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Fuzzy Laplace Transforms of the Fuzzy Caputo Fractional Derivatives about Order $2<\beta<3$

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Abstract: *The main aim of this paper is to find the formulas for fuzzy Caput fractional derivatives of the order* $0<\beta<3$ *for fuzzy –valued function* f *and the formulas of fuzzy Laplace Transforms for fuzzy Caputo fractional derivatives of the order* $2<\beta<3$ *under Hukuhara difference(H- difference).*

Keywords- fuzzy Laplace Transforms , Caputo fractional derivatives

1. INTRODUCTION

Fractional calculus and fractional differential equations have undergone expanded study in recent years as a considerable interest both in mathematics and in applications. They were applied in modeling of many physical and chemical processes and in engineering [7] .

There are many researchers have been worked on the field of fuzzy fractional differential equations (FFDEs) for example: Mohammad OH et al. [4] present an approximate analytical solution for fuzzy fractional initial value problems (FFIVs) using differential transform method, Jafarian A et al. [3] used fractional fuzzy Laplace transformation to solve the fuzzy fractional eigenvalue differential equation .

This paper is arranged as follows: Basic concepts are given in Section 2. In Section 3, the formulas for fuzzy Caputo fractional derivatives of the order $0<\beta<3$ for fuzzy –valued function f and the formulas of fuzzy Laplace Transforms for fuzzy Caputo fractional derivatives of the order $2<\beta<3$. In Section 4, example of FFIV of order $2<\beta<3$ is given. In Section 5, conclusions are drawn.

2. Basic Concepts

Definition 2.1 [1] A fuzzy number u in parametric form is a pair $(\underline{u},\bar{u})$ of functions $\underline{u}(r),\bar{u}(r)$, $0\le r\le 1$, which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded non-decreasing left continuous function in $(0,1]$,and right continuous at 0 .
2. $\bar{u}(r)$ is a bounded non-increasing left continuous function in $(0,1]$,and right continuous at 0 .
3. $\underline{u}(r)\le\bar{u}(r),\quad 0\le r\le 1$.

We denote the set of all real numbers by R and the set of all fuzzy numbers on is indicated by E .

Definition 2.2 [6] Let $x,y\in E$. If there exists $z\in E$ such that $x+y=z$, then z is called the H – difference of x and y, and it is denoted by $x\ominus y$. The sign " $\ominus$ " always stands for H–difference and also note that $x\ominus y\neq x+(-1)y$.

Definition 2.3[Submitted in partial fulfillment of the requirements of the Honors Program University of Massachusetts Lowell, 2011] :The Caputo fractional derivative of order $\beta>0$ of a function $f(x)$ is defined as:

$$\left({}^{C}D_{a^+}^{\beta}f\right)(x)=D_{a^+}^{\beta-n}\frac{d^n}{dx^n}f(x) \tag{1.7}$$

or can be written as:

$$\left({}^{C}D_{a^+}^{\beta}f\right)(x)=\frac{1}{\Gamma(n-\beta)}\int_a^x\frac{f^{(n)}(t)dt}{(x-t)^{1-n+\beta}},\ n-1<\beta<n,\ n\in N,\ x>a \tag{1.8}$$

Definition 2.4 [5] Let $f(x)\in C^F[0,b]\cap L^F[0,b]$

$$G(x)=\frac{1}{\Gamma[\lceil\beta\rceil-\beta]}\int_0^x\frac{f(t)dt-\sum_{k=0}^{\lfloor\beta\rfloor}\frac{t^k}{k!}f_0^{(k)}}{(x-t)^{1-\lceil\beta\rceil+\beta}}dt,$$

$$H(x_0)=\lim_{h\to0^+}\frac{G(x_0+h)\ominus G(x_0)}{h}=\lim_{h\to0^+}\frac{G(x_0)\ominus G(x_0-h)}{h}$$

and

$$L(x_0)=\lim_{h\to0^+}\frac{G(x_0)\ominus G(x_0+h)}{-h}=\lim_{h\to0^+}\frac{G(x_0-h)\ominus G(x_0)}{-h}.$$

$f(x)$ is the Caputo - type fuzzy fractional differentiable function of order $0<\beta<2$, $\beta\neq1$ at $x_0\in(0,b)$, if there exists an element $\left({}^{C}D^{\beta}f\right)(x_0)\in C^F$ such that for all $0\le\alpha\le1$ and for $h>0$ sufficiently near zero, either.

