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Prime Ideal in Q-Algebra

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***Abstract**: In this paper ,we presented a summary of the definition of prime ideal in Q-algebra and what is related to the generated ideal and the finite* $\cap - steructer$ *in Q-algebra.*

.**Keywords**: Fuzzy, Prime, Ideal, Q-Algebra

1 Introduction

BCK-algebra and BCI-algebra are two classes of abstract algebras introduced by Y. Imai and K. Iseki [4, 3] In 2001 H.S.Kim([1]) introduced a new notion, known as Q-algebra, which is BCH /BCI / BCK-algebra generalization. Iseki [5], introduced the concept of prime ideal in commutative BCK-algebras and Palasinski[6] , generalized this definition for any lower BCK-semi lattices. In this paper , we presented a definition of the generated on Q-algebra with an example that fulfills the definition and some proposition of the generated , as well as the effect of the definition of $(\wedge)$ on the generator and some properties that are realized and which are not realized and give some solutions to the proposition that are not realized ,especially when X is commutative bounded Q-algebra ,we also presented the definition of the prime ideal in Q-algebra with an example that fulfills the definition and gave a proposition if the number of element of Q-algebra (n) , then $(n-1)$ is always a prime ideal . Likewise, we dealt with the relationship of intersection with prime ideal , as well as the converse when X is a commutative bounded Q-algebra and give the solution that give an equivalent between the intersection and definition of the prime ideal , then we present the relationship of the prime ideal to the generated. We also presented the definition of the finite $\cap -$ stricter on the Q- algebra of and gave some of its proposition and its relationship to the prime ideal .

2 Background

In this section , we recalled the definitions of Q-algebra , bounded Q-algebra , commutative , and ideal in Q-algebra , and some of the features we need in the paper .

Definition (2.1) [1]
A Q- algebra is a set X with a binary operation $*$ and a constant 0 that fulfilled the following axioms:

1. $x * x = 0, \quad \forall x \in X.$
2. $x * 0 = x, \quad \forall x \in X.$
3. $(x * y) * z = (x * z) * y, \quad \forall x, y, z \in X.$

Remark (2.2)[1]
In a Q-algebra X , we can define a binary relation $\leq$ on X by $x \leq y$ if and only if $x * y = 0$, $\forall x, y \in X.$

Definition (2.3) [2]
A Q-algebra $(X,*,0)$ is called bounded if there is an element $\in X$, that satisfies $x \leq e, \forall x \in X.$ then e is said to be a unit .We denoted $e * x$ by x^* , for each $x \in X$ in bounded Q-algebra.

Definition(2.4)[7]
a Q-algebra $(X,*,0)$is said to be commutative if it satisfies
$\forall x; y \in X, (x * y) * y = (y * x) * x$ such that $x \neq 0, y \neq 0$ $(That\ is\ x \wedge y = y \wedge x)$.

Definition (2.5) [8]
Let $(X,*,0)$ be a Q-algebra and I be a none empty subset of X . Then I is called an ideal of X if for any $x, y \in X.$

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1. $0 \in I$
2. $x * y \in I \text{ and } y \in I \text{ imply } x \in I$

Proposition(2.6)
Let I be an ideal of a bounded Q-algebra $(X,*,0)$. If $x \in I$ then
$x \wedge y \in I, \quad \forall y \in X.$

Proof:

$Let\ x \in I\ and\ y \in X$
$(x \wedge y) * x = ((x * y) * y) * x$
$= ((x * y) * x) * y$
$= ((x * x) * y) * y$
$= (0 * y) * y$
$= 0 * y$
$= 0$
$Since\ 0 \in I\ and\ x \in I\ then\ x \wedge y \in I$

Corollary(2.7)
If I is an ideal of a commutative Q-algebra $(X ,* , 0)$ and $x \in I$, then $y \wedge x \in I$, $\forall y \in X$.

