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Statistically Convergent Difference -Double Sequence Spaces of Fuzzy Real Numbers Defined by Double Orlicz Function

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Abstract : *In this study we introduced some classes of statistically convergent difference double sequence spaces of fuzzy real numbers defined by double Orlicz functions. Some properties of these sequence spaces like completeness, solidness, symmetricity, convergence-free are studied.*

Keywords : Orlicz functions; double sequence; symmetric space; solid space; convergence-free; completeness.

1. Introduction

The notion of convergence of sequences, statistical convergence of sequences was introduced by Fast [1] and Schoenberg [2] independently. Further it was studied from sequence space point of view and linked with summability theory by Tripathy [4,5], Tripathy and Sen [6],Rath and Tripathy [7], Fridy [8], Kwon [9], Nuray and Savas [10], Šalàt [11], Altin [12] and many others. The earlies works on double sequence of real terms is found in Bromwich [14]. Further the double sequence was investigated by Moricz [15], Basarir and Sonalncan [16], Tripathly and Sarma [17] and many others.

The concept of fuzzy set theory was introduced by Zadeh (1965).Fuzzy probability theory is known as Possibility theory. The notion of statistical convergence of sequences has relationship with possibility theory. The distribution that is used in case of statistical convergence is uniform distribution.The notion of statistical convergence is same as the notion of almost sure convergence of probability theory. The results on almost sure convergence are of single sequence type.

Savas [3] introduced and discussed double convergent sequence of fuzzy numbers and showed that the set of all double convergent sequences of fuzzy numbers is complete. Quite recently, Savas and Mursaleen [13] introduced of statistically convergent and statistically Cauchy for double sequence of fuzzy numbers, many others also discussed these.

Every double sequence is defined as a function $X: N \times N \to \mathbb{R}(\mathbb{C})$. For double sequences, the statistical convergence depends on the density of the subsets of N, the set of natural numbers. Every double sequence is defined as a function $X: N \times N \to R(C)$.For double sequences, the statistical convergence depends on the density of the subsets of $N \times N$. Tripathy [27] introduced the notion of density for subset of $N \times N$, as follows, A subset K of $N \times N$ is said to have *density* $\delta_2(K)$ if

$$\delta_2(K) = \lim_{p,q\to\infty} \frac{1}{pq} \sum_{n\le p} \sum_{k\le q} \mathcal{X}_E(n,k) \text{ exists,}$$

where $\mathcal{X}_K$ is the characteristic function of K. Clearly all-finite subsets of $N \times N$ have zero density, also $\delta_2(K^c) = \delta_2(N \times N - K) = 1 - \delta_2(K)$.

Battor and Neamah [23] defined a double sequence as follows: a double sequence $(X, Y) = (X_{rs}, Y_{rs})$ is a double infinite array of elements (X_{rs}, Y_{rs}), where $X = X_{rs}$ is a double infinite array of elements X_{rs} and $Y = Y_{rs}$ is infinite a double array of elements Y_{rs} , which means (X_{rs}, Y_{rs}) is complex double sequences and they defined a double Orlicz function on double sequence space as follows:

$M:[0,\infty) \times [0,\infty) \to [0,\infty) \times [0,\infty)$ such that $M(X,Y)=(M_1(X), M_2(Y))$ where $M_1:[0,\infty) \to [0,\infty)$ and $M_2:[0,\infty) \to [0,\infty)$, such that M_1 , M_2 be two Orlicz functions which are continuous, non-decreasing, even, convex and satisfy the following conditions:

(i) $M_1(0) = 0$, $M_2(0) = 0 \Rightarrow M(0,0) = (M_1(0), M_2(0)) = (0,0)$

(ii) $M_1(X) > 0, M_2(Y) > 0 \Rightarrow M(X,Y) = (M_1(X), M_2(Y)) > (0,0)$, for all $X, Y > 0$. So, $M(X,Y) > (0,0)$ means that $M_1(X) > 0, M_2(Y) > 0$.

(iii) $M_1(X) \to \infty$, $M_2(Y) \to \infty$ as $X, Y \to \infty \Rightarrow M(X,Y) \to (\infty,\infty)$,as $(X,Y) \to (\infty,\infty)$. So, $M(X,Y) \to (\infty,\infty)$ means that $M_1(X) \to \infty, M_2(Y) \to \infty$.

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Also, a double Orlicz function $M(X,Y) = (M_1(X), M_2(Y))$, satified Δ_2-condition for all values of X, Y, if there exist a constant $k > 0$ such that $M_1(2X) \leq k\, M_1(X)$, $M_2(2Y) \leq k\, M_2(Y)$,for all $X, Y \geq 0$,

$$(i.e)\quad M(2X,2Y) = \left(M_1(2X), M_2(2Y)\right) \leq \left(kM_1(X), k\,M_2(Y)\right)$$
$$= k\left(M_1(X), M_2(Y)\right)$$
$$= k\,M(X,Y), \quad \text{for all } X, Y \geq 0.$$

Remark 1.1 [23] If M is a double Orlicz function, then $M_1(\lambda X) \leq \lambda\, M_1(X), M_2(\lambda Y) \leq \lambda M_2(Y)$, for all $X \geq 0,\ Y \geq 0$ with $0 < \lambda < 1$, therefore $M(\lambda X, \lambda Y) = \left(M_1(\lambda X), M_2(\lambda Y)\right) \leq \left(\lambda M_1(X), M_2(Y)\right) = \lambda M(X,Y)$, for all $(X,Y) \geq (0,0)$,thus $M(\lambda X, \lambda Y) \leq \lambda M(X,Y)$,for all $(X,Y) \geq (0,0)$.

