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Linear Transformations of $Type(I)$ and Linear Transformations of $Type(II)$

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Abstract: *In this paper, we introduce certain types of linear transformations such that linear transformations of $type(I)$ and of $type(II)$ with respected to ordered bases. We prove that these linear transformations are invertible and we discuss the composition of two linear transformations of each type. We give a formula to find the inverse of the matrix representation of each type.*

Keywords: Linear transformation, orthogonal of $type(I)$ matrix, orthogonal of $type(II)$ matrix.

1 Introduction

A linear transformation is an important tool in many branch of mathematics and other sciences such as engineering, physics, etc. The study of system of linear equations first introduced by the Babylonians around at 1800 BC. For solving of system of linear equations, Cardan constructed a simple rule for two linear equations with unknowns around at 1550 AD [6]. The idea of representation of a linear substitution (i.e., a linear transformation) by the square array of its defining coefficients is already found in Gauss's treatment of the arithmetical theory of quadratic forms in 1801[4].

The linear transformations were represented as rectangular arrays of numbers matrices, although Gauss did not use matrix terminology. He also defined implicitly the product of matrices (for the 2×2 and 3×3 cases only); he had in the composition of the corresponding linear transformations.

Linear transformations appeared prominently in the analytic geometry of the seventeenth and eighteenth centuries. linear transformations also show up in projective geometry, founded in the seventeenth century and described analytically in the early nineteenth.[1]

In geometrical format, vector spaces in R^2 and R^3 were first represented by Descartes and Fermat. Italian mathematician Peana defined the known modern definition of vector space concept in 1900's (Katza, 1995). After Peana, another Italian mathematician Weyl used vector spaces more efficiently and attractively in his studies that registered 'vector space and transformations view' [6]. We used orthogonal of $type(I)$ and orthogonal of $type(II)$ matrices [4][1], to introduce certain types of linear transformations with respected to ordered bases.

In this paper, T, T^{-1} , $\mathcal{M}_{\beta}^{\alpha}(T)$, A^{-1}, A^* , A^T, $\mathbb{C}^n$, $\mathbb{R}^n$, $\mathcal{F}$, $\mathcal{M}_{n\times n}(\mathcal{F})$, N, I_n, $I-O\mathcal{M}_{n\times n}(\mathcal{F})$,

$II-O\mathcal{M}_{n\times n}(\mathcal{F})$ *and* lcm means a linear transformation, the inverse of linear transformation, matrix representation of T with respect to the ordered bases α *and* β , the inverse of A , the transpose conjugate(conjugate transpose) of A, the transpose of A, the set of all $n \times 1$ complex matrices, the set of all $n \times 1$ real matrices, the field of real (or complex) numbers, the set of all $n \times n$ real(complex) matrices, the Neutral numbers, the $n \times n$ identity matrix, the set of all $n \times n$ orthogonal of $type(I)$ matrices, the set of all $n \times n$ orthogonal of $type(II)$ matrices, and lower common divisor, respectively.

2 preliminaries

Definition 2.1 [2]: A matrix A *of* $\mathcal{M}_{n\times n}(\mathcal{F})$ is called an orthogonal of $type(I)$ matrix if $A^l(A^T)^l = I_n$ *for some* $l \in N$. The smallest positive integer l with

$A^l(A^T)^l = I_n$ is called the index of A.

Theorem 2.2 [2] If A belong to $I-O\mathcal{M}_{n\times n}(\mathcal{F})$ of index , then its inverse is given by $A^{-1} = A^{l-1}(A^T)^l$.

Definition 2.3[3] A matrix A *of* $\mathcal{M}_{n\times n}(\mathcal{F})$ is called an orthogonal of $type(II)$ matrix if $A^r(A^*)^r = I_n$, *for some* $r \in N$. The smallest positive integer r with

$A^r(A^*)^r = I_n$ is called the index of A.

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Theorem 2.4 [3] If A $belong\ to\ II - O\mathcal{M}_{n\times n}(\mathcal{F})$ of index r, then its inverse is given by

$A^{-1} = A^{r-1}(A^*)^r$.

Theorem 2.5[4] Suppose that $T_1: X_1 \longrightarrow X_2 and\ T_2: X_2 \longrightarrow X_3$ be linear transformations. Let X_1 be an n-dimensional, X_2 be an m-dimensional and X_3 be an k-dimensional. If the ordered bases $\alpha, \beta, and\ \gamma$ are chosen for $X_1, X_2 and\ X_3$, respectively , then $\mathcal{M}_\gamma^\alpha(T_1 \circ T_2) = \mathcal{M}_\beta^\alpha(T_1)\mathcal{M}_\gamma^\beta(T_2)$.

Theorem 2.6[7] Let X and Y be finite-dimensional vector spaces with ordered bases $\alpha\ and\ \beta$, respectively. Let $T: X \rightarrow Y$ be linear transformation. Then T is invertible if and only if $[T]_\beta^\alpha$ is invertible. Furthermore, $[T^{-1}]_\beta^\alpha = ([T]_\beta^\alpha)^{-1}$.

Theorem 2.7 [7] Let X and Y be finite-dimensional vector spaces with ordered bases $\alpha\ and\ \beta$, respectively, and let $T, U: X \rightarrow Y$ be linear transformations. Then

i-$\mathcal{M}_\beta^\alpha\ [T + U] = \mathcal{M}_\beta^\alpha(T) + \mathcal{M}_\beta^\alpha(U)$

ii- $(\delta\mathcal{M})_\beta^\alpha(T) = \delta\mathcal{M}_\beta^\alpha(T)$ for all scalars a.

Theorem 2.8[4] Let $T_1 and\ T_2$ be linear transformations. Then $\mathcal{M}_\gamma^\alpha(T_1 + T_2) = \mathcal{M}_\beta^\alpha(T_1) + \mathcal{M}_\gamma^\beta(T_2)$.

Theorem 2.9[3] If $B_{n\times n}$ and $H_{n\times n}$ are commute orthogonal of $type(I)$ of indices $l_1 and\ l_2$ respectively, then BH is orthogonal of $type(I)$ matrix of index $lcm\{l_1, l_2\}$.

