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On cub Q-algebra

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Abstract: *In this paper , we will introduced new concept of Q-algebra called cub Q-algebra and we have introduced and illustrated several ideas that evaluate their relationship in Q-algebra and we will study some new results of positive implicative and commutative in Q-algebra .*

Keywords: Q-algebra, cub Q-algebra

1 Introduction

BCK-algebra and BCI-algebra are two classes of abstract algebras introduced by Y. Imai and K. Iseki [4, 3] In 2001 H.S.Kim([1]) introduced a new notion, known as Q-algebra, which is BCH / BCI/ BCK-algebra generalization. .In this paper we examined some important results of the positive implicative , commutative , cub Q-algebra on Q-algebra and we define the binary operation (∧) on Q-algebra and we will clarify the relationship between them.

2 Background

In this section , we recalled the definition of Q-algebra and bounded Q-algebra.

Definition (2.1) [1]
A Q- algebra is a set X with a binary operation $*$ and a constant 0 that fulfilled the following axioms:

1. $x * x = 0 \quad , \quad \forall x \in X.$
2. $x * 0 = x , \qquad \forall x \in X.$
3. $(x * y) * z = (x * z) * y , \quad \forall x, y, z \in X.$

Remark (2.2)[1]
In a Q-algebra X , we can define a binary relation $\leq$ on X by $x \leq y$ if and only if $x * y = 0, \ \forall x, y \in X.$

Definition (2.3) [2]
A Q-algebra $(X, *, 0)$ is called bounded if there is an element $\in X$, that satisfies $x \leq e, \forall x \in X.$
then e is said to be a unit .We denoted $e * x$ by x^* , for each $x \in X$ in bounded Q-algebra.

PROPOSITION(2.4) [5]
If X is a bounded Q-algebra then $0 * x = 0 , \forall x \in X.$

3 Positive Implicative

In this section we define a binary operation ∧ on a Q-algebra and we define positive implicative of Q-algebra and study the relation between them and some result of positive implicative Q-algebra and study positive implicative of bounded Q-algebra and give some result of them also we study commutative concept of positive implicative Q-algebra.

Definition(3.1)
A Q-algebra $(X; *; 0)$ is called positive implicative if it satisfies:

$(x * y) * z = (x * z) * (y * z) , \ \forall x, y, z \in X.$

Example(3.2)
$Let\ X = \{ 0 , a , b, c \}$ be a set with the following table :

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$*$	**0**	a	b	c
0	**0**	**0**	**0**	**0**
a	a	**0**	**0**	**0**
b	b	**0**	**0**	**0**
c	c	c	c	**0**

Then $(X;*;\ 0)$ is a Q-algebra and it is clear that X is positive implicative Q-algebra .

Remark(3.3)
If X is a Q-algebra we define $x \wedge y = (x * y) * y, \quad \forall x, y \in X.$

PROPOSITION(3.4)
In a positive implicative Q-algebra the following axioms are satisfies $\forall x,\ y,\ z \in X$.
1. $x \wedge y = x * y$
2. $e \wedge x = x^*$
3. $x \wedge 0 = x$
4. $x \wedge x = 0$
5. $(x \wedge y) \wedge z = (x \wedge z) \wedge y$
6. $(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z)$
7. $(y \wedge x) \wedge x = y \wedge x$
8. $(x \wedge z) \wedge (y \wedge z) = (x \wedge y) \wedge (z \wedge y)$

PROOF
1. $x \wedge y = (x * y) * y$
$= (x * y) * (y * y)$
$= (x * y) * 0$
$= x * y$

2. $e \wedge x = e * x = x^* \qquad by(1)$

3. $x \wedge 0 = x * 0 = x \qquad by(1)$

4. $x \wedge x = x * x$
$= 0 \qquad by(1)$
5. $(x \wedge y) \wedge z = (x * y) * z$
$= (x * z) * y$
$= (x \wedge y) \wedge z$

