International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 4, Issue 3 , March – 2020, Pages: 28-37

Fuzzy Set in AT-algebras

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Abstract— *the aim of this paper is to introduce a concept of AT-subalgebras, AT-ideals, and investigated some related properties of them. And study fuzzy AT-subalgebras, fuzzy AT-ideals and fuzzy AT-filters of AT-algebras and investigate some of its properties. The notions of upper t-(strong) level subsets and lower t-(strong) level subsets are introduced from some fuzzy sets, and its characterizations are given. We define complement of fuzzy set in AT-algebras and related properties are investigated.*

Keywords: AT-algebra, AT-subalgebras, AT-ideal, Fuzzy AT-subalgebras, Fuzzy AT-ideal, Fuzzy AT-filter, Upper t-(strong) level subset, Lower t-(strong) level subset, complement.

1. Introduction

Among the important and intensively studied classes of algebras are the algebras of logic. Examples of these are BCI-algebras and BCK-algebras [3,4], KU-algebras [5,6] and others. They are strongly connected with logic. For example, the BCI-algebras introduced by Iséki [6] in 1966 have connections with BCI-logic, being the BCI-system in combinatory logic, which has applications in the language of functional programming. The notion of AT-algebras was introduced by Areej Tawfeeq Hameed [1,2] introduced AT-algebra, They have studied a few properties of these algebras, the notion of AT-ideals on AT-algebras was formulated and some of its properties are investigated. The notion of fuzzy subsets of a set was first considered by Zadeh [7] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.

2. Preliminaries

In this section, we give some basic definitions and preliminaries lemmas of AT-ideals and fuzzy AT-ideals of AT-algebra

Definition 2.1[1,2]. An **AT-algebra** is a nonempty set X with a constant (0) and a binary operation ($*$) satisfying the following axioms: for all $x, y, z \in X$,

(i) $(x*y)*((y*z)*(x*z))=0$,
(ii) $0* x =x$,
(iii) $x* 0 =0$.

In X we can define a binary relation ($\leq$) by : $x \leq y$ if and only if , $y * x = 0$.

In AT-algebra (X ;*, 0), the following properties are satisfied: for all $x, y, z \in X$,

(i') $(y*z)*(x*z) \leq (x*y)$,
(ii') $0 \leq x$.

Proposition 2.2 [1,2]. In any AT-algebra (X ;*, 0), the following properties holds: for all $x, y, z \in X$;

a) $x * x = 0$,
b) $z*(x *z) = 0$,
c) $y * ((y* z) * z) = 0$,
d) $x * y = 0$ implies that $x * 0 = y * 0$,
e) $x = 0 *(0*x)$,
f) $0*x=0*y$ implies that $x=y$.

Proposition 2.3[1,2]. In any AT-algebra (X ;*, 0), the following properties holds: for all $x, y, z \in X$;

a) $x \leq y$ implies that $y * z \leq x *z$,
b) $x \leq y$ implies that $z * x \leq z *y$,
c) $z * x \leq z *y$ implies that $x \leq y$, (left cancellation law).
d) $x* y \leq z$ imply $z * y \leq x$.

Definition 2.4 [1,2]. A nonempty subset S of an AT-algebra X is called **an AT-subalgebra of AT-algebra X** if $x*y \in S$, whenever $x, y \in S$.

Definition 2.5 [1,2]. A nonempty subset I of an AT-algebra X is called **an AT-ideal of AT-algebra X** if it satisfies the following conditions: for all $x, y, z \in X$;

AT_1) $0 \in I$;

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AT_2) x * (y *z) ∈ I and y ∈ I imply x*z∈ I .

Definition 2.6 [1,2]. Let (X ; ∗ ,0) and (Y; ∗ `,0`) be two AT-algebras , the mapping *f*:(X; ∗ ,0) → (Y; ∗ `,0`) is called a **homomorphism** if it satisfies: *f* (x∗ y) = *f* (x) ∗ ` *f* (y), for all x , y ∈ X.

Remark 2.7 [1,2]. Let $f:(X;*,0)\to(Y;*',0')$ be a homomorphism from an AT-algebra X into an AT-algebra Y. A is a nonempty subset of X and B is a nonempty subset of Y. The image of A of X under *f* is $f(A)=\{f(a):a\in A\}$, and the inverse image of B of Y under *f* is $f^{-1}(B)=\{y\in Y: y=f(x)\in B, x\in X\}$.

In particular, $f^{-1}(\{0'\})$ is called the kernel of f. $\ker f=\{x\in X: f(x)=0'\}=f^{-1}(\{0'\})$.

Definition 2.8 [1,2]. Let X be an AT-algebra. A fuzzy set μ in X is called **a fuzzy AT-subalgebra of X** if , for all x, y ∈ X, μ (x ∗ y) ≥ min { μ (x), μ (y)}.

Lemma 2.9. Let B be a nonempty subset of X. Then B is an AT-subalgebra of X if and only if, the characteristic function μ_F is a fuzzy AT-subalgebra of X.

Proof.

Assume that B is an AT-subalgebra of X and let x, y ∈ X , then

Case 1: Suppose x, y ∈ S. Then μ(x)=1 or μ(y)=1. Thus min{μ(x), μ(y)}=min {1,1} =1. Since B is an AT-subalgebra of X , we have x ∗ y ∈ B. So μ(x ∗ y) = 1 ≥ 1 = min{ μ(x), μ(y)}.

Case 2: Suppose x, y ∉ S. Then μ (x)=0 or μ(y)=0. Thus min{μ(x), μ(y)}=min {0,0} =0. So μ (x ∗ y) ≥ 0 = min{μ (x), μ(y)}. Thus μ is a fuzzy AT-subalgebra of X.

Conversely, assume that μ is a fuzzy AT-subalgebra of X. Let x, y ∈B, then μ(x)=1 or μ(y)=1. Thus μ (x ∗ y) ≥ min{μ(x), μ (y)} =1. So μ (x ∗ y) = 1 , thus x ∗ y ∈ B. Hence B is an AT-subalgebra of X·

Definition 2.10 [1,2]. Let X be an AT-algebra. A fuzzy set μ in X is called **a fuzzy AT-ideal of X** if it satisfies the following conditions: for all x, y and z ∈ X,

(AT_1) μ (0) ≥ μ (x).

(AT_2) μ (x ∗ z) ≥ min { μ (x*(y∗ z)), μ (y)}.

Lemma 2.11. Let B be a nonempty subset of X. Then the constant 0 of X is in B if and only if μ(0) ≥ μ(x), for all x ∈ X.

Proof.

If 0 ∈ B, then μ(0) = 1. Thus μ(0) = 1 ≥ μ(x), for all x ∈ X.

Conversely, assume that μ(0) ≥ μ(x), for all x ∈ X. Since B is a nonempty subset of X, we have a ∈ B, for some a ∈ X. Then μ(0) ≥ μ(a) = 1. Thus μ(0) = 1. So 0∈ B.

Theorem 2.12. Let I be a nonempty subset of X. Then I is an AT-ideal of X if and only if, the characteristic function μ is a fuzzy AT-ideal of X.

Proof.

Assume that I is an AT-ideal of X. Since 0 ∈ I, it follows from Lemma (2.11) that μ (0) ≥ μ (x), for all x ∈ X.

Next, let x, y ∈ X.

Case 1: Suppose (x ∗ (y ∗ z)) ∈I and y ∈I. Then μ(x ∗ (y ∗ z)) = 1 and μ(y) = 1. Thus min{ μ(x ∗ (y ∗ z)), μ (y)} = min{1,1} = 1. Since I is an AT-ideal of X, we have x ∗ z ∈ I. So μ(x ∗ z) = 1.

Hence μ( ) = 1 ≥ 1= min{ μ(x ∗ (y ∗ z)), μ (y)}.

Case 2: Suppose (x ∗ (y ∗ z)) ∉I and y ∉I. Then μ(x ∗ (y ∗ z)) = 0 and μ(y) = 0. Thus min{ μ(x ∗ (y ∗ z)), μ (y)} = min{0,0} = 0. Since I is an AT-ideal of X, we have x ∗ z ∈ I.