Proof :
it's clear by above Proposition

Proposition(2.7)
Let $(X,*,0)$ be a bounded Q-algebra and I be an ideal in Q-algebra then if $e \in I$ then $I = X$.
Proof:
Let $x \in X$,
since $e \in I$ and $x * e = 0 \in I$
then $x \in I$ $\quad(\boldsymbol{by\ Definition\ (2.5)})$
Hence $I = X$.

3 Ideal generated by a Set
In this section , we will introduce the definition for the generated ideal in Q-algebra ,
and some of their properties .

Definition (3.1)
Let $(X ,*, 0)$ be a Q-algebra . If $A \subseteq X$, the set $< A >$ can be defined as follows :
$< A >= \cap \{ I : I\ is\ an\ ideal\ of\ X$ contain $A \}$ or the least ideal of X containing A ,
it is called generated by A . If $A = \{x\}$ we call $< A >$
generated by x , and we write $< A > = < x >$.

Example(3.2)
Let $X = \{ 0, a, b, c, \}$ define the binary operation $*$ on X by the following table :

$*$	**0**	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$
0	**0**	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$
$\boldsymbol{a}$	$\boldsymbol{a}$	**0**	$\boldsymbol{c}$	$\boldsymbol{b}$
$\boldsymbol{b}$	$\boldsymbol{b}$	$\boldsymbol{c}$	**0**	$\boldsymbol{a}$
$\boldsymbol{c}$	$\boldsymbol{c}$	$\boldsymbol{b}$	$\boldsymbol{a}$	**0**

Then $(X ,* , 0)$ is a Q-algebra and
$\{X , \{0\} , \{0, a\} , \{0, b\} , \{0, c\}\}$ is the set of all ideals of X.

Let $S1 = \{0, a, b\}$, $S2 = \{b\}$ and $S3 = \{a\}$ be set of X . Then

$< S1 >= X$
$< S2 >= \{0, b\}$
$< S3 >= \{0, a\}$

Proposition(3.3)
Let $(X, *, 0)$ be a bounded Q-algebra and , $B \subseteq X$. Then

1. $A \subseteq < A >$
2. $<< A >>=< A >$
3. $if\ A\ is\ ideal\ then < A >= A\ and << A >>= A$
4. $if\ A \subseteq B\ then < A > \subseteq < B >$
5. $< A > \cup < B > \subseteq < A \cup B >$
6. $if\ e \in A\ then < A >= X$

PROOF
1. by Definition (3.1)
2. Since $< A >$ is ideal then $<< A >>=< A >$
3. by Definition (3.1)
4. Let $\in < A >$. Since $A \subseteq B$,
   then $x \in I$ for any I is an ideal containing B ,
   so $\in < B >$, then $< A > \subseteq < B >$.
5. Since $A \subseteq A \cup B$ $then\ by\ (4)$ $< A > \subseteq < A \cup B >$
   $and\ B \subseteq A \cup B$ $then\ by\ (4)$ $< B > \subseteq < A \cup B >$
   Hence $< A > \cup < B > \subseteq < A \cup B >$
6. Since $e \in A$
   $then\ e \in < A >$ $\boldsymbol{by\ (Definition\ (3.1))}$
   $thus\ < A > = X$ $\boldsymbol{by\ (Proposition (2.7))}$

Proposition(3.4)
Let $(X, *, 0)$ be a bounded Q-algebra then $< x \wedge y > \subseteq < x >$

Proof
$Let\ x \in X.$
$Since < x > is\ an\ ideal\ then\ x \wedge y \in < x >$ $(\boldsymbol{by\ Proposition\ (2.6\ )})$
$hanse < x \wedge y > \subseteq < x >.$

Corollary (3.5)
Let $(X, *, 0)$ be a commutative bounded Q-algebra then $< y \wedge x > \subseteq < x >$

Proof :
it's clear by above proposition

Remark (3.6)
The converse of Proposition (3,4) is not true in general as shown in the following example.