6. $(x \wedge y) \wedge z = (x \wedge y) \wedge z \qquad by\ (1)$
$and\ (x * y) * z = (x * z) * (y * z)$
$= (x \wedge z) \wedge (y \wedge z) \quad (X\ is\ positive\ implactive\ )$
$Hance\ (x \wedge y) \wedge z = (x \wedge z) * (y \wedge z)$

7. $(y \wedge x) \wedge x = (y * x) * x$
$= (y * x) * (x * x)$
$= (y * x)$
$= y \wedge x$

8. $since\ (x \wedge y) \wedge z = (x \wedge z) \wedge y \qquad by(5)$
$and\ (x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z) \qquad by(6)$
$and\ (x \wedge z) \wedge y = (x \wedge y) \wedge (z \wedge y) \qquad by(6)$
$Hence\ (x \wedge z) \wedge (y \wedge z) = (x \wedge y) \wedge (z \wedge y)$

PROPOSITION(3.5)
In positive implicative Q-algebra if $x \leq y$ then the following axioms are hold $\forall\ x, y, z \in X$.

1. $(y \wedge x) \wedge y = 0$

2. $If\ x * z \leq y\ then\ 0 \leq z * y \quad \forall x, y, z \in X$

3. $x^* * y = y^*$

4. $y * [(y * x) * y] = y.$

5. $(z * y) * x = z * y$

PROOF

1. $(y \wedge x) \wedge y = (y \wedge y) \wedge (x \wedge y)$

$= 0 \wedge 0$

2. $since\ (x * z) * y = 0$

$= (x * y) * (z * y)$

$= 0 * (z * y)$
$= 0$

$hence\ 0 \leq z * y$

3. $x^* * y = (e * x) * y$

$= (e * y) * (x * y)$

$= (e * y) * 0$

$= e * y$

$= y^*$

4. $y * [(y * x) * y] = y * ((y * y) * (x * y))$

$= y * (0 * 0)$

$= y * 0$
$= y$

5. $(z * y) * x = (z * x) * y$

$= (z * y) * (x * y)$

$= (z * y) * 0$

$= z * y$

$= 0$

REMARK(3.6)
Positive implicative and bounded Q-algebra are independent concepts as in the following examples.

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EXAMPLE(3.7)
Let $X = \{0, a, b, c, d, h\}$ and a binary operation * is defined by

$*$	**0**	a	b	c	d	h
0	**0**	**0**	**0**	**0**	**0**	**0**
a	a	**0**	a	**0**	**0**	**0**
b	b	b	**0**	**0**	**0**	**0**
c	c	**0**	c	**0**	**0**	c
d	d	d	d	d	**0**	d
h	h	b	a	**0**	**0**	**0**

then its clear $(X; *; 0)$ is bounded Q-algebra but not positive implicative, since

$$(c * h) * b = c * b$$
$$= c$$
$$\neq (c * b) * (h * b)$$
$$= c * a$$
$$= 0$$

EXAMPLE(3.8)
Let $X = \{0, 1, 2, 3\}$ and the opertion $*$ is as follows .

$*$	**0**	a	b	c
0	**0**	**0**	**0**	**0**
a	a	**0**	**0**	a
b	b	b	**0**	b
c	c	c	c	**0**

Then $(X, *, 0)$ is a positive implactive Q-algebra , but not bounded Q-algebra
Since $\nexists e \in X \ s.t \ x * e = 0, \ \forall x \in X$.

PROPOSITION (3.9)
If X is a positive implicative bounded Q-algebra then $\forall x, y, z \in X$ the following rustles hold

1. $(x \wedge y) \wedge x = 0$

2. $x \wedge e = 0$

3. $((y * z) * (x * z)) \leq (y * x)$

4. $y * [(y * x) * y] = y$.