3 Main Results

Definition 3.1 Let $T: X \longrightarrow Y$ be a linear transformation. Let $s_X\ and\ s_Y$ be ordered bases of $X\ and\ Y$, respectively. Then T is called a linear transformation of $type(I)$ with respect to ordered bases $s_X\ and\ s_Y$, if the matrix representation of T with respect to $s_X\ and\ s_Y$ is an orthogonal of $type(I)$ matrix.

Example 3.2 Let $X = \mathbb{C}^2\ over\ \mathbb{C}\ and\ Y = \mathbb{C}^2\ over\ \mathbb{C}\ , let\ T: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be the linear transformation defined by ,$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, consider two ordered bases $\alpha = \left\{\begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix}\right\} and\ \beta = \left\{\begin{bmatrix} -i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -i \end{bmatrix}\right\}$. Then

$$T\left(\begin{bmatrix} i \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} i \\ 0 \end{bmatrix} = \begin{bmatrix} i \\ 0 \end{bmatrix}$$

and

$$T\left(\begin{bmatrix} 0 \\ i \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ i \end{bmatrix}$$

Thus, $\mathcal{M}_\beta^\alpha(T)$=$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is the matrix representation of T.

I f $l = 1, \mathcal{M}_\beta^\alpha(T)(\mathcal{M}_\beta^\alpha(T))^T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

Then $\mathcal{M}_\beta^\alpha(T) \in I - O\mathcal{M}_{n\times n}(\mathcal{F})$ and its index is 1, so T is a linear transformation of $type(I)$ with respect to $\alpha\ and\ \beta$.

Example 3.3 Let $X = \mathbb{R}^3\ over\ \mathbb{R}\ and\ Y = \mathbb{C}^3\ over\ \mathbb{C}$,let $T: \mathbb{R}^3 \longrightarrow \mathbb{C}^3$ be the linear transformation defined by defined by

$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, consider the two ordered bases

$\alpha = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\} and\ \beta = \left\{\begin{bmatrix} i \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ i \end{bmatrix}\right\}$

$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

and

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\-1\\0\end{bmatrix}$$

Thus $\mathcal{M}_\beta^\alpha(T)=\begin{bmatrix}-i&0&0\\0&0&i\\0&-i&0\end{bmatrix}$ is the matrix representation of T.

If $l=1, \mathcal{M}_\beta^\alpha(T)\left(\mathcal{M}_\beta^\alpha(T)\right)^T=\begin{bmatrix}-i&0&0\\0&0&i\\0&-i&0\end{bmatrix}\begin{bmatrix}-i&0&0\\0&0&i\\0&-i&0\end{bmatrix}$

$$=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$$

Now $l=2$

$$\left(\mathcal{M}_\beta^\alpha(T)\right)^2\left(\left(\mathcal{M}_\beta^\alpha(T)\right)^T\right)^2=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Thus $\mathcal{M}_\beta^\alpha(T)\in I-O\mathcal{M}_{n\times n}(\mathcal{F})$ and its index is 2,so T is a linear transformation of $type(I)$ with respect to $\alpha\ and\ \beta$.

Example 3.4 Let $X=\mathbb{R}^2\ over\ \mathbb{R}\ and\ Y=\mathbb{C}^2\ over\ \mathbb{C}$, let $T:\mathbb{R}^2\longrightarrow\mathbb{C}^2$ be the linear transformation defined by $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=\begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$, consider two ordered bases $\alpha=\left\{\begin{bmatrix}i\\0\end{bmatrix},\begin{bmatrix}0\\i\end{bmatrix}\right\}\ and \beta=\left\{\begin{bmatrix}i\\0\end{bmatrix},\begin{bmatrix}0\\i\end{bmatrix}\right\}$

$$T\left(\begin{bmatrix}i\\0\end{bmatrix}\right)=\begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}i\\0\end{bmatrix}=\begin{bmatrix}i\\i\end{bmatrix}$$

and

$$T\left(\begin{bmatrix}0\\i\end{bmatrix}\right)=\begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}0\\i\end{bmatrix}=\begin{bmatrix}-i\\i\end{bmatrix}$$

Thus $\mathcal{M}_\beta^\alpha(T)=\begin{bmatrix}1&-1\\1&1\end{bmatrix}$ is the matrix representation of T.

If $l=1, (\mathcal{M}_\beta^\alpha(T))(\mathcal{M}_\beta^\alpha(T))^T=\begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}1&1\\-1&1\end{bmatrix}=\begin{bmatrix}2&0\\0&2\end{bmatrix}\neq I_2$

Next $l=2, (\mathcal{M}_\beta^\alpha(T))^2((\mathcal{M}_\beta^\alpha(T))^T)^2=\begin{bmatrix}0&2\\2&0\end{bmatrix}\begin{bmatrix}0&2\\2&0\end{bmatrix}=\begin{bmatrix}4&0\\0&4\end{bmatrix}\neq I_2$

Assume now $l=k,\left(\mathcal{M}_\beta^\alpha(T)\right)^k((\mathcal{M}_\beta^\alpha(T))^T)^k=\begin{bmatrix}(2)^k&0\\0&(2)^k\end{bmatrix}$.

Then $\mathcal{M}_\beta^\alpha(T)\notin I-O\mathcal{M}_{n\times n}(\mathcal{F})$, so T is not linear transformation of $type(I)$.

Theorem 3.5 Let $T:X\longrightarrow Y$ be a linear transformation of $type(I)$ with respect to $\alpha\ and\ \beta$, $\mathcal{M}_\beta^\alpha(T)$ be the matrix representation of T. Then T is invertible with $(\mathcal{M}_\beta^\alpha(T))^{-1}=\mathcal{M}_\beta^\alpha(T)^{l-1}(\mathcal{M}_\beta^\alpha(T)^T)^l$.

