,|lnt|)=2\int_t^e\frac{-1}{2}(|lnu|)^{-3}(ln(|lnu|))\left(\left|\frac{lnt}{lnu}\right|\right)^{-4}\frac{dlnu}{lnu}$$

$$=-(lnt)^{-4}\int_t^e(ln(|lnu|))\,dlnu$$

$$u=ln(|lnu|)\qquad\qquad dv=dlnu$$

$$du=\frac{1}{lnu}dlnu\qquad\qquad v=lnu$$

$$= -(lnt)^{-4}\left[lnu\ (ln(|lnu|)\Big|_t^e - \int_t^e dlnu\right]$$

$$= \ (lnt)^{-3}(ln(|lnt|)) - (lnt)^{-4}[-1 + lnt\ ]$$

$$= (lnt)^{-3}(ln(|lnt|)) + (lnt)^{-4} - (lnt)^{-3}$$

if μ is an odd number

$$(lnt)u_t(x,|lnt|) + 4\,u(x,|lnt|) = \ (lnt)^{-5}$$

Take EZ to both sides

$$-(\mu+1)EZ[u(x,|lnt|)] + 4EZ[u(x,|lnt|)] = \frac{2}{(\mu-4)}$$

$$-EZ[u(x,|lnt|)][(\mu+1)-4] = \frac{2}{(\mu-4)}$$

$$EZ[u(x,|lnt|)] = \frac{-2}{(\mu-3)(\mu-4)}$$

Take EZ^{-1} to both sides

$$u(x,|lnt|) = EZ^{-1}\left[\frac{-2}{(\mu-3)(\mu-4)}\right]$$

Now we will get the solution by the convolution method

$$u(x,|lnt|) = EZ^{-1}\left[\frac{-2}{(\mu-3)} * \frac{1}{(\mu-4)}\right]$$

$$= -(lnt)^{-4} * \frac{1}{2}(lnt)^{-5}$$

$$u(x,|lnt|) = 2\int_t^e -(|lnu|)^{-4}\frac{1}{2}\left(\left|\frac{lnt}{lnu}\right|\right)^{-5}\frac{dlnu}{lnu}$$

$$= -(lnt)^{-5}\int_t^e dlnu$$

$$= -(lnt)^{-5}[1 - lnt]$$

$$= -(lnt)^{-5} + (lnt)^{-4}$$

Example (2.2.4):

To find the solution of the partial differential equation

$$(lnt)^2 u_{tt}(x,|lnt|) + (lnt)u_t(x,|lnt|) + u(x,|lnt|) = \begin{cases} cosx\, sinln(|lnt|) & if\ \mu\ is\ an\ even\ no. \\ ln(|lnt|) & if\ \mu\ is\ an\ odd\ no. \end{cases}$$

$\boldsymbol{if \mu\ is\ an\ even\ number}$

$$u(x,1) = 0$$

Take EZ to both sides

$$-2(\mu+2)u(x,1) + (\mu+2)(\mu+1)EZ[u(x,|lnt|)] + 2u(x,1) - (\mu+1)EZ[u(x,|lnt|)] + EZ[u(x,|lnt|)] = \frac{-2cosx}{(\mu+1)^2+1}$$

$$(\mu+2)(\mu+1)EZ[u(x,|lnt|)] - (\mu+1)EZ[u(x,|lnt|)] + EZ[u(x,|lnt|)] = \frac{-2cosx}{(\mu+1)^2+1}$$

$$EZ[u(x,|lnt|)](\mu+1)[(\mu+2)-1] + EZ[u(x,|lnt|)] = \frac{-2cosx}{(\mu+1)^2+1}$$

$$EZ[u(x,|lnt|)][(\mu+1)^2+1] = \frac{-2cosx}{(\mu+1)^2+1}$$

$$EZ[u(x,|lnt|)] = \frac{-2cosx}{[(\mu+1)^2+1][(\mu+1)^2+1]}$$

$take\ EZ^{-1} to\ both\ sides$

$$u(x,|lnt|) = EZ^{-1}\left[\frac{-2cosx}{[(\mu+1)^2+1][(\mu+1)^2+1]}\right]$$

Now, we will get the solution by the convolution method:

$$u(x,|lnt|) = EZ^{-1}\left[\frac{-2cosx}{[(\mu+1)^2+1]}\cdot\frac{1}{[(\mu+1)^2+1]}\right]$$

$$= cosx\ sinln(|lnt|) * \frac{-1}{2} sinln(|lnt|)$$

Now, we will get the solution by the convolution method:

$$= 2\int_t^e cosx\ sinln(|lnu|)\frac{-1}{2} sinln\left(\left|\frac{lnt}{lnu}\right|\right)\frac{dlnu}{lnu}$$