So μ(x ∗ z) = 1. Hence μ(x ∗ z) ≥ 0 = min{ μ(x ∗ (y ∗ z)), μ (y)}. Hence μ is a fuzzy AT-ideal of X.

Conversely, assume that μ is a fuzzy AT-ideal of X. Since μ(0) ≥ μ(x), for all x ∈ X, it follows from Lemma (2.11) that 0 ∈ I. Next, let x, y, z ∈ X be such that x ∗ (y ∗ z) ∈ I and y ∈ I. Then μ (x ∗ (y ∗ z)) = 1 and μ(y) = 1. To show that (x ∗ y) ∈ I, assume that (x ∗ z) ∉ I. Then μ(x ∗ z) = 0. Thus 0 = μ(x ∗ z) ≥ min{ μ(x ∗ (y ∗ z)), μ (y)} = min{1, 1} = 1, a contradiction. So x ∗ z ∈ I. Hence I is an AT-ideal of X.

Theorem 2.13 [1,2]. Let *f* : (X; ∗ ,0) → (Y; ∗ `,0`) be into homomorphism of AT-algebras, then :

A) *f* (0) = 0' .

B) *f* is injective if and only if, ker *f* = {0}.

C) x ≤ y implies f (x) ≤ f (y) .

Theorem 2.14 [1,2]. Let *f*: (X; ∗ ,0) → (Y; ∗ `,0`) be into homomorphism of an AT-algebras, then :

(F_1) If S is an AT-subalgebra of X, then *f* (S) is an AT-subalgebra of Y, where *f* is onto .

(F_2) If I is an AT-ideal of X, then *f* (I) is an AT-ideal in Y, where *f* is onto .

(F_3) If B is an AT-subalgebra of Y, then f^{-1}(B) is an AT-subalgebra of X .
(F_4) If J is an AT- ideal in Y, then f^{1} (J) is an AT-ideal in X .
(F_5) ker f is AT-ideal of X.
(F_6) Im(f) is an AT-subalgebra of Y.

3. Fuzzy AT-filter of AT-algebras

In this section, we introduce the notions of an AT-filter of AT-algebra and a fuzzy AT-filter of AT-algebra and study some of their basic properties.

Definition 3.1. A nonempty subset F of X is called **an AT-filter of X**, if it satisfies the following properties: for any x, y ∈ X,

1- $0 \in F$,
2- $x * y \in F$ and $x \in F$ implies $y \in F$.

We can easily show the following example.

Example 3.2. Let X = {0, 1, 2, 3} be a set with a binary operation * defined by the following Cayley table:

∗	0	1	2	3
0	0	1	2	3
1	0	0	0	3
2	0	1	0	3
3	0	1	0	0

Hence (X; ∗,0) is an AT-algebra. Then {0,1,3} and {0,1,2} are AT-ideals of X and an AT-filter of an AT-algebra.

Example 3.3. Let X = {0, a, b, c, d} be a set with a binary operation * defined by the following Cayley table:

∗	0	a	b	c	d
0	0	a	b	c	d
1	0	0	b	c	d
b	0	0	0	c	d
c	0	0	b	0	d
d	0	0	0	0	0

Hence (X; ∗,0) is an AT-algebra. Then {0,a,c} and {0,a,b} are AT-ideals of X and {0,a,b} is an AT-filter of an AT-algebra.

Definition 3.4. A fuzzy set μ in X is called **a fuzzy AT-filter of X**, if it satisfies the following properties: for any x, y ∈X,

(FAT_1) $\mu(0) \geq \mu(x)$.
(FAT_2) $\mu(y) \geq \min\{\mu(x * y), \mu(x)\}$.

Example 3.5. By Example (3.3), we get F= {0, a, b} is an AT-filter of X. Then it can be easily verified that

$\mu_F(x) = \begin{cases} 1 & if\ x \in \{0,a,b\} \\ 0 & if\ x \in \{c,d\} \end{cases}$, is a fuzzy AT-filter of X.

Definition 3.6. A fuzzy set in a nonempty set X (or a fuzzy subset of X) is an arbitrary function μ : X→ [0,1], where [0, 1] is the unit segment of the real line.

If A⊆ X, the characteristic function μ_A of X is a function of X into {0, 1} defined as follows:

$$\mu_A(x) = \begin{cases} 1 & if\ x \in A \\ 0 & if\ x \notin A \end{cases}$$

By the definition of characteristic function, μ_A is a function of X into {0, 1} ⊂ [0, 1]. Then μ_A is a fuzzy set in X.

Theorem 3.7. Let F be a nonempty subset of X. Then F is an AT-filter of X if and only if the characteristic function μ_F is a fuzzy AT-filter of X.

Proof.

Assume that F is an AT-filter of X. Since $0 \in F$, it follows from Lemma (2.11) that $\mu_F(0) \geq \mu_F(x)$, for all x ∈ X. Next, let x, y ∈ X.

Case 1: Suppose x, y ∈ F . Then $\mu_F(x) = 1$ and $\mu_F(y) = 1$. Thus
$\mu_F(y) = 1 \geq \mu_F(x * y) = \min\{1, \mu_F(x * y)\} = \min\{\mu_F(x), \mu_F(x * y)\}$.

Case 2: Suppose $x \notin F$ or $y \notin F$. Then $\mu_F(x) = 0$ or $\mu_B(y) = 0$.

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 4, Issue 3 , March – 2020, Pages: 28-37

Case 2.1: If $x \notin F$, then $\mu_F(x) = 0$. Thus $\mu_F(y) \geq 0 = \min\{0, \mu_F(x \cdot y)\} = \min\{\mu_F(x), \mu_F(x * y)\}$.

Case 2.2: If $y \notin F$, then $\mu_F(y) = 0$. Since F is an AT-filter of X, we have $x \notin F$ or $x * y \notin F$. Thus $\mu_F(x) = 0$ or $\mu_F(x * y) = 0$. So $\mu_F(y) = 0 = \min\{\mu_F(x), \mu_F(x * y)\}$. Hence μ_F is a fuzzy AT-filter of X.

Conversely, assume that μ_F is a fuzzy AT-filter of X. Since $\mu_F(0) \geq \mu_F(x)$, for all $x \in X$, it follows from Lemma (2.11) that $0 \in F$. Next, let $x, y \in X$ be such that $x \in F$ and $x * y \in F$. Then $\mu_F(x) = 1$ and $\mu_F(x \cdot y) = 1$. To show that $y \in F$, assume that $y \notin F$. Then $\mu_F(y) = 0$. Thus $0 = \mu_F(y) \geq \min\{\mu_F(x), \mu_F(x * y)\} = \min\{1, 1\} = 1$, a contradiction. So $y \in F$. Hence F is an AT-filter of X.

Definition 3.8. A nonempty subset B of X is called **a prime subset of** X, if for any $x, y \in X$, $x * y \in B$ implies $x \in B$ or $y \in B$.

Remark 3.9. An AT-subalgebra (resp. AT-ideal, AT-filter) B of X is called **a prime AT-subalgebra (resp. prime AT-ideal, prime AT-filter) of X**, if B is a prime subset of X.

Definition 3.10. A fuzzy set μ in X is called **a prime fuzzy set in X**, if for any $x, y \in X$, $\mu(x * y) \leq \max\{\mu(x), \mu(y)\}$.

Remark 3.11. A fuzzy AT-subalgebra (resp. fuzzy AT-ideal, fuzzy AT-filter) B of X is called **a prime fuzzy AT-subalgebra (resp. prime fuzzy AT-ideal, prime fuzzy AT-filter) of X**, if B is a prime fuzzy set in X.

Theorem 3.12. Let B be a nonempty subset of X. Then B is a prime subset of X if and only if the characteristic function μ_F is a prime fuzzy set in X.

Proof.

Assume that B is a prime subset of X and let $x, y \in X$ if and only if, the characteristic function μ_B is a prime fuzzy set in X.