Example(3.7)
Let $X = \{ 0, a, b \}$ and a binary operation * is defined by :

$*$	**0**	$\boldsymbol{a}$	$\boldsymbol{b}$
0	**0**	**0**	**0**
$\boldsymbol{a}$	$\boldsymbol{a}$	**0**	**0**
$\boldsymbol{b}$	$\boldsymbol{b}$	$\boldsymbol{a}$	**0**

Then X is a Q-algebra and also commutative, and
$< a \wedge b >=< (a * b) * b >=< 0 * b >=< 0 >= \{0\}$

and $< b >= X$
$< a \wedge b > \subseteq < b > \; but \; < b >= X \; \nsubseteq < a \wedge b >= \{0\}$.

Corollary (3.8)
Let $(X,*,0)$ be a commutative bounded Q-algebra then $< x \wedge y > \subseteq < x > \cap < y >$
PROOF
clear by **Proposition (3.4)** and **Corollary (3.5)** .

Remark(3.9)
The converse of Corollary (3.8) is not true in general as shown in the following example.

Example(3.10)
In Example (3.7) since $< a \wedge b >=< (a * b) * b >=< 0 * b >=< 0 >= \{0\}$
and
$< a >= X \; also \; < b >= X$
$< a > \cap < b = X$
$but \; < a > \cap < b > \nsubseteq < a \wedge b >$

$< x > \cap < y > \nsubseteq < x \wedge y >$

The converse of Proposition (3.4) and corollary (3.8) is true in commutative bounded Q-algebra if it $\boldsymbol{satisfy \; x * y = x \; s.t \; x \neq y \quad \forall x, y, \in X}$ as following Theorems ((3.11), (3.12)) .

Theorem(3..11)
Let X be a Q-algebra and $x * y = x \; s.t \; x \neq y \; \forall x, y \in X$. Then
$< x \wedge y >=< x >$

Proof
$Let \; x, y \in X$
$x \wedge y = (x * y) * y = x * y = x$
$then \; < x \wedge y >=< x >.$

Proposition(3.12)
Let X be a commutative bounded Q-algebra and $x * y = x \; s.t \; x \neq y \quad \forall x, y \in X$. Then
$< x \wedge y >=< x > \cap < y > \quad s.t \; x \neq y \quad \forall x, y \in X$

Proof
$Let \; x, y \in X$
since $< x >=< x \wedge y > \subseteq < x > \cap < y >$ by **Corollary(3.8)** and **Theorem(3.11)**
and $< x > \cap < y > \subseteq < x >=< x \wedge y >$
hence $< x \wedge y >=< x > \cap < y >.$

4 Prime ideal

In this section , we presented a definition of the prime ideal and its relationship to generated , as well as the definition of the finite $\cap -\boldsymbol{steructer}$ and gave some proposition of it , as well as its relationship to the prime ideal.

Definition(4.1)
A proper ideal P of a Q-algebra X is said to be Prime ideal , denoted by [P-ideal]
if $x \wedge y \in P$ impels $x \in P \; or \; y \in P \quad s.t \; x \wedge y \neq 0 \quad for \; any \; x, y \in X.$

Example(4.2) :

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Let $X = \{0, a, b, c\}$ define binary operation $*$ on X by the following table :

*	0	a	b	e	c
0	0	0	0	0	0
a	a	0	a	0	e
b	b	b	0	0	b
e	e	b	0	0	b
c	c	0	c	0	0

Then X is a Q-algebra and $I = \{0, a, c\}$ is prime ideal.

Proposition(4.3)
Let $(X,*,0)$ be a Q-algebra and if the number of elements in X is equal to n such that $n \geq 3$, then any ideal having to the number $n-1$ is a prime ideal.
Proof
Let $x, y \in X$ $s.t$ $x \wedge y \neq 0$ and $x \wedge y \in I$
Since $x \neq y$ then either $x \in I$ or $y \in I$.