Proof:

By Theorem 2.2, $\mathcal{M}_\beta^\alpha(T) \in I - O\mathcal{M}_{n\times n}(\mathcal{F})$, then it is invertible with $(\mathcal{M}_\beta^\alpha(T))^{-1} = \mathcal{M}_\beta^\alpha(T)^{l-1}(\mathcal{M}_\beta^\alpha(T)^T)^l$

Therefore T is invertible (Theorem 2.6). ■

Example 3.6 Let $X = \mathbb{C}^2\ over\ \mathbb{C}\ and\ Y = \mathbb{C}^2\ over\ \mathbb{C}$, let $T: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be the linear transformation defined by ,$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, consider two ordered bases $\alpha = \left\{\begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix}\right\}\ and\ \beta = \{\begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$

$$T\left(\begin{bmatrix} i \\ 0 \end{bmatrix}\right) = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} i \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ i \end{bmatrix}\right) = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ i \end{bmatrix}$$

Thus $\mathcal{M}_\beta^\alpha(T) = \begin{bmatrix} -1 & 0 \\ 1 & -i \end{bmatrix}$ is the matrix representation of T.

Firstly $l = 1$, $\mathcal{M}_\beta^\alpha(T)(\mathcal{M}_\beta^\alpha(T))^T = \begin{bmatrix} -1 & 0 \\ 1 & i \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$

Next $l = 2$,

$$\left(\mathcal{M}_\beta^\alpha(T)\right)^2\left(\left(\mathcal{M}_\beta^\alpha(T)\right)^T\right)^2 = \begin{bmatrix} 1 & 0 \\ -1-i & -1 \end{bmatrix}\begin{bmatrix} 1 & -1-i \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1-i \\ -1-i & 1+2i \end{bmatrix}$$

Suppose now that $l = 3$

$$(\mathcal{M}_\beta^\alpha(T))^3\left((\mathcal{M}_\beta^\alpha(T))^T\right)^3 = \begin{bmatrix} -1 & 0 \\ i & i \end{bmatrix}\begin{bmatrix} -1 & i \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & -i \\ -i & -2 \end{bmatrix}$$

Finally $l = 4$, $\left(\mathcal{M}_\beta^\alpha(T)\right)^4\left(\left(\mathcal{M}_\beta^\alpha(T)\right)^T\right)^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $\mathcal{M}_\beta^\alpha(T) \in I - O\mathcal{M}_{n\times n}(\mathcal{F})$ and its index is 4, so T is invertible linear transformation of $type(I)$ with respect to $\alpha\ and\ \beta$.

$$\mathcal{M}_\beta^\alpha(T^{-1}) = \left(\mathcal{M}_\beta^\alpha(T)\right)^{-1}$$

$$= (\mathcal{M}_\beta^\alpha(T))^3((\mathcal{M}_\beta^\alpha(T))^T)^4 = \begin{bmatrix} -1 & 0 \\ i & i \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ i & i \end{bmatrix}$$

Theorem 3.7 Let $T_1\,and\ T_2$ be linear transformations of $ype(I)$, Let $\mathcal{M}_\beta^\alpha(T_1)\ and\ \mathcal{M}_\gamma^\beta(T_2)$ be the matrices representation of $T_1 and\ T_2\ respectively\ and\ \mathcal{M}_\beta^\alpha(T_1), \mathcal{M}_\gamma^\beta(T_2)$ are commute. Then

$T_1 \circ T_2$ is a linear transformation of $type(I)$ with respect to $\alpha\ and\ \gamma$..

Proof:

By Theorem 2.5, $\mathcal{M}_\gamma^\alpha(T_1 \circ T_2) = \mathcal{M}_\beta^\alpha(T_1)\mathcal{M}_\gamma^\beta(T_2)$

Since $\mathcal{M}_{\beta}^{\alpha}(T_1)$ *and* $\mathcal{M}_{\gamma}^{\beta}(T_2)$ are orthogonal of $type(I)$ and commute, by Theorem 2.9, $\mathcal{M}_{\beta}^{\alpha}(T_1)\mathcal{M}_{\gamma}^{\beta}(T_2)$ is orthogonal of $type(I)$. Hence $T_1 \circ T_2$ is a linear transformation of $type(I)$ with respect to α *and* γ. ■

Example 3.8: Let $X = \mathbb{C}^2 = Y$ *over* $\mathbb{C}$ *and* $W = \mathbb{R}^2$ *over* $\mathbb{R}$, let T_1 *and* T_2 be the linear transformation defined bys, $T_1 : \mathbb{C}^2 \longrightarrow \mathbb{R}^2$, $T_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, consider two ordered bases

$$\alpha = \left\{\begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix}\right\} \text{ and } \beta = \left\{\begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -i \end{bmatrix}\right\}$$

and $T_2 : \mathbb{R}^2 \longrightarrow \mathbb{C}^2, T_2\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, consider two ordered bases

$$\beta = \left\{\begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -i \end{bmatrix}\right\}, \gamma = \left\{\begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right\}$$

$$T_1\left(\begin{bmatrix} i \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}\begin{bmatrix} i \\ 0 \end{bmatrix} = \begin{bmatrix} i \\ 0 \end{bmatrix}$$

and

$$T_1\left(\begin{bmatrix} 0 \\ i \end{bmatrix}\right) = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ i \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix}$$

Thus $\mathcal{M}_{\beta}^{\alpha}(T_1) = \begin{bmatrix} 1 & i \\ 0 & -1 \end{bmatrix}$ *is the matrix representation of* T_1.

If $l = 1$, $\mathcal{M}_{\beta}^{\alpha}(T_1)\left(\mathcal{M}_{\beta}^{\alpha}(T_1)\right)^T = \begin{bmatrix} 1 & i \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ i & -1 \end{bmatrix}$

$$= \begin{bmatrix} 0 & -i \\ -i & 1 \end{bmatrix}$$

Next $l = 2$, $\left(\mathcal{M}_{\beta}^{\alpha}(T_1)\right)^2((\mathcal{M}_{\beta}^{\alpha}(T_1))^T)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $\mathcal{M}_{\beta}^{\alpha}(T_1) \in I - O\mathcal{M}_{n\times n}(\mathcal{F})$ and its index is 2, so T_1 is a linear transformation of $type(I)$ with respect to α *and* β.

$$T_2\left(\begin{bmatrix} i \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} i \\ 0 \end{bmatrix} = \begin{bmatrix} -i \\ 0 \end{bmatrix}$$

and

$$T_2\left(\begin{bmatrix} 0 \\ -i \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -i \end{bmatrix} = \begin{bmatrix} 0 \\ -i \end{bmatrix}$$

Thus $\mathcal{M}_{\gamma}^{\beta}(T_2) = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ *is the matrix representation of* T_2.