If the convexity of M is replaced by $M(X+Y) \leq M(X) + M(Y)$,then it is called a modulus function [24].

In this paper $2\ell_\infty, 2c, 2c_o$ denote the spaces of absolutely summable, convergent, and null double sequences, consecutively. For a double Orlicz $M(X,Y) = (M_1(X), M_2(Y))$ we define, a double sequence $(X,Y) = (X_{rs}, Y_{rs})$ is said to be *statistically convergent* to the number (L_1, L_2), if for every $\varepsilon > 0$, we have

$$\delta_2(\{(r,s) \in N \times N : |X_{rs} - L_1| \geq \varepsilon\}) = 0,$$

and

$$\delta_2(\{(r,s) \in N \times N : |Y_{rs} - L_2| \geq \varepsilon\}) = 0,$$

and accordingly, $\delta_2(\{(r,s) \in N \times N : (|X_{rs} - L_1|, |Y_{rs} - L_2|) \geq \varepsilon\}) = 0$, we write $(X_{rs}, Y_{rs}) \xrightarrow{stat} (L_1, L_2)$ or $stat - \lim_{n,k\to\infty}(X_{rs}, Y_{rs}) = (L_1, L_2)$.

For any two double sequences (X_{rs}, Y_{rs}) and (F_{rs}, H_{rs}), we say that $(X_{rs}, Y_{rs}) \neq (F_{rs}, H_{rs})$ for almost all r, s if $\delta_2(\{(r,s) \in N \times N : (X_{rs}, Y_{rs}) = (F_{rs}, H_{rs})\}) = 0$.

Let $(X,Y) \in 2w$ and let p be a positive real number. The double sequence (X,Y) is said to be *strongly p -Cesàro summable* if there is a complex number L_1, L_2 such that

$$\lim_{u,v\to\infty} \frac{1}{uv} \sum_{r=1}^{u} \sum_{s=1}^{v} |X_{rs} - L_1|^p = 0.$$
$$\lim_{u,v\to\infty} \frac{1}{uv} \sum_{r=1}^{u} \sum_{s=1}^{v} |Y_{rs} - L_2|^p = 0.$$

and accordingly, $\lim_{u,v\to\infty} \frac{1}{uv} \sum_{r=1}^{u} \sum_{s=1}^{v} |(X_{rs}, Y_{rs}) - (L_1, L_2)|^p = 0$. We say that (X_{rs}, Y_{rs}) is *strongly p-Cesàro summable* to (L_1, L_2).

The notion of difference sequence spaces was studied by Kizmaz [18] at the initial stage, who introduced and investigated the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$, for crisp sets. The idea of Kizmaz [18] was applied to introduce different type of difference sequence spaces and study their different properties by Tripathy [5], Tripathy and Sen [6], Tripathy et al.[19], Tripathy and Mahanta [20] and many others. Tripathy and Esi [21] introduced the new type of difference sequence spaces, further Tripathy et al.[22] generalized difference sequence spaces, for $m \geq 1, n \geq 1$,

$$Z(\Delta_m^n) = \{X = (X_k) : (\Delta_m^n X_k) \in Z\}, \quad \text{for } Z = \ell_\infty, c, c_o$$

where

$$\Delta_m^n X_k = \Delta_m^{n-1} X_k - \Delta_m^{n-1} X_{k+m}, \qquad \text{for all } k \in N.$$

which is equivalent to the following binomial representation:

$$\Delta_m^n X_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} X_{k+vm}.$$

we use this idea to generalized the notion of difference double sequence space for a double Orlicz function as follows, for $m \geq 1, n \geq 1$,

$Z(\Delta_m^n) = \{(X,Y) = (X_{rs}, Y_{rs}) : (\Delta_m^n X_{rs}, \Delta_m^n Y_{rs}) \in Z\}$, for $Z = 2\ell_\infty, 2c, 2c_o$, where $\Delta_m^n X_{rs} = \Delta_m^{n-1} X_{rs} - \Delta_m^{n-1} X_{r+1,s} - \Delta_m^{n-1} X_{r,s+1} + \Delta_m^{n-1} X_{r+1,s+1}$, for all $r, s \in N$,

$$\Delta_m^n Y_{rs} = \Delta_m^{n-1} Y_{rs} - \Delta_m^{n-1} Y_{r+1,s} - \Delta_m^{n-1} Y_{r,s+1} + \Delta_m^{n-1} Y_{r+1,s+1}, \quad \text{for all } r, s \in N,$$

which implies

$$(\Delta_m^n X_{rs}, \Delta_m^n Y_{rs}) = \big(\Delta_m^{n-1} X_{rs} - \Delta_m^{n-1} X_{r+1,s} - \Delta_m^{n-1} X_{r,s+1} + \Delta_m^{n-1} X_{r+1,s+1}, \Delta_m^{n-1} Y_{rs} - \Delta_m^{n-1} Y_{r+1,s} - \Delta_m^{n-1} Y_{r,s+1} + \Delta_m^{n-1} Y_{r+1,s+1}\big), \text{ for all } r,s \in N.$$