(a) $$\left({}^{C}D^{\beta}f\right)(x_0)=\lim_{h\to0^+}\frac{G(x_0+h)\ominus G(x_0)}{h}=\lim_{h\to0^+}\frac{G(x_0)\ominus G(x_0-h)}{h}$$

(b) $$\left({}^{C}D^{\beta}f\right)(x_0)=\lim_{h\to0^+}\frac{G(x_0)\ominus G(x_0+h)}{-h}=\lim_{h\to0^+}\frac{G(x_0-h)\ominus G(x_0)}{-h}$$

for $0<\beta<1$ and either

(c) $$\left({}^{C}D^{\beta}f\right)(x_0)=\lim_{h\to0^+}\frac{H(x_0+h)\ominus H(x_0)}{h}=\lim_{h\to0^+}\frac{H(x_0)\ominus H(x_0-h)}{h}$$

(d) $$\left({}^{C}D^{\beta}f\right)(x_0)=\lim_{h\to0^+}\frac{H(x_0)\ominus H(x_0+h)}{-h}=\lim_{h\to0^+}\frac{H(x_0-h)\ominus H(x_0)}{-h}$$

(e) $$\left({}^{C}D^{\beta}f\right)(x_0)=\lim_{h\to0^+}\frac{L(x_0+h)\ominus L(x_0)}{h}=\lim_{h\to0^+}\frac{L(x_0)\ominus L(x_0-h)}{h}$$

(f) $$\left({}^{C}D^{\beta}f\right)(x_0)=\lim_{h\to0^+}\frac{L(x_0)\ominus L(x_0+h)}{-h}=\lim_{h\to0^+}\frac{L(x_0-h)\ominus L(x_0)}{-h}$$

for $1<\beta<2$.

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If the fuzzy valued function $f(x)$ is differentiable, as in definition 1.9 cases (a,c,e) , it is the Caputo-type differentiable in the first form and denoted by $\left({}^{C}D_{1}^{\beta}f\right)(x_0)$, $\left({}^{C}D_{1,1}^{\beta}f\right)(x_0)$ and $\left({}^{C}D_{2,1}^{\beta}f\right)(x_0)$ respectively. If $f(x)$ is differentiable, as in definition 2.4 cases (b,d,f), it is the Caputo-type differentiable in the second form and denoted by $\left({}^{C}D_{2}^{\beta}f\right)(x_0)$, $\left({}^{C}D_{1,2}^{\beta}f\right)(x_0)$ and $\left({}^{C}D_{2,2}^{\beta}f\right)(x_0)$ respectively

3. **Fuzzy Laplace Transforms of the Fuzzy Caputo Fractional Derivatives of Order** $2<\beta<3$.

In this section, we define Caputo fractional derivatives of order $0<\beta<3$ for fuzzy –valued function f and also we find fuzzy Laplace transform for Caputo fractional derivatives of order $2<\beta<3$ under H-differentiability.

Definition 3.1. Let $f(x)\in C^{F}[0,b]\cap L^{F}[0,b]$,

$$G(x)=\frac{1}{\Gamma(\lceil\beta\rceil-\beta)}\int_0^x\frac{f(t)dt}{(x-t)^{1-\lceil\beta\rceil+\beta}}\ominus\sum_{k=0}^{\lfloor\beta\rfloor}\frac{D^{k}f(0)x^{\lceil\beta\rceil-\beta+k}}{\Gamma(1+\lceil\beta\rceil-\beta+k)}$$

where $G_1(x_0)$ and $G_2(x_0)$ are equal to the limits defined in a1 and a2 respectively and $G_{1,1}(x_0),G_{1,2}(x_0),G_{2,1}(x_0)$ and $G_{2,2}(x_0)$ are the limits defined in b1, b2, b3 and b4 respectively. $f(x)$ is the Caputo-type fuzzy fractional differentiable function of order $0<\beta<3$, $\beta\neq 1,2$ at $x_0\in(0,b)$, if there exists an element $\left({}^{C}D^{\beta}f\right)(x_0)\in C^{F}$ such that for all $0\le r\le 1$ and for $h>0$ sufficiently near zero, either:

a1. $$\left({}^{C}D^{\beta}f\right)(x_0)=\lim_{h\to0^+}\frac{G(x_0+h)\ominus G(x_0)}{h}=\lim_{h\to0^+}\frac{G(x_0)\ominus G(x_0-h)}{h}$$ or

