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The Characteristics of Triangular Congruence in Normed Space With Wilson's Angles

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Abstract: *Congruence of two triangles and the similarity of two triangles is known in Euclid's space. This paper will discuss the congruence of two triangles and the similarity of the two triangles in the normed space. Will be defined congruence of two triangles and the similarity of the two triangles in the normed space. Next will be discussed the properties associated with it. Finally, it will be discussed about the application using Wilson's angle.*

Keywords: Normed Space, Triangeles, Congruence, similarity, Wilson's angle.

1. INTRODUCTION

The angle between the two vectors in the Euclid $\mathbb{R}^2$ space is well known. In the Euclid space the angle between two vectors is defined using the product of the dot [8]. Furthermore, the angle between the two vectors in the inner product space has also been developed in [7, 11]. Likewise in normed space, angles between two vectors are also known, including angles P, I, g ([1], [2], [3], [4]), Thy angle [2] and Wilson's angle ([5],[6]).

The angle in the normed space discussed in this paper is the Wilson's angle introduced by Valentine and Wayment (1971). The study of Wilson's angle is discussed as follows :

Let $(V, \|.\|)$ be the normed space over the field $\mathbb{R}$, for any $x, y \in V$ is defined as a nonlinear functional :

$$2\langle x, y\rangle := \|x\|^2 + \|y\|^2 - \|x-y\|^2 \quad (1)$$

From the properties of the norm it belongs:

$$\big|\|x\| - \|y\|\big|^2 \le \|x-y\|^2$$
$$\Leftrightarrow \|x\|^2 - 2\|x\|\cdot\|y\| + \|y\|^2 \le \|x-y\|^2$$
$$\Leftrightarrow \langle x, y\rangle \le \|x\|\cdot\|y\| \quad (2)$$

Meanwhile :

$$\|x-y\|^2 \le (\|x\| + \|y\|)^2$$
$$\Leftrightarrow \|x-y\|^2 - \|x\|^2 - \|y\|^2 \le 2\|x\|\cdot\|y\|$$
$$\Leftrightarrow -\langle x, y\rangle \le \|x\|\cdot\|y\| \quad (3)$$

From equations (2) and (3) obtained :

$$|\langle x, y\rangle| \le \|x\|.\|y\|, \quad \forall\ x, y \in V \quad (4)$$

fulfill the inequality Cauchy-Schwarz [8]. Wilson's angle is defined as the angle between two vectors x and y that satisfy

$$\angle(x, y) := \arccos\left(\frac{\|x\|^2 + \|y\|^2 - \|x-y\|^2}{2\|x\|\cdot\|y\|}\right) \quad (5)$$

With Wilson's angle the cosine rule is obtained :

$$\|z\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\cdot\|y\|\cos\angle(x, y) \quad (6)$$

Furthermore, from equation (5) sine rules are obtained:

$$\|x\|.\|y\|\sin\angle(x, y) = K \quad (7)$$

With $K = 2\sqrt{s(s-\|x\|)(s-\|y\|)(s-\|z\|)}$ and

$$2s = \|x\| + \|y\| + \|z\|$$

Definition. 1.1. Let $(V, \|.\|)$ be Normed space for $a, b, c \in V\setminus\{0\}$, is defined $\Delta[a,b,c]$ as $\{a, b, c\}$ that fulfills $a + c = b$, which is completed by Wilson's angle $\angle(a,b)$, $\angle(-a,c)$, and $\angle(b,c)$.

Definition. 1.2. Let $(V, \|.\|)$ be Normed space for $d \in V\setminus\{0\}$, called the Ceva vector for $\Delta[a_1, a_2, a_3]$ if there is $\alpha \in (0,1)$ such that it fulfills $\alpha a_i + d = a_j$ with $i \ne j$.