This generalized difference has the following binomial representation:

$$\Delta_m^n X_{rs} = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} (-1)^{k_1+k_2} \binom{n}{k_1}\binom{n}{k_2} X_{r+k_1 m, s+k_2 m} - X_{r+m,s} - X_{r,s+m}$$

and

$$\Delta_m^n Y_{rs} = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} (-1)^{k_1+k_2} \binom{n}{k_1}\binom{n}{k_2} Y_{r+k_1 m, s+k_2 m} - Y_{r+m,s} - Y_{r,s+m}$$

Then we have,

$$(\Delta_m^n X_{rs}, \Delta_m^n Y_{rs}) = \Big(\sum_{k_1=0}^{n} \sum_{k_2=0}^{n} (-1)^{k_1+k_2} \binom{n}{k_1}\binom{n}{k_2} X_{r+k_1 m, s+k_2 m} - X_{r+m,s} - X_{r,s+m}, \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} (-1)^{k_1+k_2} \binom{n}{k_1}\binom{n}{k_2} Y_{r+k_1 m, s+k_2 m} - Y_{r+m,s} - Y_{r,s+m}\Big) \quad \dots (1.1)$$

Throughout this paper $(2\omega)^F$, $(2\ell)^F$, $(2\ell_\infty)^F$ is denote to the classes of *all*, *absolutely summable*, and *bounded* double sequences of fuzzy real numbers, respectively.

2. Preliminaries

A fuzzy real number X is a function $X : R \to I = [0,1]$ associating each real number t, with its grade of membership $X(t)$.We denoted to the class of all fuzzy real numbers by $R(I)$. For $0 < \alpha \leq 1$, the $\alpha -$ level set $X^\alpha = \{t \in R : X(t) \geq \alpha\}$. The 0-level set $X^0 = \{t \in R: X(t) > 0\}$ is the closure of strong 0-cut.

The additive and multiplicative identities of $R(I)$ are denoted by $\bar{0}$ and $\bar{1}$, respectively and the zero sequence of fuzzy real numbers is denoted by $\bar{\theta} = \{\bar{\theta}, \bar{\theta}, \dots\}$.

Let D the set of all closed bounded intervals $X = [v_1, v_2]$ and define $d : D \times D \to R$ by $d(X,Y) = \max\{|v_1 - w_1|, |v_2 - w_2|\}$, where $Y = [w_1, w_2]$. Then clearly (D, d) is a complete metric space. Define a map $\bar{d} : R^2(I) \times R^2(I) \to R$ by

$$\bar{d}\big((X,Y),(F,H)\big) = sup_{0<\alpha\leq 1} d\big((X^\alpha, Y^\alpha), (F^\alpha, H^\alpha)\big),$$

for $(X,Y),(F,H) \in R^2(\mathrm{I})$. Clearly, $\big(R^2(I), \bar{d}\big)$ is a complete metric space. The additive and multiplicative identities of $R^2(I)$ are denoted by $(\bar{0},\bar{0})$ and $(\bar{1},\bar{1})$, respectively and the zero double sequence of fuzzy real numbers is denoted by $\overline{2\theta} = \{(\bar{\theta},\bar{\theta}),(\bar{\theta},\bar{\theta}),\dots\}$

On a double Orlicz function we define a double sequence $(X,Y) = (X_{rs}, Y_{rs})$ of fuzzy real numbers is a double infinite array of elements (X_{rs}, Y_{rs}) for all $r,s \in N$, where $X = X_{rs}$ is a double infinite array of elements X_{rs} and $Y = Y_{rs}$ is infinite a double array of elements Y_{rs}, where $X_{rs}, Y_{rs} \in R(I)$, which means (X_{rs}, Y_{rs}) is double sequence of fuzzy real numbers.

A double sequence $(X,Y) = (X_{rs}, Y_{rs})$ of fuzzy real numbers is said to be *converge* to the fuzzy numbers X_{00}, Y_{00}, if for every $\varepsilon > 0$, there exists r_o, $s_o \in N$ such that $\bar{d}(X_{rs}, X_{00}) < \varepsilon$ and $\bar{d}(Y_{rs}, Y_{00}) < \varepsilon$ for all $r \geq r_o$, $s \geq s_o$, and consequently $\bar{d}\big((X_{rs}, X_{00}),(Y_{rs}, Y_{00})\big) < \varepsilon$, for all $r \geq r_o$, $s \geq s_o$.