If $l = 1, \mathcal{M}_{\gamma}^{\beta}(T_2)(\mathcal{M}_{\gamma}^{\beta}(T_2))^T = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Assume now $l = 2$, $\left(\mathcal{M}_{\gamma}^{\beta}(T_2)\right)^2((\mathcal{M}_{\gamma}^{\beta}(T_2))^T)^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Then $\mathcal{M}_{\gamma}^{\beta}(T_2) \in I - O\mathcal{M}_{n\times n}(\mathcal{F})$ and its index is 2, so T_2 is a linear transformation of $type(I)$ with respect to β *and* γ

Thus $\mathcal{M}_{\gamma}^{\alpha}(T_1 \circ T_2) = \mathcal{M}_{\beta}^{\alpha}(T_1)\mathcal{M}_{\gamma}^{\beta}(T_2) = C = \mathcal{M}_{\gamma}^{\beta}(T_2)\mathcal{M}_{\beta}^{\alpha}(T_1)$

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Then $C = \begin{bmatrix} 1 & i \\ 0 & -1 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} i & -1 \\ 0 & -i \end{bmatrix}$

Firstly $k = 1, CC^T = \begin{bmatrix} i & -1 \\ 0 & -i \end{bmatrix}\left[\begin{bmatrix} i & 0 \\ -1 & -i \end{bmatrix}\right] = \begin{bmatrix} 0 & i \\ i & -1 \end{bmatrix}$

Now $k = 2, C^2(C^T)^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then $C \in I - O\mathcal{M}_{n\times n}(\mathcal{F})$ and its index is 2. Thus $T_1 \circ T_2$ is a linear transformation of $type(I)$ with respect to $\alpha \, and \, \gamma$.

Remark 3. If $T_1 \, and \, T_2$ are $linear \, transformations \, of type(I)$ with respect to $\alpha \, and \, \beta$ and with respect to $\beta \, and \, \gamma$, respectively, then $T_1 + T_2$ is not necessarily be $linear \, transformation \, of \, type(I)$. The following example shown that.

Example 3.10: : Let $X = \mathbb{C}^2 = Y = W \, over \, \mathbb{C} \, T_1 \, and \, T_2$ be the linear transformation defined bys , $T_1: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$, $T_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ consider two ordered bases

$$\alpha = \left\{\begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix}\right\} and \; \beta = \left\{\begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix}\right\}$$

$and \;\; T_2: \mathbb{C}^2 \longrightarrow \mathbb{C}^2, T_2\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, consider two ordered bases

$$\beta = \left\{\begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix}\right\} and \; \gamma = \left\{\begin{bmatrix} 0 \\ i \end{bmatrix}, \begin{bmatrix} i \\ 0 \end{bmatrix}\right\}.\text{ Then}$$

$$T_1\left(\begin{bmatrix} i \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} i \\ 0 \end{bmatrix} = \begin{bmatrix} i \\ -1 \end{bmatrix}$$

and

$$T_1\left(\begin{bmatrix} 0 \\ i \end{bmatrix}\right) = \begin{bmatrix} 1 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 \\ i \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Thus $\mathcal{M}_\beta^\alpha(T_1) = \begin{bmatrix} i & -1 \\ 0 & 1 \end{bmatrix}$ is the matrix representation of T_1.

Firstly let $l = 1$

$\mathcal{M}_\beta^\alpha(T_1)(\mathcal{M}_\beta^\alpha(T_1))^T = \begin{bmatrix} i & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} i & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$

Next $l = 2$

$(\mathcal{M}_\beta^\alpha(T_1))^2((\mathcal{M}_\beta^\alpha(T_1))^T)^2 = \begin{bmatrix} -1 & -1-i \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ -1-i & 1 \end{bmatrix} = \begin{bmatrix} 1+2i & -1-i \\ -1-i & 1 \end{bmatrix}$

Suppose now that $l = 3$

$(\mathcal{M}_\beta^\alpha(T_1))^3((\mathcal{M}_\beta^\alpha(T_1))^T)^3 = \begin{bmatrix} -i & -i \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -i & 0 \\ -i & 1 \end{bmatrix} = \begin{bmatrix} -2 & -i \\ -i & 1 \end{bmatrix}$

Finally $l = 4$

$(\mathcal{M}_\beta^\alpha(T_1))^4((\mathcal{M}_\beta^\alpha(T_1))^T)^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Then $\mathcal{M}_\beta^\alpha(T_1) \in I - O\mathcal{M}_{n\times n}(\mathcal{F})$ and its index is 4, so

T_1 is a $linear \, transformation \, of \, type(I)$ with respect to $\alpha \, and \, \beta$.

Now $T_2\left(\begin{bmatrix}1\\ i\end{bmatrix}\right)=\begin{bmatrix}i & 0\\ 0 & i\end{bmatrix}\begin{bmatrix}1\\ i\end{bmatrix}=\begin{bmatrix}i\\ -1\end{bmatrix}$

and

$$T_2\left(\begin{bmatrix}0\\ i\end{bmatrix}\right)=\begin{bmatrix}i & 0\\ 0 & i\end{bmatrix}\begin{bmatrix}0\\ i\end{bmatrix}=\begin{bmatrix}0\\ -1\end{bmatrix}$$

Thus $\mathcal{M}_{\gamma}^{\beta}(T_2)=\begin{bmatrix}1 & 0\\ -1 & i\end{bmatrix}$ is the matrix representation of T_2.