Definition. 1.3. Let $(V, \|.\|)$ be Normed space for $d, e, f \in V\setminus\{0\}$, called allied ceva vector of $\Delta[a,b,c]$ if there is $\alpha_i \in (0,1)$, $i = 1,2,3,4,5,6$ such that it fulfills $(1-\alpha_6)f + a = (1-\alpha_5)e$, $(1-\alpha_5)e + c = (1-\alpha_4)d$, $(1-\alpha_6)f + b = (1-\alpha_4)d$

Definition. 1.4. Let $(V, \|.\|, \angle_w)$ be, Normed space over the field $\mathbb{R}$. Vector $d \in V$ called the Ceva vector of triangles $\Delta[a_1, a_2, a_3]$ if there is $\alpha \in (0,1)$ such that it fulfills $\alpha a_i + d = a_j$ with $i \ne j$ for some $i, j = 1,2,3$.

Definition. 1.5. Let $(V, \|.\|, \angle_w)$ be, Normed space over the field $\mathbb{R}$. Vector $d \in V$ called the high Ceva vector of $\Delta[a_1, a_2, a_3]$ if there is $\alpha \in (0,1)$ such that it fulfills $\alpha a_i + d = a_j$ with $i \neq j$ and $\angle_W(\alpha a_i, d) = \frac{\pi}{2}$.

Definition 1.6. Let $(V, \|.\|, \angle_w)$ be, Normed space over the field $\mathbb{R}$. Vector $d \in V$ called the divided Ceva vector $\Delta[a_1, a_2, a_3]$ if there is $\alpha \in (0,1)$ such that it fulfills $\alpha a_i + d = a_j$ with $i \neq j$ and $\angle_W(\alpha a_i, d) = \angle_W(\alpha a_k, d)$

Definition 1.7. Let $(V, \|.\|, \angle_w)$ be, Normed space over the field $\mathbb{R}$. Vector $d \in V$ called the heavy Ceva vector of $\Delta[a_1, a_2, a_3]$ if there is $\alpha \in (0,1)$ such that it fulfills $\alpha a_i + d = a_j$ with $i \neq j$ and $\alpha = \frac{1}{2}$.

2. MAIN RESULT

In space Euclid is known as congruence and two triangles, then in the normed space will also be introduced harmony and congruence of two triangles in the normed space.

In normed space two triangles are called congruent if for each corresponding side have the same norm and for each corresponding angle are equal. More details are defined as follows:

Definition. 2.1. Let $(V, \|\cdot\|, \angle_w)$ be triangle $\Delta[a, b, c]$ and $\Delta[p, q, r]$ called congruent if $\|a\| = \|p\|$, $\|b\| = \|q\|$, $\|c\| = \|r\|$ and $\angle_W(-a, c) = \angle_W(-p, r)$, $\angle_W(a, b) = \angle_W(p, q)$, $\angle_W(b, c) = \angle_W(q, r)$. Next symbolized $\Delta[a, b, c] \cong \Delta[p, q, r]$.

then defined as the similarity of the two triangles in the normed space, complete as follows.

Definition. 2.2. Let $(V, \|\cdot\|, \angle_w)$ be triangle $\Delta[a, b, c]$ and $\Delta[p, q, r]$ called similarity when the corresponding angles are equal. Next symbolized $\Delta[a, b, c] \approx \Delta[p, q, r]$.

To facilitate the proving theorem, the following axioms are given.

Axiom 2.1. Let $(V, \|\cdot\|, \angle_w)$ be triangle $\Delta[a, b, c]$ and $\Delta[p, q, r]$ which fulfills the nature $\|a\| = \|p\|$, $\|b\| = \|q\|$, $\|c\| = \|r\|$ then $\Delta[a, b, c] \cong \Delta[p, q, r]$.

Axiom 2.2. Let $(V, \|\cdot\|, \angle_w)$ be triangle $\Delta[a, b, c]$ and $\Delta[p, q, r]$ which fulfills the nature $\angle_W(-a, c) = \angle_W(-p, r)$, $\angle_W(a, b) = \angle_W(p, q)$ and , $\angle_W(b, c) = \angle_W(q, r)$ then $\Delta[a, b, c] \cong \Delta[p, q, r]$.