Case 1: Suppose $x * y \in B$. Since B is a prime subset of X, we have $x \in B$ or $y \in B$. Then $\mu_B(x)=1$ or $\mu_B(y)=1$. Thus $\max\{\mu_B(x), \mu_B(y)\}=1$. So $\mu_B(x * y) \leq 1 = \max\{\mu_B(x), \mu_B(y)\}$.

Case 2: Suppose $x * y \notin B$. Then $\mu_B(x * y) = 0 \quad \max \mu_B(x), \mu_B(y)$. Thus μ_B is a prime fuzzy set in X.

Conversely, assume that μ_B is a prime fuzzy set in X. Let $x, y \in X$ be such that $x*y \in B$. Then $\mu_B(x * y)=1$. Thus $1= \mu_B(x * y) \leq \max\{\mu_B(x), \mu_B(y)\}$. So $\max \mu_B(x), \mu_B(y)=1$. Hence $\mu_B(x)=1$ or $\mu_B(y)= 1$. Therefore $x \in B$ or $y \in B$ and thus B is a prime subset of X.

Theorem 3.13. Let B be a nonempty subset of A. Then B is a prime AT-subalgebra of X if and only if, the characteristic function μ_B is a prime fuzzy AT-subalgebra of X.

Proof. It is straightforward by Theorem (2.9) and Theorem (3.12) .

Theorem 3.14. Let B be a nonempty subset of A. Then B is a prime AT-ideal of X if and only if, the characteristic function μ_B is a prime fuzzy AT-ideal of X.

Proof. It is straightforward by Theorem (2.12) and Theorem (3.12) .

Theorem 3.15. Let F be a nonempty subset of X. Then F is a prime AT-filter of X if and only if the characteristic function μ_F is a prime fuzzy AT-filter of X.

Proof. It is straightforward by Theorem (3.7) and Theorem (3.12) .

4. Upper *t*-(strong) level subsets and lower *t*-(strong) level subsets of AT-algebras

In this section, we introduce the notions of a Upper *t*-(strong) level subsets and lower *t*-(strong) level subsets are derived from some fuzzy sets.

Definition 4.1. Let μ be a fuzzy set in X. For any $t \in [0, 1]$, the set $U(\mu ; t) = \{x \in X \mid \mu(x) \geq t\}$ and $U^+(\mu ; t) = \{x \in X \mid \mu(x) > t\}$ are called **an upper t-level subset and an upper t-strong level subset of** μ, respectively.

The set $L(\mu; t) = \{x \in X \mid \mu(x) \leq t\}$ and $L^-(\mu; t) = \{x \in X \mid \mu(x) < t\}$ are called **a lower t-level subset and a lower t-strong level subset of** μ , respectively.

Theorem 4.2. Let μ be a fuzzy set in X. Then μ is a fuzzy AT-subalgebra of X if and only if for all $t \in [0, 1]$, $U(\mu ; t)$ is an AT-subalgebra of X, if $U(\mu ; t)$ is nonempty.

Proof. Assume that μ is a fuzzy AT-subalgebra of X. Let $t \in [0, 1]$ be such that $U(\mu;t) \neq \emptyset$ and let $x, y \in U(\mu; t)$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$, so t is a lower bound of $\{\mu(x), \mu(y)\}$. Since μ is a fuzzy AT-subalgebra of X, we have $\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \geq t$. Thus $x * y \in U(\mu ; t)$. So $U(\mu ; t)$ is an AT-subalgebra of X.

Conversely, assume that for all $t \in [0, 1]$, $U(\mu ; t)$ is AT-subalgebra of X, if $U(\mu ; t)$ is nonempty. Let $x, y \in X$. Then $\mu(x), \mu(y) \in [0, 1]$. Choose $t = \min\{\mu(x), \mu(y)\}$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. Thus $x, y \in U(\mu ; t) \neq \emptyset$. By assumption, we have $U(\mu ; t)$ is AT-subalgebra of X. So $x * y \in U(\mu; t)$. Hence $\mu(x * y) \geq t = \min\{\mu(x), \mu(y)\}$. Therefore μ is a fuzzy AT-subalgebra of X.

Theorem 4.3. Let μ be a fuzzy set in X. Then μ is a fuzzy AT-ideal of X if and only if for all $t \in [0, 1]$, $U(\mu; t)$ is an AT-ideal of A, if $U(\mu; t)$ is nonempty.

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 4, Issue 3 , March – 2020, Pages: 28-37

Proof.

Assume that μ is a fuzzy AT-ideal of X. Let $t \in [0, 1]$ be such that $U(\mu;t) \neq \emptyset$ and let $a \in U(\mu; t)$. Then $\mu(a) \geq t$. Since μ is a fuzzy AT-ideal of X, we have $\mu(0) \geq \mu(a) \geq t$. Thus $0 \in U(\mu;t)$.

Next, let $x, y, z \in A$ be such that $x * (y * z) \in U(\mu;t)$ and $y \in U(\mu;t)$. Then $\mu(x * (y * z)) \geq t$ and $\mu(y) \geq t$. Thus t is a lower bound of $\{\mu(x * (y * z)), \mu(y)\}$. Since μ is a fuzzy AT-ideal of X, we have $\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\} \geq t$.

So $x * z \in U(\mu;t)$. Hence $U(\mu;t)$ is an AT-ideal of X.

Conversely, assume that for all $t \in [0, 1]$, $U(\mu;t)$ is an AT-ideal of X, if $U(\mu;t)$ is nonempty. Let $x \in X$. Then $\mu(x) \in [0, 1]$. Choose $t = \mu(x)$. Then $\mu(x) \geq t$. Thus $x \in U(\mu;t) \neq \emptyset$. By assumption, we have $U(\mu;t)$ is an AT-ideal of X. So $0 \in U(\mu;t)$. Hence $\mu(0) \geq t = \mu(x)$.

Next, let $x, y, z \in X$. Then $\mu(x * (y * z)), \mu(y) \in [0, 1]$. Choose $t = \min\{\mu(x * (y * z)), \mu(y)\}$. Then $\mu(x * (y * z)) \geq t$ and $\mu(y) \geq t$. Thus $x * (y * z), y \in U(\mu;t) \neq \emptyset$. By assumption, we have $U(\mu;t)$ is an AT-ideal of X. So $x * z \in U(\mu;t)$.

Hence $\mu(x * z) \geq t = \min\{\mu(x * (y * z)), \mu(y)\}$. Therefore μ is a fuzzy AT-ideal of X.

Theorem 4.4. Let μ be a fuzzy set in X. Then μ is a fuzzy AT-filter of X if and only if for all $t \in [0, 1]$, $U(\mu; t)$ is an AT-filter of X, if $U(\mu; t)$ is nonempty.

Proof.

Assume that μ is a fuzzy AT-filter of X. Let $t \in [0, 1]$ be such that $U(\mu;t) \neq \emptyset$ and let $a \in U(\mu; t)$. Then $\mu(a) \in t$. Since μ is a fuzzy AT-filter of X, we have $\mu(0) \geq \mu(a) \geq t$. Thus $0 \in U(\mu; t)$.

Next, let $x, y \in X$ be such that $x \in U(\mu;t)$ and $x * y \in U(\mu;t)$. Then $\mu(x) \geq t$ and $\mu(x * y) \geq t$. Thus t is a lower bound of $\{\mu(x * y), \mu(x)\}$. Since μ is a fuzzy AT-filter of X, we have $\mu(y) \geq \min\{\mu(x * y), \mu(x)\} \geq t$. So $y \in U(\mu;t)$.

Hence $U(\mu;t)$ is an AT-filter of X.

Conversely, assume that for all $t \in [0, 1]$, $U(\mu;t)$ is an AT-filter of X, if $U(\mu;t)$ is nonempty. Let $x \in X$. Then $\mu(x) \in [0, 1]$. Choose $t = \mu(x)$. Then $\mu(x) \geq t$. Thus $x \in U(\mu;t) \neq \emptyset$. By assumption, we have $U(\mu;t)$ is an AT-filter of X. So $0 \in U(\mu;t)$. Hence $\mu(0) \geq t = \mu(x)$.