Firstly $l=1$, $\mathcal{M}_{\gamma}^{\beta}(T_2)\left(\mathcal{M}_{\gamma}^{\beta}(T_2)\right)^T=\begin{bmatrix}1 & 0\\ -1 & i\end{bmatrix}\begin{bmatrix}1 & -1\\ 0 & i\end{bmatrix}$

$$=\begin{bmatrix}1 & -1\\ -1 & 0\end{bmatrix}$$

Next $l=2$,

$$\left(\mathcal{M}_{\gamma}^{\beta}(T_2)\right)^2((\mathcal{M}_{\gamma}^{\beta}(T_2))^T)^2=\begin{bmatrix}1 & 0\\ -1-i & -1\end{bmatrix}\begin{bmatrix}1 & -1-i\\ 0 & -1\end{bmatrix}$$

$$=\begin{bmatrix}1 & -1-i\\ -1-i & 1+2i\end{bmatrix}$$

Assume now that $l=3$,

$$\left(\mathcal{M}_{\gamma}^{\beta}(T_2)\right)^3((\mathcal{M}_{\gamma}^{\beta}(T_2))^T)^3=\begin{bmatrix}1 & 0\\ -i & -i\end{bmatrix}\begin{bmatrix}1 & -i\\ 0 & -i\end{bmatrix}$$

$$=\begin{bmatrix}1 & -i\\ -i & -2\end{bmatrix}$$

Finally $l=4$, $\left(\mathcal{M}_{\gamma}^{\beta}(T_2)\right)^4((\mathcal{M}_{\gamma}^{\beta}(T_2))^T)^4=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

Then $\mathcal{M}_{\gamma}^{\beta}(T_2)\in I-O\mathcal{M}_{n\times n}(\mathcal{F})$ and its index is 4 , so T_2 is a $linear\ transformation\ of\ type(I)$ with respect to $\beta\ and\ \gamma$.

$$C=\mathcal{M}_{\beta}^{\alpha}(T_1)+\mathcal{M}_{\gamma}^{\beta}(T_2)=\begin{bmatrix}i & -1\\ 0 & 1\end{bmatrix}+\begin{bmatrix}1 & 0\\ -1 & i\end{bmatrix}$$

$$=\begin{bmatrix}1+i & -1\\ -1 & 1+i\end{bmatrix}$$

$C\notin I-O\mathcal{M}_{n\times n}(\mathcal{F})$. Thus T_1+T_2 needs not be $linear\ transformation\ of\ type(I)$.

Remark 3.11 If T is $linear\ transformation\ of\ type(I)$ with respect to $\alpha\ and\ \beta$ and let α be scalar, then σT needs not be $linear\ transformation\ of\ type(I)$ with respect to $\alpha\ and\ \beta$ as in the following example.

Example 3.12: Let $X=\mathbb{C}^2\ over\ \mathbb{C}\ and\ Y=\mathbb{C}^2\ over\ \mathbb{C}\ , let\ T:\mathbb{C}^2\longrightarrow\mathbb{C}^2$ be the linear transformation defined by

$T\left(\begin{bmatrix}x_1\\ x_2\end{bmatrix}\right)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$, consider two ordered bases

$\alpha=\left\{\begin{bmatrix}i\\ 0\end{bmatrix},\begin{bmatrix}0\\ i\end{bmatrix}\right\}\ and\ \beta=\left\{\begin{bmatrix}-i\\ 0\end{bmatrix},\begin{bmatrix}0\\ -i\end{bmatrix}\right\}$

$$T\left(\begin{bmatrix}i\\ 0\end{bmatrix}\right)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}i\\ 0\end{bmatrix}=\begin{bmatrix}i\\ 0\end{bmatrix}$$

and

$$T\left(\begin{bmatrix}0\\i\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}0\\i\end{bmatrix}=\begin{bmatrix}0\\i\end{bmatrix}$$

Thus $\mathcal{M}_\beta^\alpha(T)=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$ is the matrix representation of T.

If $l=1, \mathcal{M}_\beta^\alpha(T)\left(\mathcal{M}_\beta^\alpha(T)\right)^T=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$

$$=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I_2$$

Then $\mathcal{M}_\beta^\alpha(T)\in I-O\mathcal{M}_{n\times n}(\mathcal{F})$ and its index is 1, so T is a $linear\ \ transformation\ of\ type(I)$ with respect to $\alpha\ and\ \beta$.

Let $\sigma=2$, then $\sigma\mathcal{M}_\beta^\alpha(T)=2\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$

$$=\begin{bmatrix}-2&0\\0&-2\end{bmatrix}=C$$

If $l=1, CC^T=\begin{bmatrix}-2&0\\0&-2\end{bmatrix}\begin{bmatrix}-2&0\\0&-2\end{bmatrix}$

$$=\begin{bmatrix}4&0\\0&4\end{bmatrix}\neq I_2\ ,$$

$C\notin I-O\mathcal{M}_{n\times n}(\mathcal{F})$, Thus σT needs not be $lineartransformation$

$of\ type(I)$.

4 Linear transformation of $type(II)$

Definition 4.1 Let $T:X\longrightarrow Y$ be a linear transformation and $s_X\ and\ s_Y$ be ordered bases of $X\ and\ Y$ respectively Then T is called a $linear\ transformation\ of\ type(II)$ with respect to $s_X\ and\ s_Y$ if the matrix representation of T with respect to $s_X\ and\ s_Y$ is an $orthogonal\ of\ type(II)$ matrix.

Example 4.2 : Let $X=\mathbb{R}^2\ over\ \mathbb{R}\ and\ Y=\mathbb{C}^2\ over\ \mathbb{C}\ , let\ T:\mathbb{R}^2\longrightarrow\mathbb{C}^2$ be the linear transformation defined by

$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$, consider two ordered bases

$\alpha=\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ and $\beta=\left\{\begin{bmatrix}\sqrt{i}\\0\end{bmatrix},\begin{bmatrix}0\\\sqrt{i}\end{bmatrix}\right\}$,

$$T\left(\left[\begin{bmatrix}1\\0\end{bmatrix}\right]\right)=\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

and

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

Thus $\mathcal{M}_\beta^\alpha(T)=\begin{bmatrix}1/\sqrt{i}&0\\0&1/\sqrt{i}\end{bmatrix}$ is the matrix representation of T.