PROOF

1. $(x \wedge y) \wedge x = (x \wedge x) \wedge (y \wedge x)$ $\quad (by\ proposition (3.4), 6)$
$= 0 \wedge (y \wedge x)$
$= 0$

2. $x \wedge e = x \wedge e$ $\quad (by\ proposition (3.4), 1)$
$= 0$

3. $\big((y * z) * (x * z)\big) * (y * x) = \big((y * x) * z\big) * (y * x)$ $\quad (X\ is\ Positive\ implicative)$
$= \big((y * x) * (y * x)\big) * (z * (y * x))$
$= 0 * (z * (y * x))$

$= 0$

4. $y * [(y * x) * y] = y * ((y * y) * (x * y))$
$= y * (0 * (x * y))$
$= y * 0$
$= y$

PROPOSITION(3.10)
If X is positive implicative bounded Q-algebra and if $x \leq y$ then

1. $x * z \leq y * z, \qquad \forall x, y, z \in X.$
2. $((y * x) * x) \leq y, \qquad \forall x, y \in X.$

PROOF
1. $(x * z) * (y * z) = (x * y) * z$
$= 0 * z$
$= 0$

2. $((y * x) * x) * y = ((y * x) * y) * (x * y)$
$= ((y * x) * y) * 0$
$= (y * x) * y$
$= (y * y) * x$
$= 0 * x$
$= 0$

DEFINITION(3.11)
A Q-algebra $(X, *, 0)$ is said to be commutative if it satisfies
$\forall x, y \in X, \quad (x * y) * y = (y * x) * x. \quad s.t\,(x \neq 0, y \neq 0) \quad (That\ is\ x \wedge y = y \wedge x)$

EXAMPLE(3.12)

Let $X = \{0, a, b\}$ with the table as follows :

*	**0**	***a***	***b***
0	**0**	***b***	***a***
a	***a***	**0**	***b***
b	***b***	***a***	**0**

Then X is Q-algebra and also is commutative .

REMARK(3.13)
Positive implicative and commutative Q-algebra are independent concepts as in the following example.

EXAMPLE(3.14)
In Example (3.2) , X is positive implicative Q-algebra but not commutative since
$(c * b) * b = c * b = c \neq 0 = (b * c) * c$
and in example (3.12) X is commutative Q-algebra but not positive implicative since

$(a * b) * a = b * a$

$= a$

$\neq (a * a) * (b * a) = 0 * a$

$= b$

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THEOREM(3.15)
If X is commutative positive implicative Q-algebra then $x * y = y * x \quad \forall x, \in X$

PROOF :

$Let\ x, y \in X\ then$
$since\ x \wedge y = y \wedge x$
$then\ x * y = y * x$

4 cub Q-algebra

In this section , we presented the concept of cub Q-algebra in Q-algebra. And we showed that not every Q-algebra is cub Q-algebra , and gave us some theories , examples , and connections a among them.

Definition(4.1)
A Q-algebra $(X, *, 0)$ is called cub Q-algebra if $(X, \wedge, 0)$ is a Q-algebra.

Example (4.2)

Let $X = \{0, a, b, c\}$ be a Q-algebra with the table :

$*$	**0**	***a***	***b***	***c***
0	**0**	**0**	**0**	**0**
a	***a***	**0**	**0**	***a***
b	***b***	***b***	**0**	**0**
c	***c***	***c***	***c***	**0**

Then $(X, *, 0)$ is Q-algebra and X is cub Q-algebra since

$\wedge$	**0**	***a***	***b***	***c***
0	**0**	**0**	**0**	**0**
a	***a***	**0**	**0**	***a***
b	***b***	***b***	**0**	**0**
c	***c***	***c***	**0**	**0**

$(X, \wedge . 0)$ is Q-algebra.

REMARK (4.3)

In general not every Q-algebra is cub Q-algebra.