A double sequence space $2E$ is said to be *solid* if $(F_{rs}, H_{rs}) \in 2E$, whenever $(X_{rs}, Y_{rs}) \in 2E$ and $|(F_{rs}, H_{rs})| \leq |(X_{rs}, Y_{rs})|$, for all $r,s \in N$.

A double sequence space $2E$ is said to be *symmetric* if $S(X_{rs}, Y_{rs}) \subset 2E$ whenever $(X_{rs}, Y_{rs}) \in 2E$, where $S(X_{rs}, Y_{rs})$ denotes the set of all permutations of the elem- ents of (X_{rs}, Y_{rs}), that is $S(X_{rs}, Y_{rs}) = \{\big(X_{\pi(r),\pi(s)}, Y_{\pi(r),\pi(s)}\big): \pi$ is a permutation of N$\}$. A double sequence space $2E$ is said to be *convergence-free* if $(\mathrm{F}_{rs}, \mathrm{H}_{rs}) \in 2E$ whenever $(X_{rs}, Y_{rs}) \in 2E$ and $(X_{rs}, Y_{rs}) = (\bar{0},\bar{0})$ implies that $(\mathrm{F}_{rs}, \mathrm{H}_{rs}) = (\bar{0},\bar{0})$.

A double sequence space $2E$ is said to be *monotone* If $2E$ contains the canonical pre-images of all its step spaces.

Remark 2.1. A double sequence space $2E$ is *solid* implies that $2E$ is *monotone*.

Lindenstrauss and Tzafriri [25] used the notion of Orlicz function and introduced the sequence space L_M, later Battor and Neamah [23] used that idea to define a double sequence space:

$$2L_M = \left\{(X_{rs}, Y_{rs}) \in 2\omega: \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\left[\left(M_1\left(\frac{|X_{rs}|}{\rho}\right)\right) \vee \left(M_2\left(\frac{|Y_{rs}|}{\rho}\right)\right)\right] < \infty, \text{ for some } \rho > 0\right\},$$

which becomes a Banach space with the norm:

$$\|(X_{rs}, Y_{rs})\|_M = \inf\left\{\rho > 0 : \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\left[\left(M_1\left(\frac{|X_{rs}|}{\rho}\right)\right) \vee \left(M_2\left(\frac{|Y_{rs}|}{\rho}\right)\right)\right] \le 1\right\},$$

which is called a double Orlicz of a double sequence space where 2ω is a family of all $\mathbb{R}^2$ or $\mathbb{C}^2$ double sequence, that is, (X_{rs}) and (Y_{rs}) are real or complex double sequence. They conclusion that the double Orlicz of a double sequence space $2L_M$ be closely related to the space $2L_p$, which is a double Orlicz double sequence space with $M(X,Y) = (M_1(X), M_2(Y)) = (X^p, Y^p)$, for $(1,1) \le (p,p) < (\infty,\infty)$ where $M_1(X) = X^p$, for $1 \le p < \infty$ and $M_2(Y) = Y^p$, for $1 \le p < \infty$.

Tripathy et al. [19], Tripathy and Mahanta [20], Tripathy and Borgohain [26] Tripathy[27] introduced and investigated different classes of Orlicz sequence spaces.

In this paper we introduce the following double sequence spaces, for a double Orlicz function $M(X,Y) = (M_1(X), M_2(Y))$:

$$\overline{2c}(M,\Delta_m^n)^F = \left\{(X_{rs}, Y_{rs}) \in (2\omega)^F: \text{stat-lim}\left[\left(M_1\left(\frac{\bar{d}(\Delta_m^n X_{rs}, \mathrm{L}_1)}{\rho}\right)\right) \vee \left(M_2\left(\frac{\bar{d}(\Delta_m^n Y_{rs}, \mathrm{L}_2)}{\rho}\right)\right)\right] = 0, \text{ for some } \rho > 0,\ \mathrm{L}_1, \mathrm{L}_2 \in R(I)\right\},$$

$$\overline{2c_0}(M,\Delta_m^n)^F = \left\{(X_{rs}, Y_{rs}) \in (2\omega)^F: \text{stat-lim}\left[\left(M_1\left(\frac{\bar{d}(\Delta_m^n X_{rs}, \bar{0})}{\rho}\right)\right) \vee \left(M_2\left(\frac{\bar{d}(\Delta_m^n Y_{rs}, \bar{0})}{\rho}\right)\right)\right] = 0, \text{ for some } \rho > 0\right\}, \text{ and}$$

$$2W(M,\Delta_m^n, p)^F = \left\{\begin{array}{c}(X_{rs}, Y_{rs}) \in (2\omega)^F: \lim_{n,k\to\infty}\frac{1}{nk}\sum_{r=1}^{n}\sum_{s=1}^{k}\left[\left(M_1\left(\frac{\bar{d}(\Delta_m^n X_{rs}, \mathrm{L}_1)}{\rho}\right)\right) \vee \left(M_2\left(\frac{\bar{d}(\Delta_m^n Y_{rs}, \mathrm{L}_2)}{\rho}\right)\right)\right]^p = 0, \\ \text{for some } \rho > 0,\ \mathrm{L}_1, \mathrm{L}_2 \in R(I)\end{array}\right\}$$

Furthermore, we define

$2m^F(M) = \overline{2c}(M,\Delta_m^n)^F \cap 2\ell_\infty(M,\Delta_m^n)^F$,

$2m_0^F(M) = \overline{2c_0}(M,\Delta_m^n)^F \cap 2\ell_\infty(M,\Delta_m^n)^F$.