Proposition(4.4)
Let P be a P-ideal of commutative bounded Q-algebra $(X, *, 0)$. If $A \cap B \subseteq P$ then $A \subseteq P$ or $B \subseteq P$ for any ideal A, B of X .

Proof
If P is a P-ideal and A , B are ideals of X such that $A \cap B \subseteq P$,
suppose that $A \nsubseteq P$, $B \nsubseteq P$, $then \; \exists\, x \in A - P$
and $\exists y \in B - P$ then $x \wedge y \in A$, $x \wedge y \in B$ (**by Proposition (2.6)**)
So $x \wedge y \in A \cap B \subseteq P$ $then$ $x \in P$ or $y \in P$ [since P is a P-ideal]
is a contradiction , this completes the proof.
.

Remark(4.5)
The converse of Proposition(4.4) is not true in general as shown in the following example.

Example(4.6)
In Example(3.7) all ideals of X are X and $\{0\}$
Notice that $X \cap \{0\} \subseteq \{0\}$, and $\{0\} \subseteq \{0\}$ but $\{0\}$ is not a P-ideal ,
since $a \wedge b = (a * b) * b = 0 * b = 0 \in \{0\}$ but $a \notin \{0\}$ and $b \notin \{0\}$.

Corollary(4.7)
Let $(X,*,0)$ be a Q-algebra and P be an ideal of X . If P is a prime ideal and $< x > \cap < y > \subseteq P$ implies $x \in P$ or $y \in P$.

Proof
it's clear by **Proposition(4.4).**

Remark(4.8)
The converse of **Corollary(4.7)** is not true in general as shown in the following example.

Example(4.9)
In Example (3.7) $<$ **0** $>$ = **{0}** and $< a > = < b > = X$
$< 0 > \cap < a > \subseteq \{0\}$
but $\{0\}$is not prime ideal.

The converse of Corollary (4.7) is true in commutative bounded Q-algebra if it satisfy $x * y = x$ $s.t$ $x \neq y$ $\forall x, y \in X$ as following theorem ((4.10) .

Theorem(4.10)
Let X be a commutative bounded Q-algebra and $x * y = x \quad s.t\ x \neq y \quad \forall x,\ y \in X$ then P is prime ideal if and only if $< x > \cap < y > \subseteq P$ implies $x \in P$ or $y \in P$.
Proof
$\Rightarrow$ by Corollary(4.7)
$\Leftarrow$ Let $x \wedge y \in P$
$< x \wedge y > \subseteq P$
since $< x \wedge y > = < x > \cap < y >$ by **Proposition (3.11)**
then $< x > \cap < y > \subseteq P$ so
either $x \in P\ or\ y \in P$

Remark(4.11)
If $x * y = x \quad s.t \quad x \neq y \quad \forall x, y \in X$ then Definition(4.1) and Proposition (4.4) are equivalent.

Definition(4.12)
Let X be a Q-algebra then a none empty set F of X is called finite $\cap - steructer$ if
$(< x > \cap < y >) \cap F \neq \phi$, $\forall\ x, y \in F$.

Example(4.13)
In Example(3.2) if $F = \{0, a, b\}$ then F is finite $\cap - steructer$ since
$(< a > \cap < b >) \cap F \neq \phi$ and
$(< a > \cap < 0 >) \cap F \neq \phi$
and if $F = \{a, b\}$ then F is not finite $\cap - steructer$ since
$(< a > \cap < b >) \cap F = (\{0, a\} \cap \{0, b\}) \cap F = \{0\} \cap F = \phi$

Proposition(4.14)
Let $(X, *, 0)$ is Q-algebra. If $0 \in F$ then F is finite $\cap - steructer$
Proof
Let $x,\ y \in$ X.
$< x >, < y >$ are ideals
then $0 \in < x >, < y >$
then $(< x > \cap < y >) \cap F \neq \phi$ for all $x,\ y \in F$
hence F is finite $\cap - steructer$.