Next, let $x, y \in X$. Then $\mu(x), \mu(x * y) \in [0, 1]$. Choose $t = \min\{\mu(x), \mu(x * y)\}$. Then $\mu(x) \geq t$ and $\mu(x * y) \geq t$. Thus $x, x * y \in U(\mu;t) \neq \emptyset$. By assumption, we have $U(\mu;t)$ is an AT-filter of X. So $y \in U(\mu;t)$. Hence $\mu(y) \geq t = \min\{\mu(x), \mu(x * y)\}$. Therefore μ is a fuzzy AT-filter of X.

Theorem 4.5. Let μ be a fuzzy set in X. Then μ is a prime fuzzy set in X if and only if for all $t \in [0, 1]$, $U(\mu; t)$ is a prime subset of X, if $U(\mu; t)$ is nonempty.

Proof.

Assume that μ is a prime fuzzy set in X. Let $t \in [0, 1]$ be such that $U(\mu; t) \neq \emptyset$. Let $x, y \in X$ be such that $x * y \in U(\mu;t)$. Assume that $x \notin U(\mu;t)$ and $y \notin U(\mu;t)$. Then $\mu(x) < t$ and $\mu(y) < t$. Thus t is an upper bound of $\{\mu(x), \mu(y)\}$. Since μ is a prime fuzzy set in X, we have $\mu(x * y) \leq \max\{\mu(x), \mu(y)\} < t$. So $x * y \notin U(\mu;t)$, a contradiction. Hence $x \in U(\mu;t)$ or $y \in U(\mu;t)$. Therefore $U(\mu;t)$ is a prime subset of X.

Conversely, assume that for all $t \in [0, 1]$, $U(\mu;t)$ is a prime subset of X if $U(\mu;t)$ is nonempty. Let $x, y \in X$. Then $\mu(x * y) \in [0, 1]$. Choose $t = \mu(x * y)$. Then $\mu(x * y) \geq t$. Thus $x * y \in U(\mu;t) \neq \emptyset$. By assumption, we have $U(\mu;t)$ is a prime subset of A. So $x \in U(\mu;t)$ or $y \in U(\mu;t)$. Hence $t \leq \mu(x)$ or $t \leq \mu(y)$, so $\mu(x * y) = t \leq \max\{\mu(x), \mu(y)\}$. Therefore μ is a prime fuzzy set in X.

Theorem 4.6. Let μ be a fuzzy set in X. Then μ is a prime fuzzy AT-subalgebra of X if and only if, for all $t \in [0, 1]$, $U(\mu;t)$ is a prime AT-subalgebra of X, if $U(\mu;t)$ is nonempty.

Proof. It is straightforwardby Theorem (4.2) and Theorem (4.5).

Theorem 4.7. Let μ be a fuzzy set in X. Then μ is a prime fuzzy AT-ideal of X if and only if for all $t \in [0, 1]$, $U(\mu;t)$ is a prime AT-ideal of X, if $U(\mu;t)$ is nonempty.

Proof. It is straightforwardby Theorem (4.3) and Theorem (4.5).

Theorem 4.8. Let μ be a fuzzy set in X. Then μ is a prime fuzzy AT-filter of X if and only if for all $t \in [0, 1]$, $U(\mu;t)$ is a prime AT-filter of X, if $U(\mu;t)$ is nonempty.

Proof. It is straightforwardby Theorem (4.4) and Theorem (4.5).

Theorem 4.9. Let μ be a fuzzy set in X. Then μ is a fuzzy AT-subalgebra of X if and only if for all $t \in [0, 1]$, $U^{+}(\mu;t)$ is an AT-subalgebra of X, if $U^{+}(\mu;t)$ is nonempty.

Proof.

Assume that μ is a fuzzy AT-subalgebra of X. Let $t \in [0, 1]$ be such that $U^+(\mu;t) \neq \emptyset$ and let $x, y \in U^+(\mu; t)$. Then $\mu(x) > t$ and $\mu(y) > t$, so t is a lower bound of $\{\mu(x), \mu(y)\}$. Since μ is a fuzzy AT-subalgebra of X, we have $\mu(x * y) \geq \min\{\mu(x), \mu(y)\} > t$. Thus $x * y \in U^+(\mu;t)$. So $U^+(\mu;t)$ is an AT-subalgebra of X.

Conversely, assume that for all $t \in [0, 1]$, $U^+(\mu;t)$ is AT-subalgebra of X, if $U^+(\mu;t)$ is nonempty. Let $x, y \in X$. Then $\mu(x), \mu(y) \in [0, 1]$. Choose $t = \min\{\mu(x), \mu(y)\}$. Then $\mu(x) > t$ and $\mu(y) > t$. Thus $x, y \in U^+(\mu;t) \neq \emptyset$. By assumption, we have $U^+(\mu; t)$ is AT-subalgebra of X. So $x * y \in U^+(\mu; t)$. Hence $\mu(x * y) \geq t = \min\{\mu(x), \mu(y)\}$. Therefore μ is a fuzzy AT-subalgebra of X.

Theorem 4.10. Let μ be a fuzzy set in X. Then μ is a fuzzy AT-ideal of X if and only if for all $t \in [0, 1]$, $U^+(\mu; t)$ is an AT-ideal of A, if $U^+(\mu; t)$ is nonempty.

Proof.

Assume that μ is a fuzzy AT-ideal of X. Let $t \in [0, 1]$ be such that $U^+(\mu;t) \neq \emptyset$ and le $a \in U^+(\mu; t)$. Then $\mu(a) > t$. Since μ is a fuzzy AT-ideal of X, we have $\mu(0) \geq \mu(a) > t$. Thus $0 \in U^+(\mu;t)$.

Next, let $x, y, z \in A$ be such that $x * (y * z) \in U^+(\mu;t)$ and $y \in U^+(\mu;t)$. Then $\mu(x * (y * z)) > t$ and $\mu(y) > t$. Thus t is a lower bound of $\{\mu(x * (y * z)), \mu(y)\}$. Since μ is a fuzzy AT-ideal of X, we have $\mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\} > t$. So $x * z \in U^+(\mu;t)$. Hence $U^+(\mu;t)$ is an AT-ideal of X.

Conversely, assume that for all $t \in [0, 1]$, $U^+(\mu;t)$ is an AT-ideal of X, if $U^+(\mu;t)$ is nonempty. Let $x \in X$. Then $\mu(x) \in [0, 1]$. Choose $t = \mu(x)$. Then $\mu(x) > t$. Thus $x \in U^+(\mu;t) \neq \emptyset$. By assumption, we have $U^+(\mu;t)$ is an AT-ideal of X. So $0 \in U^+(\mu;t)$. Hence $\mu(0) \geq t = \mu(x)$.

Next, let $x, y, z \in X$. Then $\mu(x * (y * z)), \mu(y) \in [0, 1]$. Choose $t = \min\{\mu(x * (y * z)), \mu(y)\}$. Then $\mu(x * (y * z)) > t$ and $\mu(y) > t$. Thus $x * (y * z), y \in U^+(\mu;t) \neq \emptyset$. By assumption, we have $U^+(\mu;t)$ is an AT-ideal of X. So $x * z \in U^+(\mu;t)$.

Hence $\mu(x * z) \geq t = \min\{\mu(x * (y * z)), \mu(y)\}$. Therefore μ is a fuzzy AT-ideal of X.

Theorem 4.11. Let μ be a fuzzy set in X. Then μ is a fuzzy AT-filter of X if and only if for all $t \in [0, 1]$, $U^+(\mu; t)$ is an AT-filter of X, if $U^+(\mu; t)$ is nonempty.

Proof.

Assume that μ is a fuzzy AT-filter of X. Let $t \in [0, 1]$ be such that $U(\mu;t) \neq \emptyset$ and let $a \in U^+(\mu; t)$. Then $\mu(a) \in t$. Since μ is a fuzzy AT-filter of X, we have $\mu(0) \geq \mu(a) > t$. Thus $0 \in U^+(\mu; t)$.