3. Main Results

Theorem 3.1 The spaces $2m^F(M)$ and $2m_0^F(M)$ are complete metric spaces defined by the metric,

$$g_M\big((X,Y),(F,H)\big) = \sum_{k=1}^{mn}\sum_{l=1}^{mn}\bar{d}\big((X_{kl}, Y_{kl}), (F_{kl}, H_{kl})\big) + \inf\left\{\rho > 0: \sup_{r,s}\left[M_1\left(\frac{\bar{d}(\Delta_m^n X_{rs}, \Delta_m^n Y_{rs})}{\rho}\right) \vee M_2\left(\frac{\bar{d}(\Delta_m^n F_{rs}, \Delta_m^n H_{rs})}{\rho}\right)\right] \le 1\right\},$$

for $M\big((X,Y),(F,H)\big) = \big(M_1(X,Y), M_2(F,H)\big)$, for $(X,Y),(F,H) \in 2m^F(M)$, and $2m_0^F(M)$.

Proof. We prove the result for $2m^F(M)$. Let (X^i, Y^i) be a double Cauchy sequence in $2m^F(M)$, such that $(X^i), (Y^i)$ be a Cauchy sequence in $2m^F(M_1)$, $2m^F(M_2)$, respectively, such that $(X^i) = \left(X_{nq}^i\right)_{n,q=1}^{\infty}$, $(Y^i) = \left(Y_{nq}^i\right)_{n,q=1}^{\infty}$.

Let $\varepsilon > 0$ be given, for a fixed $\eta > 0$, choose $t > 0$ such that $M_1\left(\frac{t\eta}{2}\right) \ge 1$, $M_2\left(\frac{t\eta}{2}\right) \ge 1$, that is, $M\left(\frac{t\eta}{2}, \frac{t\eta}{2}\right) = \left(M_1\left(\frac{t\eta}{2}\right), M_2\left(\frac{t\eta}{2}\right)\right) \ge (1,1)$.

Then there exists a positive integer $n_0(\varepsilon)$ such that $g_{M_1}(X^i, X^j) < \frac{\varepsilon}{t\eta}$, for all $i,j \ge n_0$ and $g_{M_2}(Y^i, Y^j) < \frac{\varepsilon}{t\eta}$, for all $i,j \ge n_0$ and consequently $g_M\left((X^i, X^j),(Y^i, Y^j)\right) < \frac{\varepsilon}{t\eta}$, for all $i,j \ge n_0$. Then By definition of g, we obtain that, $\sum_{k=1}^{mn}\sum_{l=1}^{mn}\bar{d}\left(\left(X_{kl}^i, X_{kl}^j\right), \left(Y_{kl}^i, Y_{kl}^j\right)\right) +$

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$\inf\left\{\rho > 0: \sup_{r,s}\left[M_1\left(\frac{\bar{d}\left(\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs}^j\right)}{\rho}\right) \vee M_2\left(\frac{\bar{d}\left(\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs}^j\right)}{\rho}\right)\right] \leq 1\right\} < \varepsilon,$ ... (3.1), for all $i, j \geq n_0$,which implies

$\sum_{k=1}^{mn}\sum_{l=1}^{mn} \bar{d}\left(\left(X_{kl}^i, X_{kl}^j\right), \left(Y_{kl}^i, Y_{kl}^j\right)\right) < \varepsilon$, for all $i, j \geq n_0$.

$$\Rightarrow \bar{d}\left(\left(X_{kl}^i, X_{kl}^j\right), \left(Y_{kl}^i, Y_{kl}^j\right)\right) < \varepsilon, \ \forall i, j \geq n_0, \quad k, l = 1,2,3, \dots, mn.$$

Hence $\left(X_{kl}^i, Y_{kl}^i\right)$, for $k, l = 1,2,3, \dots, mn$ is a double Cauchy sequence in $R^2(I)$, so it is convergent in $R^2(I)$ by the completeness property of $R^2(I)$.