Axiom 2.3. Let $(V, \|\cdot\|, \angle_w)$ be triangle $\Delta[a, b, c]$ and $\Delta[p, q, r]$ which fulfills the nature $\angle_W(-a, c) = \angle_W(-p, r)$, $\angle_W(a, b) = \angle_W(p, q)$ and $\|a\| = \|p\|$ then $\Delta[a, b, c] \cong \Delta[p, q, r]$.

Theorem 2.1. Let $(V, \|\cdot\|, \angle_w)$ be triangle $\Delta[a, b, c]$ which fulfills the nature $\|a\| = \|c\|$ if and only if $\angle_W(a, b) = \angle_W(b, c)$.

Proof.

($\Longleftarrow$) Let $\angle_W(a, b) = \angle_W(b, c)$ be then , there is $\alpha \in (0,1)$ such that it applies $a + d = \alpha b$ dan $d + (1-\alpha)b = c$ that fulfills $\angle_W(d, \alpha b) = \angle_W(d, (1-\alpha)b) = \pi/2$ then obtained $\angle_W(a, d) = \pi/2 - \angle_W(a, b) = \pi/2 - \angle_W(b, c) = \angle_W(d, -c)$ as a result $\Delta[a, \alpha b, d] \cong [d, c, (1-\alpha)b]$ then $\|a\| = \|c\|$.

($\Longrightarrow$) Let $\|a\| = \|c\|$ then was obtained $\|b\| \cos \angle_W(a, b) +$
$\|c\| \cos \angle_W(-a, c) = \|b\| \cos \angle_W(b, c) + \|a\| \cos \angle_W(-a, c)$
this matter $\cos \angle_W(a, b) = \cos \angle_W(b, c)$. Because $\angle_W(a, b), \angle_W(b, c) \in [0, \pi]$ then it must be $\angle_W(a, b) = \angle_W(b, c)$. ■

Furthermore, in the two triangles in the normed space that have two corresponding sides of the same shape and two angles that correspond to the same size, then the two triangles are congruent. For the more complete following theorem.

Theorem 2.2. Let $(V, \|\cdot\|, \angle_w)$ be in triangle $\Delta[a, b, c]$ and triangle $\Delta[p, q, r]$ that fulfills $\angle_W(-a, c) = \angle_W(-p, r)$, $\angle_W(b, c) = \angle_W(q, r)$ and $\|a\| = \|p\|$ then $\Delta[a, b, c] \cong \Delta[p, q, r]$.

Proof.

Clear that :

$$\angle_W(a, b) = \pi - \angle_W(-a, c) - \angle_W(b, c) = \pi - \angle_W(-p, r) - \angle_W(q, r)$$

$$= \angle_W(p, q)$$

which means $\Delta[a, b, c] \cong \Delta[p, q, r]$. ■

In an isosceles triangle in a normed space it will have two heavy Ceva vectors and apply vice versa. The following is more clear theorem.

Theorem 2.3. Let $(V, \|\cdot\|, \angle_w)$ be, in triangle $\Delta[a, b, c]$ isosceles, if and only if $\Delta[a, b, c]$ has two ceva vectors of the same weight.

Proof.

($\Longrightarrow$) Let $\Delta[a, b, c]$ be, isosceles triangle, Let $\|a\| = \|c\|$. Suppose a heavy ceva vector $d_1, d_2 \in V$ such that it fulfills $\frac{1}{2}c + d_1 = a$ and $\frac{1}{2}a + d_2 = b$ which results in $\Delta\left[\frac{1}{2}c, d_1, a\right] \cong \Delta\left[\frac{1}{2}a, d_2, b\right]$ so that $\|d_1\| = \|d_2\|$.