Next, let $x, y \in X$ be such that $x \in U^+(\mu;t)$ and $x * y \in U^+(\mu;t)$. Then $\mu(x) \geq t$ and $\mu(x * y) > t$. Thus t is a lower bound of $\{\mu(x * y), \mu(x)\}$. Since μ is a fuzzy AT-filter of X, we have $\mu(y) \geq \min\{\mu(x * y), \mu(x)\} > t$. So $y \in U^+(\mu;t)$. Hence $U^+(\mu;t)$ is an AT-filter of X.

Conversely, assume that for all $t \in [0, 1]$, $U^+(\mu;t)$ is an AT-filter of X, if $U^+(\mu;t)$ is nonempty. Let $x \in X$.

Then $\mu(x) \in [0, 1]$. Choose $t = \mu(x)$. Then $\mu(x) > t$. Thus $x \in U^+(\mu;t) \neq \emptyset$. By assumption, we have $U^+(\mu;t)$ is an AT-filter of X. So $0 \in U^+(\mu;t)$. Hence $\mu(0) > t = \mu(x)$.

Next, let $x, y \in X$. Then $\mu(x), \mu(x * y) \in [0, 1]$. Choose $t = \min\{\mu(x), \mu(x * y)\}$. Then $\mu(x) > t$ and $\mu(x * y) > t$. Thus $x, x * y \in U^+(\mu;t) \neq \emptyset$. By assumption, we have $U^+(\mu;t)$ is an AT-filter of X. So $y \in U^+(\mu;t)$.

Hence $\mu(y) \geq t = \min\{\mu(x), \mu(x * y)\}$. Therefore μ is a fuzzy AT-filter of X.

Theorem 4.12. Let μ be a fuzzy set in X. Then μ is a prime fuzzy set in X if and only if for all $t \in [0, 1]$, $U^+(\mu; t)$ is a prime subset of X, if $U^+(\mu; t)$ is nonempty.

Proof.

Assume that μ is a prime fuzzy set in X. Let $t \in [0, 1]$ be such that $U^+(\mu; t) \neq \emptyset$. Let $x, y \in X$ be such that $x * y \in U^+(\mu;t)$. Assume that $x \notin U^+(\mu;t)$ and $y \notin U^+(\mu;t)$. Then $\mu(x) \leq t$ and $\mu(y) \leq t$. Thus t is an upper bound of $\{\mu(x), \mu(y)\}$. Since μ is a prime fuzzy set in X, we have $\mu(x * y) \leq \max\{\mu(x), \mu(y)\} \leq t$. So $x * y \notin U^+(\mu;t)$, a contradiction. Hence $x \in U^+(\mu;t)$ or $y \in U^+(\mu;t)$. Therefore $U^+(\mu;t)$ is a prime subset of X.

Conversely, assume that for all $t \in [0, 1]$, $U^+(\mu;t)$ is a prime subset of X, if $U^+(\mu;t)$ is nonempty. Assume that there exist $x, y \in X$ such that $\mu(x * y) > \max\{\mu(x), \mu(y)\}$. Then $\max\{\mu(x), \mu(y)\} \in [0, 1]$. Choose $t = \max\{\mu(x), \mu(y)\}$.

Then $\mu(x \cdot y) > t$. Thus $x * y \in U^+(\mu;t) \neq \emptyset$. By assumption, we have $U^+(\mu;t)$ is a prime subset of X and thus $x \in U^+(\mu;t)$ or $y \in U^+(\mu;t)$. So $\mu(x) > t = \max\{\mu(x), \mu(y)\}$ or $\mu(y) > t = \max\{\mu(x), \mu(y)\}$, a contradiction.

Hence $\mu(x * y) \leq \max\{\mu(x), \mu(y)\}$, for all $x, y \in X$. Therefore μ is a prime fuzzy set in X.

Theorem 4.13. Let μ be a fuzzy set in X. Then μ is a prime fuzzy AT-subalgebra of X if and only if, for all $t \in [0, 1]$, $U^+(\mu;t)$ is a prime AT-subalgebra of X, if $U^+(\mu;t)$ is nonempty.

Proof. It is straightforwardby Theorem (4.9) and Theorem (4.12).

Theorem 4.14. Let μ be a fuzzy set in X. Then μ is a prime fuzzy AT-ideal of X if and only if for all $t \in [0, 1]$, $U^+(\mu;t)$ is a prime AT-ideal of X, if $U^+(\mu;t)$ is nonempty.

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 4, Issue 3 , March – 2020, Pages: 28-37

Proof. It is straightforwardby Theorem (4.10) and Theorem (4.12).

Theorem 4.15. Let μ be a fuzzy set in X. Then μ is a prime fuzzy AT-filter of X if and only if for all t ∈ [0, 1], $U^+(\mu;t)$ is a prime AT-filter of X, if $U^+(\mu;t)$ is nonempty.

Proof. It is straightforwardby Theorem (4.11) and Theorem (4.12).

5. Complement fuzzy sets in AT-algebras

In this section, we introduce the notions of a complement of fuzzy set in AT-algebra and related properties are investigated.

Definition5.1. Let μ be a fuzzy set in X. The fuzzy set μ^c defined by $\mu^c(x) = 1-\mu(x)$ for all x ∈ X is called **the complement of μ in X**.

Lemma 5.2. Let μ be a fuzzy set in X. Then the following statements hold for any x, y ∈ X,

(1) $1 - \max\{\mu(x), \mu(y)\} = \min\{1-\mu(x), 1-\mu(y)\}$,

(2) $1 - \min\{\mu(x), \mu(y)\} = \max\{1-\mu(x), 1-\mu(y)\}$.

Proof.

(1) If $\max\{\mu(x), \mu(y)\} = \mu(x)$, then $\mu(y) \le \mu(x)$. Thus $1-\mu(y) \ge 1-\mu(x)$. So $\min\{1-\mu(x), 1-\mu(y)\} = 1-\mu(x) = 1-\max\{\mu(x), \mu(y)\}$. Similarly, if $\max\{\mu(x), \mu(y)\} = \mu(y)$, then $\min\{1-\mu(x), 1-\mu(y)\} = 1-\mu(y) = 1-\max\{\mu(x), \mu(y)\}$.

(2) If $\min\{\mu(x), \mu(y)\} = \mu(x)$, then $\mu(x) \le \mu(y)$. Thus $1-\mu(x) \ge 1-\mu(y)$. So $\max\{1-\mu(x), 1-\mu(y)\} = 1-\mu(x) = 1-\min\{\mu(x), \mu(y)\}$. Similarly, if $\min\{\mu(x), \mu(y)\} = \mu(y)$, then $\max\{1-\mu(x), 1-\mu(y)\} = 1-\mu(y) = 1-\min\{\mu(x), \mu(y)\}$.

Theorem5.3. Let μ be a fuzzy set in X. Then μ^c is a fuzzy AT-subalgebra of X if and only if, for all t ∈[0, 1], L(μ ;t) is an AT-subalgebra of X, if L(μ ;t) is nonempty.

Proof.

Assume that μ^c is a fuzzy AT-subalgebra of X. Let t ∈ [0, 1] be such that L(μ ;t) ≠ ∅ and let x, y ∈ L(μ ;t). Then $\mu(x) \le t$ and $\mu(y) \le t$. Thus t is an upper bound of {μ (x), μ (y)}. Since μ is a fuzzy AT-subalgebra of X, we have $\mu(x * y) \ge \min\{\mu(x), \mu(y)\}$.

By Lemma (5.2(1)), we have $1-\mu(x * y) \ge \min\{1-\mu(x), 1-\mu(y)\} = 1-\max\{\mu(x), \mu(y)\}$.

Thus $\mu(x * y) \le \max\{\mu(x), \mu(y)\} \le t$. So x ∗ y∈L(μ ;t). Hence L(μ ;t) is an AT- subalgebra of X.