Let $\lim_{i\to\infty}\left(X_{kl}^i, Y_{kl}^i\right) = (X_{kl}, Y_{kl})$, for $k, l = 1,2,3, \dots, mn$. …(3.2)

Also,

$$\sup_{r,s}\left[M_1\left(\frac{\bar{d}\left(\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs}^j\right)}{\rho}\right) \vee M_2\left(\frac{\bar{d}\left(\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs}^j\right)}{\rho}\right)\right] \leq 1, \ \forall i, j \geq n_0, \quad \dots (3.3)$$

$$\Rightarrow \left[M_1\left(\frac{\bar{d}\left(\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs}^j\right)}{g_{M_1}(X^i, X^j)}\right), M_2\left(\frac{\bar{d}\left(\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs}^j\right)}{g_{M_2}(Y^i, Y^j)}\right)\right] \leq (1,1) \leq \left(M_1\left(\tfrac{t\eta}{2}\right), M_2\left(\tfrac{t\eta}{2}\right)\right),$$

for all $i, j \geq n_0$. By continuity of M so M_1, M_2 , we get

$\bar{d}\left(\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs}^j\right) \leq \frac{t\eta}{2} . g_{M_1}(X^i, X^j)$ and $\bar{d}\left(\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs}^j\right) \leq \frac{t\eta}{2} . g_{M_2}(Y^i, Y^j)$,

for all $i, j \geq n_0$

$\Rightarrow \bar{d}\left(\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs}^j\right) \leq \frac{t\eta}{2} . \frac{\varepsilon}{t\eta} = \frac{\varepsilon}{2}$ and $\bar{d}\left(\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs}^j\right) < \frac{t\eta}{2} . \frac{\varepsilon}{t\eta} = \frac{\varepsilon}{2}$, for all $i, j \geq n_0$

$\Rightarrow \bar{d}\left(\left(\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs}^j\right), \left(\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs}^j\right)\right) < \frac{\varepsilon}{2}$, for all $i, j \geq n_0$.

That implies $(\Delta_m^n X_{rs}^i, \Delta_m^n Y_{rs}^i)$ is a double Cauchy sequence in $R^2(I)$ and so it is convergent in $R^2(I)$ by the completeness property of $R^2(I)$.

Next, we have to prove that, $\lim_i (X^i, Y^i) = (X, Y)$, $(X, Y) \in 2m^F(M)$.

Let, $\lim_i (\Delta_m^n X_{rs}^i, \Delta_m^n Y_{rs}^i) = (F_{rs}, H_{rs})$ (say) in $R^2(I)$, for each $r, s \in N$.

For r, s =1, we get from (1.1) and (3.2),

$$\lim_{i\to\infty}\left(X_{mn+1,mn+1}^i, Y_{mn+1,mn+1}^i\right) = \left(X_{mn+1,mn+1}, Y_{mn+1,mn+1}\right).$$

Proceeding in this way inductively, we get

$$\lim_{i\to\infty}(X_{rs}^i, Y_{rs}^i) = (X_{rs}, Y_{rs}), \ \text{ for each } r, s \in N.$$

Also,

$$\lim_{i\to\infty}(\Delta_m^n X_{rs}^i, \Delta_m^n Y_{rs}^i) = (\Delta_m^n X_{rs}, \Delta_m^n Y_{rs}), \quad \text{for each } r, s \in N.$$

Now, by using the continuity of M, so M_1, M_2 and taking $j \to \infty$ and fixing i, it follows from (3.3),

$\sup_{r,s}\left[M_1\left(\frac{\bar{d}(\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs})}{\rho}\right) \vee M_2\left(\frac{\bar{d}(\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs})}{\rho}\right)\right] \leq 1$, for some $\rho > 0$.

Now, on taking the infimum of such ρ's, we get

$\inf\left\{\rho > 0: \sup_{r,s}\left[M_1\left(\frac{\bar{d}(\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs})}{\rho}\right) \vee M_2\left(\frac{\bar{d}(\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs})}{\rho}\right)\right] \leq 1\right\} < \varepsilon$, for all $i \geq n_0$. Thus, we get

$$\sum_{k=1}^{mn}\sum_{l=1}^{mn} \bar{d}\left(\left(X_{kl}^i, X_{kl}\right), \left(Y_{kl}^i, Y_{kl}\right)\right) + \inf\left\{\rho > 0: \sup_{r,s}\left[M_1\left(\frac{\bar{d}(\Delta_m^n X_{rs}^i, \Delta_m^n X_{rs})}{\rho}\right) \vee M_2\left(\frac{\bar{d}(\Delta_m^n Y_{rs}^i, \Delta_m^n Y_{rs})}{\rho}\right)\right] \leq 1\right\} < \varepsilon + \varepsilon = 2\varepsilon,$$

for all $i \geq n_0$, which implies that $g\left((X^i, X), (Y^i, Y)\right) < 2\varepsilon$, for all $i \geq n_0$.

i.e., $\lim_i (X^i, Y^i) = (X, Y)$.

Now, we show that $(X, Y) \in 2m^F(M)$.