If $r=1, \mathcal{M}_\beta^\alpha(T)\left(\mathcal{M}_\beta^\alpha(T)\right)^*=\begin{bmatrix}1/\sqrt{i}&0\\0&1/\sqrt{i}\end{bmatrix}\begin{bmatrix}1/\sqrt{-i}&0\\0&1/\sqrt{-i}\end{bmatrix}$

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$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then $\mathcal{M}_\beta^\alpha(T) \in II - O\mathcal{M}_{n\times n}(\mathcal{F})$ and its index is 1, so T is a $linear\ \ transformation\ of type(II)$ with respect to $\alpha\ and\ \beta$.

Example 4.3 Let $X = \mathbb{R}^3\ over\ \mathbb{R}\ and\ Y = \mathbb{C}^3\ over\ \mathbb{C}$, let $T: \mathbb{R}^3 \longrightarrow \mathbb{C}^3$ be the linear transformation defined by

$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, consider two ordered bases

$$\alpha = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}\ and\ \beta = \left\{\begin{bmatrix} i \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ i \end{bmatrix}\right\}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

and

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$$

Thus $\mathcal{M}_\beta^\alpha(T) = \begin{bmatrix} -i & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2}i & -\frac{1}{2}i \\ 0 & -\frac{1}{2}i & \frac{\sqrt{3}}{2}i \end{bmatrix}$ $is\ the\ matrix\ representation\ of\ T.$

If $r = 1, \mathcal{M}_\beta^\alpha(T)\left(\mathcal{M}_\beta^\alpha(T)\right)^* = \begin{bmatrix} -i & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2}i & -\frac{1}{2}i \\ 0 & -\frac{1}{2}i & \frac{\sqrt{3}}{2}i \end{bmatrix}\begin{bmatrix} i & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2}i & \frac{1}{2}i \\ 0 & \frac{1}{2}i & -\frac{\sqrt{3}}{2}i \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $\mathcal{M}_\beta^\alpha(T) \in II - O\mathcal{M}_{n\times n}(\mathcal{F})$ and its index is 1, so T is a $linear\ \ transformation\ of type(II)$ with respect to $\alpha\ and\ \beta$.

Example 4.4 Let $X = \mathbb{R}^2\ over\ \mathbb{R}\ and\ Y = \mathbb{C}^2\ over\ \mathbb{C}$,let $T: \mathbb{R}^2 \longrightarrow \mathbb{C}^2$ be the linear transformation defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, consider two ordered bases

$$\alpha = \left\{\begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix}\right\} and \beta = \left\{\begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix}\right\}$$

$$T\left(\begin{bmatrix} i \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} i \\ 0 \end{bmatrix} = \begin{bmatrix} i \\ i \end{bmatrix}$$

and

$$T\left(\begin{bmatrix} 0 \\ i \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ i \end{bmatrix}$$

Thus $\mathcal{M}_\beta^\alpha(T) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is the matrix representation of T.

If $r = 1, \left(\mathcal{M}_\beta^\alpha(T)\right)\left(\mathcal{M}_\beta^\alpha(T)\right)^* = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \neq I_2$

Next $r = 2, \left(\mathcal{M}_\beta^\alpha(T)\right)^2 ((\mathcal{M}_\beta^\alpha(T))^*)^2 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \neq I_2$

Now $r = k, (\mathcal{M}_\beta^\alpha(T))^k((\mathcal{M}_\beta^\alpha(T))^*)^k = \begin{bmatrix} (2)^k & 0 \\ 0 & (2)^k \end{bmatrix}$.

Then $\mathcal{M}_\beta^\alpha(T) \notin II - O\mathcal{M}_{n\times n}(\mathcal{F})$. So T is not $linear\ transformation\ of\ type(II)$ with respect to $\alpha\ and\ \beta$.

Theorem 4.5 Let $T: X \longrightarrow Y$ be a linear transformation of $type(II)$ with respect to $\alpha\ and\ \beta$ and let $\mathcal{M}_\beta^\alpha(T)$ be the matrix representation of T. Then T is invertible with $(\mathcal{M}_\beta^\alpha(T))^{-1} = \mathcal{M}_\beta^\alpha(T)^{r-1}(\mathcal{M}_\beta^\alpha(T)^*)^r$.

Proof:

Since $\mathcal{M}_\beta^\alpha(T) \in II - O\mathcal{M}_{n\times n}(\mathcal{F})$, then it is invertible with $(\mathcal{M}_\beta^\alpha(T))^{-1} = \mathcal{M}_\beta^\alpha(T)^{r-1}(\mathcal{M}_\beta^\alpha(T)^*)^r$ by Theorem 2.2

Therefor T is invertible by Theorem 2.6.■

Example 4.6 Let $X = \mathbb{C}^2\ over\ \mathbb{C}\ and\ Y = \mathbb{R}^2\ over\ \mathbb{C}$, let $T: \mathbb{C}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 & i \\ -i & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, consider two ordered bases

$\alpha = \left\{\begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix}\right\}\ and\ \beta = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$, Then

$$T\left(\begin{bmatrix} i \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & i \\ -i & 1 \end{bmatrix}\begin{bmatrix} i \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$T\left(\begin{bmatrix} 0 \\ i \end{bmatrix}\right) = \begin{bmatrix} 0 & i \\ -i & 1 \end{bmatrix}\begin{bmatrix} 0 \\ i \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix}$$

$Thus\ \mathcal{M}_\beta^\alpha(T)=\begin{bmatrix} i & -1 \\ 0 & -i \end{bmatrix}$ is the matrix representation of T.