Conversely, assume that for all t ∈ [0, 1], L(μ ;t) is an AT-subalgebra of X, if L(μ ;t) is nonempty. Let x, y∈ X. Then μ (x), μ (y) ∈[0, 1]. Choose $t = \max\{\mu(x), \mu(y)\}$. Then $\mu(x) \le t$ and $\mu(y) \le t$. Thus x, y ∈ L(μ; t) ≠ ∅. By assumption, we have L(μ;t) is an AT-subalgebra of X and thus x ∗ y ∈ L(μ ;t). So $\mu(x * y) \le t = \max\{\mu(x), \mu(y)\}$. By Lemma (5.2(1)), we have

$$\mu^c(x * y) = 1 - \mu(x \cdot y) \ge 1-\max\{\mu(x), \mu(y)\}$$
$$= \min\{1 - \mu(x), 1 - \mu(y)\} = \min\{\mu^c(x), \mu^c(y)\}.$$

Therefore, μ is a fuzzy AT-subalgebra of X.

Theorem 5.4. Let μ be a fuzzy set in X. Then μ^c is a fuzzy AT-ideal of A if and only if, for all t ∈ [0, 1], L(μ; t) is an AT-ideal of X, if L(μ; t) is nonempty.

Proof.

Assume that μ^c is a fuzzy AT-ideal of X. Let t∈[0, 1] be such that L(μ ;t)≠∅ and let a ∈ L(μ ; t). Then μ(a) ≤ t. Since μ^c is a fuzzy AT-ideal of X, we have $\mu^c(0) \ge \mu^c(a)$. Thus $1-\mu(0) \ge 1-\mu(a)$. so $\mu(0) \le \mu(a) \le t$. Hence 0 ∈ L(μ ;t).

Next, let x, y, z ∈ X be such that x ∗ (y ∗ z) ∈ L(μ ;t) and y ∈ L(μ ;t). Then $\mu(x * (y * z)) \le t$ and $\mu(y) \le t$. Thus t is an upper bound of {μ (x ∗ (y ∗ z)), μ (y)}.

Since μ^c is a fuzzy AT-ideal of X, we have $\mu^c(x * z) \ge \min\{\mu^c(x * (y * z)), \mu^c(y)\}$. By Lemma(5.2(1)), we have $1-\mu(x * z) \ge \min\{1-\mu(x * (y * z)), 1-\mu(y)\} = 1-\max\{\mu(x * (y * z)), \mu(y)\}$.

So $\mu(x * z) \le \max\{\mu(x * (y * z)), \mu(y)\} \le t$ and thus x ∗ z ∈ L(μ ;t). Hence L(μ ;t) is an AT-ideal of X.

Conversely, assume that for all t ∈ [0, 1], L(μ ;t) is an AT-ideal of X, if L(μ ;t) is nonempty. Let x ∈ X. Then μ (x) ∈ [0, 1]. Choose t = μ(x). Then μ (x) ≤ t. Thus x ∈ L(μ ;t) ≠ ∅. By assumption, we have L(μ ;t) is an AT-ideal of X and thus 0 ∈ L(μ ;t). So $\mu(0) \le t = \mu(x)$. Hence $\mu^c(0) = 1-\mu(0) \ge 1-\mu(x) = \mu^c(x)$.

Next, let x, y, z ∈ X. Then μ (x ∗ (y ∗ z)), μ (y) ∈ [0, 1]. Choose $t = \max\{\mu(x * (y * z)), \mu(y)\}$. Then $\mu(x * (y * z)) \le t$ and $\mu(y) \le t$. Thus x ∗ (y ∗ z), y ∈ L(μ ;t) ≠ ∅. By assumption, we have L(μ;t) is an AT-ideal of X and thus x ∗ z ∈ L(μ ;t). So $\mu(x * z) \le t = \max\{\mu(x * (y * z)), \mu(y)\}$. By Lemma(5.2(1)), we have

$$\mu^c(x * z) = 1 - \mu(x * z) \ge 1-\max\{\mu(x * (y * z)), \mu(y)\}$$
$$= \min\{1 - \mu(x * (y * z)), 1 - \mu(y)\} = \min\{\mu^c(x * (y * z)), \mu^c(y)\}.$$

Hence μ^c is a fuzzy AT-ideal of X.

Theorem 5.5. Let μ be a fuzzy set in X. Then μ^c is a fuzzy AT-filter of X if and only if for all t ∈ [0, 1], L(μ ; t) is an AT-filter of X, if L(μ;t) is nonempty.

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 4, Issue 3 , March – 2020, Pages: 28-37

Proof.

Assume that μ^c is a fuzzy AT-filter of X. Let $t \in [0,1]$ be such that $L(\mu;t) \neq \emptyset$ and let $a \in L(\mu;t)$. Then $\mu(a) \le t$. Since μ^c is a fuzzy AT-filter of X, we have $\mu^c(0) \ge \mu^c(a)$. Thus $1-\mu(0) \ge 1-\mu(a)$, so $\mu(0) \le \mu(a) \le t$. So $0 \in L(\mu;t)$.

Next, let $x, y \in X$ be such that $x \in L(\mu;t)$ and $x * y \in L(\mu;t)$. Then $\mu(x) \le t$ and $\mu(x * y) \le t$. Thus t is an upper bound of $\{\mu(x), \mu(x * y)\}$. Since μ^c is a fuzzy AT-filter of X, we have $\mu^c(y) \ge \min\{\mu^c(x), \mu^c(x * y)\}$. By Lemma(5.2(1)), we have

$1-\mu(y) \ge \min\{1-\mu(x), 1-\mu(x * y)\} = 1-\max\{\mu(x), \mu(x * y)\}$.

So $\mu(y) \le \max\{\mu(x), \mu(x * y)\} \le t$ and thus $y \in L(\mu;t)$. Hence $L(\mu;t)$ is an AT-filter of X.

Conversely, assume that for all $t \in [0,1]$, $L(\mu;t)$ is an AT-filter of X if $L(\mu;t)$ is nonempty. Let $x \in X$. Then $\mu(x) \in [0,1]$. Choose $t = \mu(x)$. Then $\mu(x) \le t$. Thus $x \in L(\mu;t) \neq \emptyset$. By assumption, we have $L(\mu;t)$ is an AT-filter of X and thus $0 \in L(\mu;t)$. So $\mu(0) \le t = \mu(x)$. Hence $\mu^c(0) = 1-\mu(0) \ge 1-\mu(x) = \mu^c(x)$.

Next, let $x, y \in X$. Then $\mu(x), \mu(x * y) \in [0,1]$. Choose $t = \max\{\mu(x), \mu(x * y)\}$ Then $\mu(x) \le t$ and $\mu(x * y) \le t$. Thus $x, x * y \in L(\mu;t) \neq \emptyset$. By assumption, we have $L(\mu;t)$ is an AT-filter of X and thus $y \in L(\mu;t)$.

Thus $\mu(y) \le t = \max\{\mu(x), \mu(x * y)\}$. By Lemma(5.2(1)), we have $\mu^c(y) = 1-\mu(y) \ge 1-\max\{\mu(x), \mu(x * y)\}$
$= \min\{1-\mu(x), 1-\mu(x * y)\} = \min\{\mu^c(x), \mu^c(x * y)\}$. Hence μ^c is a fuzzy AT-filter of X.

Theorem 5.6. Let μ be a fuzzy set in X. Then μ^c is a prime fuzzy set in X if and only if for all $t \in [0,1]$, $L(\mu;t)$ is a prime subset of X, if $L(\mu;t)$ is nonempty.

Proof.