Let $(X^i, Y^i) \in 2m^F(M)$.Then there exists L_{i_1},L_{i_2}, for each $i_1, i_2 \in N$ such that

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stat- $\lim_{r,s\to\infty}\left[\left(M_1\left(\frac{\bar{d}(\Delta_m^n X_{rs},L_{i_1})}{\rho}\right)\right)\vee\left(M_2\left(\frac{\bar{d}(\Delta_m^n Y_{r,s},L_{i_2})}{\rho}\right)\right)\right]=0$, for some $\rho>0$, $L_{i_1},L_{i_2}\in R(I)$. We need to show that,

1. (L_{i_1},L_{i_2}) converges to (L_1,L_2), for $i_1,i_2\to\infty$.
2. stat- $\lim_{r,s\to\infty}\left[\left(M_1\left(\frac{\bar{d}(\Delta_m^n X_{rs},L_1)}{\rho}\right)\right)\vee\left(M_2\left(\frac{\bar{d}(\Delta_m^n Y_{rs},L_2)}{\rho}\right)\right)\right]=0$, for some $\rho>0$, $L_1,L_2\in R(I)$.

Since $(X^i,X),(Y^i,Y)$ are convergent sequence of elements from $2m^F(M)$, so for a given ε > 0, there exists $n_0\in N$ such that, $g_M\left((X^i,X^j),(Y^i,Y^j)\right)<\frac{\varepsilon}{3}$

Again, for given ε > 0, we have

$$\delta_2(B_i)=\delta_2\left(\left\{(r,s)\in N\times N: g_M\left((X^i,L_{i_1}),(Y^i,L_{i_2})\right)<\frac{\varepsilon}{3}\right\}\right)=1,$$

and $\delta_2(B_j)=\delta_2\left(\left\{(r,s)\in N\times N: g_M\left((X^j,L_{j_1}),(Y^j,L_{j_2})\right)<\frac{\varepsilon}{3}\right\}\right)=1.$

Let we take $B=B_i\cap B_j$, then $\delta_2(B)=1$. Choose $(r,s)\in B$. Then for all $i,j\geq n_0$,

$$g_M\left((L_{i_1},L_{j_1}),(L_{i_2},L_{j_2})\right)\leq$$
$$g_M\left((L_{i_1},X^i),(L_{i_2},Y^i)\right)+g_M\left((X^i,X^j),(Y^i,Y^j)\right)+g_M\left((X^j,L_{j_1}),(Y^j,L_{j_2})\right)<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

Since the double sequence (L_{i_1},L_{i_2}) fulfills the Cauchy condition for convergence, it must be convergent to fuzzy real numbers L_1,L_2(say). Thus, $\lim_{i_1,i_2\to\infty}(L_{i_1},L_{i_2})=(L_1,L_2)$.

Let $\tau>0$, we show that $\delta_2(A)=\delta_2(\{(r,s)\in N\times N: g_M((X_{rs},L_1),(Y_{rs},L_2))<\tau\})=1$. Since $(X^n,Y^n)\to(X,Y)$, there exists $d\in N$, such that

$$g_M((X^d,X),(Y^d,Y))<\frac{\tau}{3} \qquad \dots(3.4)$$

We can be chosen the numbers d_1,d_2 in such a way that together with (3.4) we have

$$g_M\left((L_{d_1},L_1),(L_{d_2},L_2)\right)<\frac{\tau}{3}$$

Since stat- $\lim_{r,s\to\infty}\left[\left(M_1\left(\frac{\bar{d}(\Delta_m^n X_{rs}^i,L_{i_1})}{\rho}\right)\right)\vee\left(M_2\left(\frac{\bar{d}(\Delta_m^n Y_{rs}^i,L_{i_2})}{\rho}\right)\right)\right]=0$, we have a subset F of N such that $\delta_2(F)=1$, where $F=\left\{(r,s)\in N\times N: g_M\left((X^d,L_{d_1}),(Y^d,L_{d_2})\right)<\frac{\tau}{3}\right\}$. Therefore, for each $r,s\in F$, we have

$$g_M((X,L_1),(Y,L_2))\leq$$
$$g_M((X,X^d),(Y,Y^d))+g_M\left((X^d,L_{d_1}),(Y^d,L_{d_2})\right)+g_M\left((L_{d_1},L_1),(L_{d_2},L_2)\right)<\frac{\tau}{3}+\frac{\tau}{3}+\frac{\tau}{3}=\tau.$$

This completes the proof. The proof of $2m_0^F(M)$ is similarly. ■

Theorem 3.2 The spaces $\overline{2c}(M,\Delta_m^n)^F$, $\overline{2c_0}(M,\Delta_m^n)^F$, $2m^F(M)$ and $2m_0^F(M)$ are neither *solid*, nor *monotone* in general.

Proof. To prove that we cite the following example.

Example 3.3 Let $m=3$, $n=2$ and $M(X,Y)=\left(M_1(X),M_2(Y)\right)=(|X|,|Y|)$, for all $(X,Y)\in[0,\infty)\times[0,\infty)$. Consider the space $\overline{2c}(M,\Delta_m^n)^F$. Let the double sequence $(X_{rs},Y_{rs})=(\bar{r},\bar{r})$, for all $r,s\in K\subset N$, with $\delta_2(K)=1$.