($\Longleftarrow$)Suppose a heavy ceva vector $d_1, d_2 \in V$ such that $\|d_1\| = \|d_2\|$,

as a result $\Delta\left[\frac{1}{2}c, d_1, a\right] \cong \Delta\left[\frac{1}{2}a, d_2, b\right]$ so that $\|a\| = \|c\|$ then $\Delta[a, b, c]$ isosceles triangle.

■

Example 2.1

Suppose that the set of functions integrated in $[0,1]$

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$$L^3([01]) = \left\{ f \;\middle|\; \int_0^1 |f(x)|^3 \;\; dx \right\}$$

Let $\Delta[a,b,c]$, $\{a,b,c\} \subseteq L^3([0,1])$ be, and fulfill $a + c = b$ with $a(t) = t^3$, $b(t) = t^2$ and $c(t) = t^2 - t^3$, and $\Delta[p,q,r]$, $\{p,q,r\} \subseteq L^3([0,1])$, and fulfill $p + r = q$ with $p(t) = 2t^3$, $q(t) = 2t^2$ and $r(t) = 2t^2 - 2t^3$ then obtained the respective norms:

$$\|a\| = \left(\int_0^1 |t^3|^3\, dt\right)^{\frac{1}{3}}$$

$$= \left(\int_0^1 t^9\, dt\right)^{\frac{1}{3}}$$

$$= \;\; 0{,}464$$

$$\|b\| = \left(\int_0^1 |t^2|^3\, dt\right)^{\frac{1}{3}}$$

$$= \left(\int_0^1 t^6\, dt\right)^{\frac{1}{3}}$$

$$= \;\; 0{,}523$$

$$\|c\| = \left(\int_0^1 |t^2 - t^3|^3\, dt\right)^{\frac{1}{3}}$$

$$= \left(\int_0^1 (t^6 - 3t^7 + 3t^8 - t^9)\, dt\right)^{\frac{1}{3}}$$

$$= 0{,}106$$

$$\|p\| = \left(\int_0^1 |2t^3|^3\, dt\right)^{\frac{1}{3}}$$

$$= \left(8\int_0^1 t^9\, dt\right)^{\frac{1}{3}}$$

$$= \;\; 0{,}928$$

$$\|q\| = \left(\int_0^1 |t^2|^3\, dt\right)^{\frac{1}{3}}$$

$$= \left(8\int_0^1 t^6\, dt\right)^{\frac{1}{3}}$$

$$= \;\; 1{,}046$$

$$\|r\| = \left(\int_0^1 |t^2 - t^3|^3\, dt\right)^{\frac{1}{3}}$$

$$= \left(8\int_0^1 (t^6 - 3t^7 + 3t^8 - t^9)\, dt\right)^{\frac{1}{3}}$$

$$= 0{,}212$$

So obtained:

$$\angle(a,b) = arccos\left(\frac{\|a\|^2 + \|b\|^2 - \|c\|^2}{2\|a\|\|b\|}\right) = 10{,}29$$

$$\angle(b,c) = arccos\left(\frac{\|b\|^2 + \|c\|^2 - \|a\|^2}{2\|b\|\|c\|}\right) = 118{,}27$$

$$\angle(-a,c) = arccos\left(\frac{\|a\|^2 + \|c\|^2 - \|b\|^2}{2\|a\|\|c\|}\right) = 51{,}44$$

and

$$\angle(p,q) = arccos\left(\frac{\|p\|^2 + \|q\|^2 - \|r\|^2}{2\|p\|\|q\|}\right) = 10{,}29$$

$$\angle(q,r) = arccos\left(\frac{\|q\|^2 + \|r\|^2 - \|p\|^2}{2\|q\|\|r\|}\right) = 118{,}27$$

$$\angle(-p,r) = arccos\left(\frac{\|p\|^2 + \|r\|^2 - \|q\|^2}{2\|p\|\|r\|}\right) = 51{,}44$$

therefore $\angle(a,b) = \angle(p,q) = 10{,}29$

$\angle(b,c) = \angle(q,r) = 118{,}27$

$\angle(-a,c) = \angle(-p,r) = 51{,}44$

Then $\Delta[a,b,c] \approx \Delta[p,q,r]$.