Remark(4.15)
The converse of Proposition(4.14) is not true in general as shown in the following example.

Example(4.16)
In example (3.2) if $F = \{a, b\}$ then F is a finite $\cap - steructer$ because
$< a > = < b > X$ and $X \cap F = \{a,\ b\} \neq \phi\ \ but\ 0 \notin F$

Corollary(4.17)
If $(X, *, 0)$ is a Q-algebra then X is a finite $\cap - steructer$.
Proof
since $0 \in X$ and by Proposition(4.14)
then X is a finite $\cap - steructer$.

Proposition(4.18)
Let X be a commutative bounded Q-algebra and $x * y = x \quad s.t\ x \neq y \quad \forall x, y \in X$
then F is a finite $\cap - steructer$ if $< x \wedge y > \cap F \neq \phi, \quad \forall\ x, y \in F$.
Proof :

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It's clear (by Proposition(3.11) and Definition(4.12)).

Proposition(4.19)
Let X be a commutative bounded Q-algebra and $x * y = x \;\; s.t\, x \neq y \;\; \forall x, y \in X$
then F is a finite $\cap - stereucter$ if $< x > \cap F \neq \phi, \;\; \forall\, x \in F.$
Proof
It's clear (by Theorem(3.10) and Definition(4.12))

Proposition (4.20)
Let X be a Q-algebra then if P is prime ideal then $F = X - P$ is a finite $\cap - stereucter$

Proof
Let P be a prime ideal of X and $x, y \in F = X - P$,
If $(< x > \cap < y >) \cap F =$, then $< x > \cap < y > \subseteq P$
then either $x \in P$ or $y \in P$ (P is prime ideal)
and which is contradiction because $x, y \in F.$

Remark(4.21)
The converse of Proposition(4.20) is not true in general as shown in the following example.

Example(4.22)
In Example (3.7) $P = \{0\}$ is an ideal
and $F = X - \{0\}$ is a finite $\cap - stereucter$
but $\{0\}$ is not prime ideal.

The converse of Proposition (4.20) is true in commutative bounded Q-algebra if it satisfy
$\boldsymbol{x * y = x \;\; s.t \;\; x \neq y \;\; \forall\, x, \; y \in X}$ **as following theorem ((4.23)**

Proposition (4.23)
Let $(X, * ,0)$ is a commutative bounded Q-algebra and $\boldsymbol{x * y = x \;\; s.t \;\; x \neq y \;\; \forall\, x, \; y \in X}$ then an a proper ideal P is prime ideal if and only if $F = X - P$ is a finite $\cap - stereucter$

Proof
$\Rightarrow$ by Proposition (4.18)
$\Leftarrow$ suppose that F be a finite $\cap - stereucter$ and $x, \; y \in X$ such that
$< x > \cap < y > \subseteq P.$
If $x \notin P$ and $y \notin P$
Then $x, y \in F$ and $(< x > \cap < y >) \cap F \neq \phi.$
Thus $< x > \cap < y > \nsubseteq P$
and this is a contradiction.
Hence either $x \in P \; or \; y \in P.$

Corollary(4.24)
Let $(X, *, 0)$ be a commutative bounded Q-algebra and $\boldsymbol{x * y = x \;\; s.t \;\; x \neq y \;\; \forall\, x, \; y \in X}$ then $\forall z \in X, z \neq 0$ there exist a prime ideal P of X , such that $z \notin P$.

Proof
Let $x \in X \; s.t \; x \neq 0$ and $P = X - \{x\}.$
Then $0 \in P$.
If $a * b \in P$ and $b \in P,$
then $a \neq x$ and $a \in P$ $\quad (by\, x * y = x)$
thus P is an ideal and

$X - P$ is a finite $\cap - steructer$
thus P is prime ideal (by Proposition(4.23))

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