Assume that μ^c is a prime fuzzy set in X. Let $t \in [0,1]$ be such that $L(\mu;t) \neq \emptyset$. Let $x, y \in X$ be such that $x * y \in L(\mu;t)$. Assume that $x \notin L(\mu;t)$ and $y \notin L(\mu;t)$. Then $\mu(x) > t$ and $\mu(y) > t$. Thus t is a lower bound of $\{\mu(x), \mu(y)\}$. Since μ is a prime fuzzy set in X, we have $\mu^c(x * y) \le \max\{\mu^c(x), \mu^c(y)\}$. By Lemma(5.2(2)), we have

$1-\mu(x * y) \le \max\{1-\mu(x), 1-\mu(y)\} = 1-\min\{\mu(x), \mu(y)\}$. So $\mu(x * y) \ge \min\{\mu(x), \mu(y)\} > t$ and thus $x * y \notin L(\mu;t)$, a contradiction. Hence $x \in L(\mu;t)$ or $y \in L(\mu;t)$. Therefore $L(\mu;t)$ is a prime subset of X.

Conversely, assume that for all $t \in [0,1]$, $L(\mu;t)$ is a prime subset of X if $L(\mu;t)$ is nonempty. Let $x, y \in X$. Then $\mu(x * y) \in [0,1]$. Choose $t = \mu(x * y)$. Then $\mu(x * y) \le t$. Thus $x * y \in L(\mu;t) \neq \emptyset$. By assumption, we have $L(\mu;t)$ is a prime subset of X and thus $x \in L(\mu;t)$ or $y \in L(\mu;t)$. So $t \ge \mu(x)$ or $t \ge \mu(y)$. Hence $\mu(x * y) = t \ge \min\{\mu(x), \mu(y)\}$. By Lemma(5.2(2)), we have $\mu^c(x * y) = 1-\mu(x * y) \le 1-\min\{\mu(x), \mu(y)\} = \max\{1-\mu(x), 1-\mu(y)\} = \max\{\mu^c(x), \mu^c(y)\}$. Therefore μ^c is a prime fuzzy set in X.

Theorem 5.7. Let μ be a fuzzy set in X. Then μ^c is a prime fuzzy AT-subalgebra of X if and only if, for all $t \in [0,1]$, $L(\mu;t)$ is a prime AT-subalgebra of X, if $L(\mu;t)$ is nonempty.

Proof. It is straightforward by Theorem (5.3) and Theorem (5.6).

Theorem 5.8. Let μ be a fuzzy set in X. Then μ^c is a prime fuzzy AT-ideal of X if and only if for all $t \in [0,1]$, $L(\mu;t)$ is a prime AT-ideal of A, if $L(\mu;t)$ is nonempty.

Proof. It is straightforward by Theorem (5.4) and Theorem (5.6).

Theorem 5.9. Let μ be a fuzzy set in A. Then μ^c is a prime fuzzy AT-filter of X if and only if for all $t \in [0,1]$, $L(\mu;t)$ is a prime AT-filter of X, if $L(\mu;t)$ is nonempty.

Proof. It is straightforward by Theorem (5.5) and Theorem (5.6).

Theorem 5.10. Let μ be a fuzzy set in X. Then μ^c is a fuzzy AT-subalgebra of X if and only if, for all $t \in [0,1]$, $L^-(\mu;t)$ is an AT-subalgebra of X, if $L^-(\mu;t)$ is nonempty.

Proof.

Assume that μ^c is a fuzzy AT-subalgebra of X. Let $t \in [0,1]$ be such that $L^-(\mu;t) \neq \emptyset$ and let $x, y \in L^-(\mu;t)$.

Then $\mu(x) < t$ and $\mu(y) < t$. Thus t is an upper bound of $\{\mu(x), \mu(y)\}$. Since μ is a fuzzy AT-subalgebra of X, we have $\mu(x * y) \ge \min\{\mu(x), \mu(y)\}$. By Lemma (5.2(1)), we have $1-\mu(x * y) \ge \min\{1-\mu(x), 1-\mu(y)\} = 1-\max\{\mu(x), \mu(y)\}$.

Thus $\mu(x * y) \le \max\{\mu(x), \mu(y)\} < t$. So $x * y \in L^-(\mu;t)$. Hence $L^-(\mu;t)$ is an AT- subalgebra of X.

Conversely, assume that for all $t \in [0,1]$, $L^-(\mu;t)$ is an AT-subalgebra of X, if $L^-(\mu;t)$ is nonempty. Assume that there exist $x, y \in X$ such that $\mu^c(x * y) < \min\{\mu^c(x), \mu^c(y)\}$. By Lemma (5.2(1)), we have $1-\mu(x * y) < \min\{1-\mu(x), 1-\mu(y)\}$

$= 1 - \max\{\mu(x), \mu(y)\}$.

Now $\mu(x * y) \in [0, 1]$, we choose $t = \mu(x * y)$. Then $\mu(x) < t$ and $\mu(y) < t$. Thus $x, y \in L^-(\mu;t) \neq \emptyset$. By assumption, we have $L^-(\mu;t)$ is an AT-subalgebra of X and thus $x * y \in L^-(\mu;t)$. So $\mu(x * y) < t = \mu(x \mu y)$, a contradiction. Hence $\mu^c(x * y) \geq \text{mim}\{\mu^c(x), \mu^c(y)\}$.x, y∈ X. Therefore, μ is a fuzzy AT-subalgebra of X. .

Theorem 5.11. Let μ be a fuzzy set in X. Then μ^c is a fuzzy AT-ideal of A if and only if, for all $t \in [0, 1]$, $L^-(\mu; t)$ is an AT-ideal of X, if $L^-(\mu; t)$ is nonempty.

Proof.

Assume that μ^c is a fuzzy AT-ideal of X. Let $t \in [0, 1]$ be such that $L^-(\mu;t) \neq \emptyset$ and let $a \in L^-(\mu;t)$. Then $\mu(a) < t$. Since μ^c is a fuzzy AT-ideal of X, we have $\mu^c(0) \geq \mu^c(a)$. Thus $1 - \mu(0) \geq 1 - \mu(a)$. so $\mu(0) \leq \mu(a) < t$. Hence $0 \in L^-(\mu;t)$.

Next, let $x, y, z \in X$ be such that $x * (y * z) \in L^-(\mu;t)$ and $y \in L^-(\mu;t)$. Then $\mu(x * (y * z)) < t$ and $\mu(y) < t$. Thus t is an upper bound of $\{\mu(x * (y * z)), \mu(y)\}$.

Since μ^c is a fuzzy AT-ideal of X, we have $\mu^c(x * z) \geq \min\{\mu^c(x * (y * z)), \mu^c(y)\}$. By Lemma(5.2(1)), we have $1 - \mu(x * z) \geq \min\{1 - \mu(x * (y * z)), 1 - \mu(y)\} = 1 - \max\{\mu(x * (y * z)), \mu(y)\}$.

So $\mu(x * z) \leq \max\{\mu(x * (y * z)), \mu(y)\} < t$ and thus $x * z \in L^-(\mu;t)$. Hence $L^-(\mu;t)$ is an AT-ideal of X.

Conversely, assume that for all $t \in [0, 1]$, $L^-(\mu;t)$ is an AT-ideal of X, if $L^-(\mu;t)$ is nonempty. Assume that there exists $x \in A$ such that $\mu^c(0) < \mu^c(x)$. Then $1 - \mu(0) < 1 - \mu(x)$. Thus $\mu(0) > \mu(x)$.

Now $\mu(0) \in [0, 1]$, we choose $t = \mu(0)$. Then $\mu(x) < t$. Thus $x \in L^-(\mu;t) \neq \emptyset$. By assumption, we have $L^-(\mu;t)$ is an AT-ideal of X and thus $0 \in L^-(\mu;t)$. So $\mu(0) < t = \mu(0)$, a contradiction. Hence $\mu^c(0) \geq \mu^c(x)$, for all $x \in X$.

Assume that there exist $x, y, z \in X$ such that $\mu^c(x * z) < \min\{\mu^c(x * (y * z)), \mu^c(y)\}$. By Lemma (5.2(1)), we have $1 - \mu(x * z) < \min\{1 - \mu(x * (y * z)), 1 - \mu(y)\} = 1 - \max\{\mu(x * (y * z)), \mu(y)\}$.Then $\mu(x * z) > \max\{\mu(x * (y * z)), \mu(y)\}$.