EXAMPLE (4.4)
In Example (3.2.2), X is Q-algebra but not cub Q-algebra because
$1 \wedge 1 = (1 * 1) * 1$
$= 0 * 1$

$= 2 \neq 0$

THEOREM(4.5)
A Q-algebra X is cub Q-algebra iff $0 * x = 0, \qquad \forall x \in X.$
PROOF
$\Rightarrow$ Let X be a cub Q-algebra. Then $x \wedge x = 0.$
Thus $(x * x) * x = 0,$
So $0 * x = 0$.
$\Leftarrow$ Let $0 * x = 0$, then
$1. x \wedge x = (x * x) * x$

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$= 0 * x$

$= 0$

2. $x \wedge 0 = (x * 0) * 0$

$= x * 0$

$= x$

3. $(x \wedge y) \wedge z = (x \wedge z) \wedge y$

$$since\ (x \wedge y) \wedge z = \Big(\big((x * y) * y\big) * z\Big) * z$$
$$= \Big(\big((x * y) * z\big) * y\Big) * z$$
$$= \Big(\big((x * z) * y\big) * z\Big) * y$$
$$= \Big(\big((x * z) * z\big) * y\Big) * y$$
$$= (x \wedge z) \wedge y$$

COROLLARY (4.6)

If X is a Q-algebra and $x * y = x \ \ s.t \ \ x \neq y \ \forall x \in X$ then X is cub Q-algebra .

Proof :

Let $x , y \in X$,

since $x * y = x$ so $0 * y = 0 \ , \forall y \in X$.

Hence X is cub Q-algebra (by above Theorem)

PROPOSITION(4.7)

If X is cub Q-algebra then $0 \wedge x = 0, \quad \forall x \in X.$

PROOF:

$Let\ x \in X. Now , 0 \wedge x = (0 * x) * x$

Since X is cup Q-algebra then $0 * x = 0$ $\big(by\ Theorem(4.5)\big)$.

Now $0 \wedge x = (0 * x) * x = 0 * x = 0$

$Thus\ 0 \wedge x = 0$

COROLLARY (4.8)

If $(X,*,0)$ is cub Q-algebra then $(X,\wedge,0)$ is cub Q-algebra.

PROOF:

It's clear by above proposition.

COROLLARY(4.9)

Every bounded Q-algebra is cub Q-algebra.

PROOF

since X is bounded then $0 * x = 0$ (by Proposition(2.4))

so X is cub Q-algebra (by Theorem (4.5))

REMARK(4.10)

The converse of Corollary (4.9) is not be true in general as show in the example.

EXAMPLE (4.11)

If $X = \{0, a, b, c\}$ define with table below

$*$	**0**	***a***	***b***	***c***
0	**0**	**0**	**0**	**0**

a	***a***	**0**	***a***	**0**
b	***b***	***b***	**0**	***b***
c	***c***	***c***	***c***	**0**

Then its clear $(X, *, 0)$ *is* Q-algebra and X is cub Q-algebra since

∧	**0**	***a***	***b***	***c***
0	**0**	**0**	**0**	**0**
a	***a***	**0**	***a***	**0**
b	***b***	***b***	**0**	***b***
c	***c***	***c***	***c***	**0**

$(X, \wedge, 0)$ is Q-algebra ,but not bounded Q-algebra .

THEOREM(4.12)
Let X be a cub Q-algebra. If $(X, *, 0)$ is bounded then $(X, \wedge, 0)$ is bounded.

PROOF
Let $(X, *, 0)$ be bounded with unit e .

$$\begin{aligned} Now,\quad x \wedge e &= (x * e) * e \\ &= 0 * e \\ &= 0 \end{aligned}$$

Thus $(X, \wedge, 0)$ is bounded .

REMARK(4.13)
the converse of Theorem (4.12) is not true in general as shown in Exam-

ple (4.2), X is bounded cub Q-algebra but not bounded Q-algebra.

The converse of Theorem (4.12) is true in Q-algebra if it is satisfy
$\boldsymbol{((x * y) * y) * y = x * y, \quad \forall x, y \in X}$.
as The following theorem (4 . 14).