Then, $\bar{d}\left((\Delta_3^2X_{rs},\bar{0}),(\Delta_3^2Y_{rs},\bar{0})\right)=(0,0)$, for $r,s\in K\subset N$. Hence, we have

$$\text{stat-}\lim_{r,s\to\infty}\left[\left(M_1\left(\frac{\bar{d}(\Delta_3^2X_{rs},\bar{0})}{\rho}\right)\right)\vee\left(M_2\left(\frac{\bar{d}(\Delta_3^2Y_{rs},\bar{0})}{\rho}\right)\right)\right]=0, \text{ for some } \rho>0.$$

That implies, $(X_{rs},Y_{rs})\in\overline{2c}(M,\Delta_3^2)^F$. Now, let (α_{rs},β_{rs}) be a double sequence of scalars defined as follows, for all $s\in N$,

$$(\alpha_{rs},\beta_{rs})=\begin{cases}(1,1), & \text{for } r=2i, i\in N,\\ (0,0), & \text{otherwise.}\end{cases}$$

Now,

$$(\alpha_{rs}X_{rs},\beta_{rs}Y_{rs})(t)=\begin{cases}(\bar{r},\bar{r}), & \text{for } r=2i\text{-}1, i\in N,\\ (0,0), & \text{otherwise.}\end{cases}$$

$$\Rightarrow \quad \bar{d}\left((\Delta\alpha_{rs}X_{rs},\bar{0}),(\Delta\beta_{rs}Y_{rs},\bar{0})\right)=\begin{cases}r, & \text{for } r \text{ odd},\\ r+1 & \text{for } r \text{ even}.\end{cases}$$

Thus $(\alpha_{rs}X_{rs}, \beta_{rs}Y_{rs}) \notin \overline{2c}(M, \Delta_3^2)^F$. Hence, $\overline{2c}(M, \Delta_3^2)^F$ is not *solid* in general.

Similarly, the other spaces can be proven. ■

Theorem 3.4. The spaces $\overline{2c}(M, \Delta_m^n)^F$, $\overline{2c_0}(M, \Delta_m^n)^F$, $2m^F(M)$ and $2m_0^F(M)$ are neither *symmetric* nor *convergence free*.

Proof. The result follows from the following example.

***Example* 3.5.** Consider the space $\overline{2c_0}(M, \Delta_m^n)^F$. Let $m = 1, n = 1$ and $M(X, Y) = \big(M_1(X), M_2(Y)\big) = (X^3, Y^3)$, for all $(X, Y) \in [0, \infty) \times [0, \infty)$.

Consider the double sequence (X_{rs}, Y_{rs}) define by, for all $s \in N$

$(X_{rs}, Y_{rs})(t) = (-(t-r)^2 + 1, -(t-r)^2 + 1)$, for $r = i^2, i \in N$ and $t \in [-(1+r), (1+r)]$.And $(X_{rs}, Y_{rs}) = (\bar{0}, \bar{0})$, otherwise.

Then, for $r = i^2$ and $i^2 - 1, i \in N$,

$(\Delta X_{rs}, \Delta Y_{rs})(t) = (-(t-r)^2 + 1, -(t-r)^2 + 1)$,

for $t \in \left[\frac{-(2r^2+4r+1)}{r^2+r}, \frac{(2r^2+4r+1)}{r^2+r}\right]$ and $(\Delta X_{rs}, \Delta Y_{rs}) = (\bar{0}, \bar{0})$,otherwise. Which shows $(X_{rs}, Y_{rs}) \in \overline{2c_0}(M, \Delta_m^n)^F$.

Now, let (F_{rs}, H_{rs}) be a re-arrangement of (X_{rs}, Y_{rs}) such that

$(F_{rs}, H_{rs})(t) = (-(t-r)^2 + 1, -(t-r)^2 + 1)$, for $r = 2i, i \in N$ and $t \in [-(1+r), (1+r)]$.And $(F_{rs}, H_{rs}) = (\bar{0}, \bar{0})$, otherwise.

Then, for r odd, and for all $s \in N$,

$(\Delta F_{rs}, \Delta H_{rs})(t) = (-(t-r)^2 + 1, -(t-r)^2 + 1)$,

for $t \in \left[\frac{-(2r^2+4r+1)}{r^2+r}, \frac{(2r^2+4r+1)}{r^2+r}\right]$ and $(\Delta F_{rs}, \Delta H_{rs}) = (\bar{0}, \bar{0})$, otherwise.

For r even, and for all $s \in N$,

$(\Delta F_{rs}, \Delta H_{rs})(t) = (-(t-r)^2 + 1, -(t-r)^2 + 1)$,

for $t \in \left[\frac{-(2r^2+4r+1)}{r^2+r}, \frac{(2r^2+4r+1)}{r^2+r}\right]$ and $(\Delta F_{rs}, \Delta H_{rs}) = (\bar{0}, \bar{0})$, otherwise. Which shows $(F_{rs}, H_{rs}) \notin \overline{2c_0}(M, \Delta_m^n)^F$. Hence the space is not *symmetric*.

The proof for convergence-free is similarly. ■

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