Let $r = 1, \mathcal{M}_\beta^\alpha(T)\left(\mathcal{M}_\beta^\alpha(T)\right)^* = \begin{bmatrix} i & -1 \\ 0 & -i \end{bmatrix}\begin{bmatrix} -i & 0 \\ -1 & i \end{bmatrix}$

$= \begin{bmatrix} 2 & -i \\ i & 1 \end{bmatrix}$

Now $r = 2, (\mathcal{M}_\beta^\alpha(T))^2\left(\left(\mathcal{M}_\beta^\alpha(T)\right)^*\right)^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

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$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then $\mathcal{M}_\beta^\alpha(T) \in II - O\mathcal{M}_{n\times n}(\mathcal{F})$ *and its index is* 2, so T is invertible *linear transformation of* $type(II)$ with respect to α *and* β.

$$\mathcal{M}_\beta^\alpha(T^{-1}) = (\mathcal{M}_\beta^\alpha(T))^{-1}$$

$$= \mathcal{M}_\beta^\alpha(T)\left(\left(\mathcal{M}_\beta^\alpha(T)\right)^*\right)^2$$

$$= \begin{bmatrix} i & -1 \\ 0 & -i \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -i & 1 \\ 0 & i \end{bmatrix}$$

Theorem 4.7 Let T_1 *and* T_2 be linear transformations of $type(II)$, Let $\mathcal{M}_\beta^\alpha(T_1)$ *and* $\mathcal{M}_\gamma^\beta(T_2)$ be the matrices representation of T_1 *and* T_2 *respectively if* $\mathcal{M}_\beta^\alpha(T_1), \mathcal{M}_\gamma^\beta(T_2)$ are commute. Then

$T_1 \circ T_2$ is a linear transformation of $type(II)$ with respect to α *and* γ.

Proof:

By Theorem 2.5 $\mathcal{M}_\gamma^\alpha(T_1 \circ T_2) = \mathcal{M}_\beta^\alpha(T_1)\mathcal{M}_\gamma^\beta(T_2)$

Since $\mathcal{M}_\beta^\alpha(T_1)$ *and* $\mathcal{M}_\gamma^\beta(T_2)$ are orthogonal of $type(II)$ and commute, then by Theorem 2.9 $\mathcal{M}_\beta^\alpha(T_1)\mathcal{M}_\gamma^\beta(T_2)$ is an orthogonal of $type(II)$. Hence $T_1 \circ T_2$ is a linear transformation of $type(II)$ with respect to α *and* γ. ■

Example 4.8: Let $X = \mathbb{C}^3$ *over* $\mathbb{C}$ *and* $Y = W = \mathbb{R}^3$ *over* $\mathbb{R}$, let T_1 *and* T_2 be the linear transformations $T_1: \mathbb{C}^3 \longrightarrow \mathbb{R}^3$ defined by $T_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, consider two ordered bases

$\alpha = \left\{\begin{bmatrix} i \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ i \end{bmatrix}\right\}$ *and* $\beta = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}\right\}$

and $T_2: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T_2\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, consider two ordered bases

$\beta = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}\right\}$ *and* $\gamma = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

$$T_1\left(\begin{bmatrix} i \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} i \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} i \\ 0 \\ 0 \end{bmatrix}$$

$$T_1\left(\begin{bmatrix} 0 \\ i \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 0 \\ i \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}}i \end{bmatrix}$$

and

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$$T_1\left(\begin{bmatrix}0\\0\\i\end{bmatrix}\right)=\begin{bmatrix}1&0&0\\0&\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}0\\0\\i\end{bmatrix}=\begin{bmatrix}0\\-\frac{1}{\sqrt{2}}i\\\frac{1}{\sqrt{2}}i\end{bmatrix}$$

Thus $\mathcal{M}_\beta^\alpha(T_1)=\begin{bmatrix}i&0&0\\0&-\frac{1}{\sqrt{2}}i&\frac{1}{\sqrt{2}}i\\0&-\frac{1}{\sqrt{2}}i&-\frac{1}{\sqrt{2}}i\end{bmatrix}$ *is the matrix representation of* T_1.

$$If\ \ r=1\,,\mathcal{M}_\beta^\alpha(T_1)(\mathcal{M}_\beta^\alpha(T_1))^*=\begin{bmatrix}i&0&0\\0&-\frac{1}{\sqrt{2}}i&\frac{1}{\sqrt{2}}i\\0&-\frac{1}{\sqrt{2}}i&-\frac{1}{\sqrt{2}}i\end{bmatrix}\begin{bmatrix}-i&0&0\\0&\frac{1}{\sqrt{2}}i&\frac{1}{\sqrt{2}}i\\0&-\frac{1}{\sqrt{2}}i&\frac{1}{\sqrt{2}}i\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$\mathcal{M}_\beta^\alpha(T_1)\in II-O\mathcal{M}_{n\times n}(\mathcal{F})$and its index is 1, so T_1 is a linear transformation of $type(II)$ with respect to $\alpha\ and\ \beta$.

Now

$$T_2\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$T_2\left(\begin{bmatrix}0\\-1\\0\end{bmatrix}\right)=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}\begin{bmatrix}0\\-1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\-1\end{bmatrix}$$

and

$$T_2\left(\begin{bmatrix}0\\0\\-1\end{bmatrix}\right)=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}\begin{bmatrix}0\\0\\-1\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}$$

Thus $\mathcal{M}_\gamma^\beta(T_2)=\begin{bmatrix}1&0&0\\0&0&1\\0&-1&0\end{bmatrix}$ is matrix representation of T_2.

$$If\ \ r=1\,,\mathcal{M}_\gamma^\beta(T_2)(\mathcal{M}_\gamma^\beta(T_2))^*=\begin{bmatrix}1&0&0\\0&0&1\\0&-1&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$\mathcal{M}_\gamma^\beta(T_2)\ \in II-O\mathcal{M}_{n\times n}(\mathcal{F})$and its index is 1, so T_2 is a linear transformation of $type(II)$ with respect to $\beta\ and\ \gamma$.