THEOREM(4.14)
Let X be a cup Q-algebra and $((x * y) * y) * y = x * y, \quad \forall x, y \in X$ *.Then* $(X, *, 0)$ *is bounded iff* $(X, \wedge, 0)$ *is bounded.*

PROOF
$\Rightarrow$ *by Theorem* (4.12)

$\Leftarrow$ *Let* $(X, \wedge, 0)$ *be bounded then*
$x \wedge e = (x * e) * e = 0$
so $\quad (x * e) * e) * e = 0 * e$
and since $(x * e) * e) * e = x * e \qquad$ (*from the hypothesis*)
And also $0 * e = 0 \qquad$ (*by* Theorem(4.5))
Hence $\quad x * e = 0 \quad$ and this leads to $(X, *, 0)$ is bounded Q-algebra .

THEOREM(4.15) :
If $(X, *, 0)$ is positive implicative Q -algebra then $(X, \wedge, 0)$ is Q-algebra

PROOF :

1. $x \wedge x = 0 \qquad$ *by* (*Proposition*(3.4), 3)
2. $x \wedge 0 = x \qquad$ *by* (*Proposition*(3.4), 2)
3. $(x \wedge y) \wedge z = (x \wedge z) \wedge y$

$$since\ (x \wedge y) \wedge z = \Big(\big((x * y) * y\big) * z\Big) * z$$
$$= \Big(\big((x * y) * z\big) * y\Big) * z$$
$$= \Big(\big((x * z) * y\big) * z\Big) * y$$
$$= \Big(\big((x * z) * z\big) * y\Big) * y$$
$$= (x \wedge z) \wedge y$$

REMARK (4.16)
The converse of theorem (4.15) does not true in general as shows in the following example

EXAMPLE(4.17)
Let $X = \{0, a, b\}$ define the binary operation $*$ by

$*$	**0**	a	b
0	**0**	**0**	**0**
a	a	**0**	**0**
b	b	a	**0**

Then $(X ,* , 0)\ is\ Q - algebra$ and also $(X ,\wedge, 0)\ is\ Q - algebra$ we can show by the table

$\wedge$	**0**	a	b
0	**0**	**0**	**0**
a	a	**0**	**0**
b	b	**0**	**0**

Then $X ,\wedge, 0)$ is Q-algebra but $(X ,* , 0)$ is not positive implicative since

$(b * a) * a = a * a$
$= 0$
$\neq$
$(b * a) * (a * a) = a$

THEOREM(4.18)
If $(X,* ,0)$ is positive implicative Q-algebra then $(X,\wedge, 0)$ is positive implicative Q-algebra.

PROOF:

Since $(X,*, 0)$ is positive implicative Q-algebra
then $x \wedge y = x * y,\ \forall x,\ y \in X.$
then $(X,\wedge, 0)$ is positive implicative Q-algebra .

REMARK (4.19)
The converse of Theorem (4.18) does not true in general as shows in the following example.

EXAMPLE(4.20)
Let $X = \{ 0 , 1 , 2,3 \}$ define the binary operation $*$ by

$*$	**0**	**1**	**2**	**3**
0	**0**	**0**	**0**	**0**
1	**1**	**0**	**0**	**0**

2	**2**	**1**	**0**	**0**
3	**3**	**3**	**3**	**0**

Then $(X, *, 0)$ is Q-algebra and $(X, \wedge, 0)$ is positive implicative as shown in the following table

$\wedge$	**0**	**1**	**2**	**3**
0	**0**	**0**	**0**	**0**
1	**1**	**0**	**0**	**0**
2	**2**	**0**	**0**	**0**
3	**3**	**3**	**3**	**0**

But $(X, *, 0)$ is not positive implicative Q-algebra .

Since $(2 * 1) * 1 = 1 * 1$

$= 0$

$\neq \ (2 * 1) * (1 * 1)$

$= 1 * 0$

$= 1$

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