Example 2.2

Suppose that the set of functions integrated in [0,1]

$$L^3([01]) = \left\{ f \;\middle|\; \int_0^1 |f(x)|^3 \;\; dx \right\}$$

Let $\Delta[a,b,c]$, $\{a,b,c\} \subseteq L^3([0,1])$ be, and fulfill $a + c = b$ with $a(t) = t^3$, $b(t) = t^2$ and $c(t) = t^2 - t^3$, and $\Delta[p,q,r]$, $\{p,q,r\} \subseteq L^3([0,1])$, and fulfill $p + r = q$ with

$p(t) = t^3$, $q(t) = t^2$ and $r(t) = t^2 - t^3$ then obtained the respective norms:

$$\|a\| = \left(\int_0^1 |t^3|^3\, dt\right)^{\frac{1}{3}}$$

$$= \left(\int_0^1 t^9\, dt\right)^{\frac{1}{3}}$$

$$= 0{,}464$$

$$\|b\| = \left(\int_0^1 |t^2|^3\, dt\right)^{\frac{1}{3}}$$

$$= \left(\int_0^1 t^6\, dt\right)^{\frac{1}{3}}$$

$$= 0{,}523$$

$$\|c\| = \left(\int_0^1 |t^2 - t^3|^3\, dt\right)^{\frac{1}{3}}$$

$$= \left(\int_0^1 (t^6 - 3t^7 + 3t^8 - t^9)\, dt\right)^{\frac{1}{3}}$$

$$= 0{,}106$$

$$\|p\| = \left(\int_0^1 |t^3|^3\, dt\right)^{\frac{1}{3}}$$

$$= \left(\int_0^1 t^9\, dt\right)^{\frac{1}{3}}$$

$$= 0{,}464$$

$$\|q\| = \left(\int_0^1 |t^2|^3\, dt\right)^{\frac{1}{3}}$$

$$= \left(\int_0^1 t^6\, dt\right)^{\frac{1}{3}}$$

$$= 0{,}523$$

$$\|r\| = \left(\int_0^1 |t^2 - t^3|^3\, dt\right)^{\frac{1}{3}}$$

$$= \left(\int_0^1 (t^6 - 3t^7 + 3t^8 - t^9)\, dt\right)^{\frac{1}{3}}$$

$$= 0{,}106$$

So obtained:

$$\angle(a,b) = arccos\left(\frac{\|a\|^2 + \|b\|^2 - \|c\|^2}{2\|a\|\|b\|}\right) = 10{,}29$$

$$\angle(b,c) = arccos\left(\frac{\|b\|^2 + \|c\|^2 - \|a\|^2}{2\|b\|\|c\|}\right) = 118{,}27$$

$$\angle(-a,c) = arccos\left(\frac{\|a\|^2 + \|c\|^2 - \|b\|^2}{2\|a\|\|c\|}\right) = 51{,}44$$

and

$$\angle(p,q) = arccos\left(\frac{\|p\|^2 + \|q\|^2 - \|r\|^2}{2\|p\|\|q\|}\right) = 10{,}29$$

$$\angle(q,r) = arccos\left(\frac{\|q\|^2 + \|r\|^2 - \|p\|^2}{2\|q\|\|r\|}\right) = 118{,}27$$

$$\angle(-p,r) = arccos\left(\frac{\|p\|^2 + \|r\|^2 - \|q\|^2}{2\|p\|\|r\|}\right) = 51{,}44$$

therefore $\angle(a,b) = \angle(p,q) = 10{,}29$

$\angle(b,c) = \angle(q,r) = 118{,}27$

$\angle(-a,c) = \angle(-p,r) = 51{,}44$

and $\|a\| = \|p\| = 0{,}464$

$\|b\| = \|q\| = 0{,}523$

$\|c\| = \|r\| = 0{,}106$

Then $\Delta[a,b,c] \cong \Delta[p,q,r]$.

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