Thus $\mathcal{M}_\gamma^\alpha(T_1\circ T_2)=\mathcal{M}_\beta^\alpha(T_1)\mathcal{M}_\gamma^\beta(T_2)=C$

Then $C=\mathcal{M}_\beta^\alpha(T_1)\mathcal{M}_\gamma^\beta(T_2)=\begin{bmatrix}i&0&0\\0&-\frac{1}{\sqrt{2}}i&\frac{1}{\sqrt{2}}i\\0&-\frac{1}{\sqrt{2}}i&-\frac{1}{\sqrt{2}}i\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\\0&-1&0\end{bmatrix}=\begin{bmatrix}i&0&0\\0&-\frac{1}{\sqrt{2}}i&-\frac{1}{\sqrt{2}}i\\0&\frac{1}{\sqrt{2}}i&-\frac{1}{\sqrt{2}}i\end{bmatrix}$

$$k=1,CC^*=\begin{bmatrix}i&0&0\\0&-\frac{1}{\sqrt{2}}i&-\frac{1}{\sqrt{2}}i\\0&\frac{1}{\sqrt{2}}i&-\frac{1}{\sqrt{2}}i\end{bmatrix}\begin{bmatrix}-i&0&0\\0&\frac{1}{\sqrt{2}}i&-\frac{1}{\sqrt{2}}i\\0&\frac{1}{\sqrt{2}}i&\frac{1}{\sqrt{2}}i\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Then $C\in II-O\mathcal{M}_{n\times n}(\mathcal{F})$ and its index is 1, so $T_1\circ T_2$ is a linear transformation of $type(II)$with respect to $\alpha\ and\ \gamma$ ..

Remark 4.9 If T_1 *and* T_2 are *linear transformations of type*(II) then $T_1 + T_2$ is not necessarily be *linear transformation of type*(II) with respect to α *and* γ. The following example shows that.

Example 4.10: Let $X = Y = W = \mathbb{C}^2$ *over* $\mathbb{C}$ T_1 *and* T_2 be linear transformations of $type(II)$,
$T_1: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$, $T_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, consider two ordered bases

$$\alpha = \left\{\begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix}\right\} \; and \; \beta = \left\{\begin{bmatrix} 0 \\ -i \end{bmatrix}, \begin{bmatrix} -i \\ 0 \end{bmatrix}\right\}$$

and $T_2: \mathbb{C}^2 \longrightarrow \mathbb{C}^2, T_2\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, consider two ordered bases

$$\beta = \left\{\begin{bmatrix} 0 \\ -i \end{bmatrix}, \begin{bmatrix} -i \\ 0 \end{bmatrix}\right\}, and \; \gamma = \left\{\begin{bmatrix} 0 \\ i \end{bmatrix}, \begin{bmatrix} i \\ 0 \end{bmatrix}\right\}$$

$$T_1\left(\begin{bmatrix} i \\ 0 \end{bmatrix}\right) = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} i \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

and

$$T_1\left(\begin{bmatrix} 0 \\ i \end{bmatrix}\right) = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 0 \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Thus $\mathcal{M}_\beta^\alpha(T_1) = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$ is the matrix representation of T_1.

If $r = 1 \Longrightarrow \mathcal{M}_\beta^\alpha(T_1)(\mathcal{M}_\beta^\alpha(T_1))^* = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then $\mathcal{M}_\beta^\alpha(T_1) \in I - O\mathcal{M}_{n\times n}(\mathcal{F})$ and its index is 2, so T_1 is a linear transformation of $type(II)$ with respect to α *and* β.

Now $T_2\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 0 \\ -i \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

and

$$T_2\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} -i \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Thus $\mathcal{M}_\gamma^\beta(T_2) = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$ is the matrix representation of T_2.

If $r = 1, \mathcal{M}_\gamma^\beta(T_2)(\mathcal{M}_\gamma^\beta(T_2))^* = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Then $\mathcal{M}_\gamma^\beta(T_2) \in I - O\mathcal{M}_{n\times n}(\mathcal{F})$ and its index is 1, so T_2 is a linear transformation of $type(II)$ with respect to β *and* γ.

$$C = \mathcal{M}_\beta^\alpha(T_1) + \mathcal{M}_\gamma^\beta(T_2) = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} + \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} -i & -i \\ -i & i \end{bmatrix}$$

$CC^* = \begin{bmatrix} -i & -i \\ -i & i \end{bmatrix}\begin{bmatrix} i & i \\ i & -i \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is not orthogonal of $type(II)$. Thus $T_1 + T_2$ needs not be linear transformation of $type(II)$.

Remark 4.11 If T is *linear transformation of type*(II) with respect to α *and* β and let σ be scalar, then σT is not a *linear transformation of type*(II) with respect to α *and* β as the following example.

Example 4.12: Let $X = \mathbb{C}^2$ *over*$\mathbb{C}$ *and* $Y = \mathbb{R}^2$ *over*$\mathbb{R}$, *let* $T: \mathbb{C}^2 \longrightarrow \mathbb{R}^2$ is the linear transformation defined by

$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, consider two ordered bases

$\alpha = \left\{\begin{bmatrix} i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix}\right\} \; and \; \beta = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

$$T\left(\begin{bmatrix} i \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} i \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

and

$$T\left(\begin{bmatrix} 0 \\ i \end{bmatrix}\right) = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 \\ i \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Thus $\mathcal{M}_\beta^\alpha(T) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ is the matrix representation of T.

Let $r = 1$ $\mathcal{M}_\beta^\alpha(T)(\mathcal{M}_\beta^\alpha(T))^* = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Then $\mathcal{M}_\beta^\alpha(T) \in II - O\mathcal{M}_{n\times n}(\mathcal{F})$ and its index is 1, so T is a linear transformation of $type(II)$ with respect to $\alpha\ and\ \beta$.

Let $\sigma = -3$, $\sigma\,\mathcal{M}_\beta^\alpha(T) = -3\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} = B$

$BB^* = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$, $B \notin II - O\mathcal{M}_{n\times n}(\mathcal{F})$. Thus σT needs not be linear transformation of $type(II)$.

5 Conclusion

Linear transformation of $type(I)(type(II))$ depend on the matrices representation of each type which are orthogonal of $type(I)$ and orthogonal of $type(II)$ matrices . These linear transformations are invertible with a formula to fined invers of each the matrix representation of them.

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