Now $\mu(x * z) \in [0, 1]$, we choose $t = \mu(x * z)$. Then $\mu(x * (y * z)) < t$ and $\mu(y) < t$. Thus $x * (y * z), y \in L^-(\mu;t) \neq \emptyset$. By assumption, we have $L^-(\mu;t)$ is an AT-ideal of A and thus $x * z \in L^-(\mu;t)$. So $\mu(x * z) < t = \mu(x * z)$, a contradiction.

Hence $\mu^c(x * z) \geq \min\{\mu^c(x * (y * z)), \mu^c(y)\}$, for all $x, y, z \in X$. Therefore μ^c is a fuzzy AT-ideal.

Theorem 5.12. Let μ be a fuzzy set in X. Then μ^c is a fuzzy AT-filter of X if and only if for all $t \in [0, 1]$, $L^-(\mu; t)$ is an AT-filter of X, if $L^-(\mu;t)$ is nonempty.

Proof.

Assume that μ^c is a fuzzy AT-filter of X. Let $t \in [0, 1]$ be such that $L^-(\mu;t) \neq \emptyset$ and let $a \in L^-(\mu;t)$. Then $\mu(a) < t$. Since μ^c is a fuzzy AT-filter of X, we have $\mu^c(0) \geq \mu^c(a)$. Thus $1 - \mu(0) \geq 1 - \mu(a)$, so $\mu(0) \leq \mu(a) < t$. So $0 \in L^-(\mu;t)$.

Next, let $x, y \in X$ be such that $x \in L^-(\mu; t)$ and $x * y \in L^-(\mu; t)$. Then $\mu(x) < t$ and $\mu(x * y) < t$. Thus t is an upper bound of $\{\mu(x), \mu(x * y)\}$. Since μ^c is a fuzzy AT-filter of X, we have $\mu^c(y) \geq \min\{\mu^c(x), \mu^c(x * y)\}$. By Lemma(5.2(1)), we have $1 - \mu(y) \geq \min\{1 - \mu(x), 1 - \mu(x * y)\} = 1 - \max\{\mu(x), \mu(x * y)\}$.

So $\mu(y) \leq \max\{\mu(x), \mu(x * y)\} < t$ and thus $y \in L^-(\mu;t)$. Hence $L^-(\mu;t)$ is an AT-filter of X.

Conversely, assume that for all $t \in [0, 1]$, $L^-(\mu;t)$ is an AT-filter of A, if $L^-(\mu;t)$ is nonempty. Assume that there exists $x \in X$ such that $\mu^c(0) < \mu^c(x)$. Then $1 - \mu(0) < 1 - \mu(x)$. Thus $\mu(0) > \mu(x)$.

Now $\mu(0) \in [0, 1]$, we choose $t = \mu(0)$. Then $\mu(x) < t$. Thus $x \in L^-(\mu;t) \neq \emptyset$. By assumption, we have $L^-(\mu;t)$ is an AT-filter of X and thus $0 \in L^-(\mu;t)$. So $\mu(0) < t = \mu(0)$, a contradiction. Hence $\mu(0) \geq \mu(x)$, for all $x \in X$.

Assume that there exist $x, y \in X$ such that $\mu^c(y) < \min\{\mu^c(x), \mu^c(x * y)\}$. By Lemma (5.2(1)), we have $1 - \mu(y) < \min\{1 - \mu(x), 1 - \mu(x * y)\} = 1 - \max\{\mu(x), \mu(x * y)\}$.

Then $\mu(y) > \max\{\mu(x), \mu(x * y)\}$.

Now $\mu(y) \in [0, 1]$, we choose $t = \mu(y)$. Then $\mu(x) < t$ and $\mu(x * y) < t$. Thus $x, x \cdot y \in L^-(\mu;t) \neq \emptyset$. By assumption, we have $L^-(\mu;t)$ is an AT-filter of X and thus $y \in L^-(\mu;t)$. So $\mu(y) < t = \mu(y)$, a contradiction. Hence $\mu^c(y) \geq \min\{\mu^c(x), \mu^c(x * y)\}$, for all $x, y \in X$. Therefore μ^c is a fuzzy AT-filter of X.

Theorem 5.13. Let μ be a fuzzy set in X. Then μ^c is a prime fuzzy set in X if and only if for all $t \in [0, 1]$, $L^-(\mu;t)$ is a prime subset of X, if $L^-(\mu;t)$ is nonempty.

Proof.

Assume that μ is a prime fuzzy set in X. Let $t \in [0,1]$ be such that $L^{-}(\mu;t) \neq \emptyset$. Let $x, y \in X$ be such that $x \cdot y \in L^{-}(\mu;t)$. Assume that $x \notin L^{-}(\mu;t)$ and $y \notin L^{-}(\mu;t)$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$. Thus t is a lower bound of $\{\mu(x), \mu(y)\}$. Since μ^c is a prime fuzzy set in X, we have $\mu^c(x \cdot y) \leq \max\{\mu^c(x), \mu^c(y)\}$. By Lemma (5.2(2)), we have $1-\mu(x * y) \leq \max\{1-\mu(x), 1-\mu(y)\} = 1-\min\{\mu(x), \mu(y)\}$. So $\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \geq t$ and thus $x * y \notin L^{-}(\mu;t)$, a contradiction. Hence $x \in L^{-}(\mu;t)$ or $y \in L^{-}(\mu;t)$. Therefore $L^{-}(\mu;t)$ is a prime subset of X.

Conversely, assume that for all $t \in [0,1]$, $L^{-}(\mu;t)$ is a prime subset of X, if $L^{-}(\mu;t)$ is nonempty. Assume that there exist x, $y \in X$ such that $\mu^c(x * y) > \max\{\mu^c(x), \mu^c(y)\}$. By Lemma (5.2(2)), we have $1-\mu(x * y) > \max\{1-\mu(x), 1-\mu(y)\} = 1-\min\{\mu(x), \mu(y)\}$. Then $\mu(x * y) < \min\{\mu(x), \mu(y)\}$.

Now $\min\{\mu(x), \mu(y)\} \in [0,1]$, we choose $t = \min\{\mu(x), \mu(y)\}$. Then $\mu(x * y) < t$. Thus $x * y \in L^{-}(\mu;t) \neq \emptyset$. By assumption, we have $L^{-}(\mu;t)$ is a prime subset of X and thus $x \in L^{-}(\mu;t)$ or $y \in L^{-}(\mu;t)$.

So $\mu(x) < t = \min\{\mu(x), \mu(y)\}$ or $\mu(y) < t = \min\{\mu(x), \mu(y)\}$, a contradiction. Hence $\mu^c(x * y) \leq \max\{\mu^c(x), \mu^c(y)\}$, for all $x, y \in X$. Therefore μ^c is a prime fuzzy set in X.

Theorem 5.14. Let μ be a fuzzy set in X. Then μ^c is a prime fuzzy AT-subalgebra of X if and only if, for all $t \in [0,1]$, $L^{-}(\mu;t)$ is a prime AT-subalgebra of X, if $L^{-}(\mu;t)$ is nonempty.

Proof. It is straightforward by Theorem (5.10) and Theorem (5.13).

Theorem 5.15. Let μ be a fuzzy set in X. Then μ^c is a prime fuzzy AT-ideal of X if and only if for all $t \in [0,1]$, $L^{-}(\mu;t)$ is a prime AT-ideal of A, if $L^{-}(\mu;t)$ is nonempty.

Proof. It is straightforward by Theorem (5.11) and Theorem (5.13).

Theorem 5.16. Let μ be a fuzzy set in A. Then μ^c is a prime fuzzy AT-filter of X if and only if for all $t \in [0,1]$, $L^{-}(\mu;t)$ is a prime AT-filter of X, if $L^{-}(\mu;t)$ is nonempty.

Proof. It is straightforward by Theorem (5.12) and Theorem (5.13).

6. Acknowledgment

The authors wish to express their sincere thanks to the referees for the valuable suggestions which lead to an improvement of this paper.

7. References

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