$$= -cosx\int_t^e sinln(|lnu|)[sin(ln\,(|lnt|) - (ln(|lnu|))]\ \frac{dlnu}{lnu}$$

$$= -cosx\int_t^e sinln(|lnu|)\ [sinln(|lnt|)cosln(|lnu|)$$

$$- cosln(|lnt|)sinln(|lnu|)]\frac{dlnu}{lnu}$$

$$= -cosx\ sinln(|lnt|)\int_t^e sinln(|lnu|)cosln(|lnu|)\frac{dlnu}{lnu}$$

$$+ cosx\ cosln(|lnt|)\int_t^e (sinln(|lnu|))^2\frac{dlnu}{lnu}$$

$$= -cosx\ sinln(|lnt|)\frac{(sinln(|lnu|))^2}{2}\Big|_t^e$$

$$+ cosx\ cosln(|lnt|)\int_t^e \frac{1 - cos2ln(|lnu|)}{2}\frac{dlnu}{lnu}$$

$$= cosx\ \frac{(sinln(|lnt|))^3}{2} + cosx\frac{cosln(|lnt|)}{2}\left[ln(|lnu|) - \frac{sin2ln(|lnu|)}{2}\Big|_t^e\right]$$

$$= \frac{cosx}{2}(sinln(|lnt|))^3 - \frac{cosx}{2}ln(|lnu|)\,cosln(|lnt|)$$

$$+ \frac{cosx}{2}sinln(|lnt|)(cosln(|lnt|))^2$$

if μ is an odd number

$$(lnt)^2u_{tt}(x,|lnt|) + (lnt)u_t(x,|lnt|) + u(x,|lnt|) = ln(|lnt|)$$

$$u_t(x,1) = 0$$

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ISSN: 2643-640X
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Take EZ to both sides

$$2u_t(x,1)+(\mu+2)(\mu+1)EZ[u(x,|lnt|)]-(\mu+1)EZ[u(x,|lnt|)] + EZ[u(x,|lnt|)]=\frac{-2}{(\mu+1)^2}$$

$$(\mu+2)(\mu+1)EZ[u(x,|lnt|)]-(\mu+1)EZ[u(x,|lnt|)]+EZ[u(x,|lnt|)] =\frac{-2}{(\mu+1)^2}$$

$$(\mu+1)[(\mu+2)-1]EZ[u(x,|lnt|)]+EZ[u(x,|lnt|)]=\frac{-2}{(\mu+1)^2}$$

$$[(\mu+1)^2+1]EZ[u(x,|lnt|)]=\frac{-2}{(\mu+1)^2}$$

$$EZ[u(x,|lnt|)]=\frac{-2}{(\mu+1)^2[(\mu+1)^2+1]}$$

$take\ EZ^{-1} to\ both\ sides$

$$u(x,|lnt|)=EZ^{-1}\left[\frac{-2}{(\mu+1)^2[(\mu+1)^2+1]}\right]$$

Now, we will get the solution by the convolution method:

$$u(x,|lnt|)=EZ^{-1}\left[\frac{1}{(\mu+1)^2}*\frac{-2}{[(\mu+1)^2+1]}\right]$$

$$=\frac{-1}{2}ln(|lnt|)*sinln(|lnt|)$$

$$u(x,|lnt|)=2\int_t^e \frac{1}{2}ln(|lnu|)sinln\left(\left|\frac{lnt}{lnu}\right|\right)\frac{dlnu}{lnu}$$

$$=\int_t^e ln(|lnu|)\,sin[ln(|lnt|)-ln(|lnu|)]\frac{dlnu}{lnu}$$

$$=\int_t^e ln(|lnu|)\,sinln(|lnt|)\frac{dlnu}{lnu}-\int_t^e ln(|lnu|)\,sinln(|lnu|)\frac{dlnu}{lnu}$$

$$u = ln(|lnu|) \qquad dv = sinln(|lnu|)\frac{dlnu}{lnu}$$

$$du = \frac{1}{lnu}dlnu \qquad v = -cosln(|lnu|)$$

$$= sinln(|lnt|)\int_t^e ln(|lnu|)\frac{dlnu}{lnu} - \left[-ln(|lnu|)\,cosln(|lnu|)\Big|_t^e + \int_t^e cosln(|lnu|)\frac{dlnu}{lnu}\right]$$

$$= sinln(|lnt|)\frac{(ln(|lnu|))^2}{2}\Big|_t^e - ln(|lnt|)\,cosln(|lnt|) - sinln(|lnu|)\Big|_t^e$$

$$= \frac{-1}{2}sinln(|lnt|)(ln(|lnt|))^2 - ln(|lnt|)\,cosln(|lnt|) + sinln(|lnt|)$$

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