a2. $$\left({}^{C}D^{\beta}f\right)(x_0)=\lim_{h\to0^+}\frac{G(x_0)\ominus G(x_0+h)}{-h}=\lim_{h\to0^+}\frac{G(x_0-h)\ominus G(x_0)}{-h}$$

for $0<\beta<1$ and either

b1. $$\left({}^{C}D^{\beta}f\right)(x_0)=\lim_{h\to0^+}\frac{G_1(x_0+h)\ominus G_1(x_0)}{h}=\lim_{h\to0^+}\frac{G_1(x_0)\ominus G_1(x_0-h)}{h}$$ or

b2. $$\left({}^{C}D^{\beta}f\right)(x_0)=\lim_{h\to0^+}\frac{G_1(x_0)\ominus G_1(x_0+h)}{-h}=\lim_{h\to0^+}\frac{G_1(x_0-h)\ominus G_1(x_0)}{-h}$$ or b3. $\left({}^{C}D^{\beta}f\right)(x_0)=\lim\limits_{h\to 0^+}\dfrac{G_2(x_0+h)\ominus G_2(x_0)}{h}=\lim\limits_{h\to 0^+}\dfrac{G_2(x_0)\ominus G_2(x_0-h)}{h}$ or

b4. $\left({}^{C}D^{\beta}f\right)(x_0)=\lim\limits_{h\to 0^+}\dfrac{G_2(x_0)\ominus G_2(x_0+h)}{-h}=\lim\limits_{h\to 0^+}\dfrac{G_2(x_0-h)\ominus G_2(x_0)}{-h}$

for $1<\beta<2$ and either

c1. $\left({}^{C}D^{\beta}f\right)(x_0)=\lim\limits_{h\to 0^+}\dfrac{G_{1,1}(x_0+h)\ominus G_{1,1}(x_0)}{h}=\lim\limits_{h\to 0^+}\dfrac{G_{1,1}(x_0)\ominus G_{1,1}(x_0-h)}{h}$ or

c2. $\left({}^{C}D^{\beta}f\right)(x_0)=\lim\limits_{h\to 0^+}\dfrac{G_{1,1}(x_0)\ominus G_{1,1}(x_0+h)}{-h}=\lim\limits_{h\to 0^+}\dfrac{G_{1,1}(x_0-h)\ominus G_{1,1}(x_0)}{-h}$ or

c3. $\left({}^{C}D^{\beta}f\right)(x_0)=\lim\limits_{h\to 0^+}\dfrac{G_{1,2}(x_0+h)\ominus G_{1,2}(x_0)}{h}=\lim\limits_{h\to 0^+}\dfrac{G_{1,2}(x_0)\ominus G_{1,2}(x_0-h)}{h}$ or

c4. $\left({}^{C}D^{\beta}f\right)(x_0)=\lim\limits_{h\to 0^+}\dfrac{G_{1,2}(x_0)\ominus G_{1,2}(x_0+h)}{-h}=\lim\limits_{h\to 0^+}\dfrac{G_{1,2}(x_0-h)\ominus G_{1,2}(x_0)}{-h}$ or

c5. $\left({}^{C}D^{\beta}f\right)(x_0)=\lim\limits_{h\to 0^+}\dfrac{G_{2,1}(x_0+h)\ominus G_{2,1}(x_0)}{h}=\lim\limits_{h\to 0^+}\dfrac{G_{2,1}(x_0)\ominus G_{2,1}(x_0-h)}{h}$ or

c6. $\left({}^{C}D^{\beta}f\right)(x_0)=\lim\limits_{h\to 0^+}\dfrac{G_{2,1}(x_0)\ominus G_{2,1}(x_0+h)}{-h}=\lim\limits_{h\to 0^+}\dfrac{G_{2,1}(x_0-h)\ominus G_{2,1}(x_0)}{-h}$ or

c7. $\left({}^{C}D^{\beta}f\right)(x_0)=\lim\limits_{h\to 0^+}\dfrac{G_{2,2}(x_0+h)\ominus G_{2,2}(x_0)}{h}=\lim\limits_{h\to 0^+}\dfrac{G_{2,2}(x_0)\ominus G_{2,2}(x_0-h)}{h}$ or

c8. $\left({}^{C}D^{\beta}f\right)(x_0)=\lim\limits_{h\to 0^+}\dfrac{G_{2,2}(x_0)\ominus G_{2,2}(x_0+h)}{-h}=\lim\limits_{h\to 0^+}\dfrac{G_{2,2}(x_0-h)\ominus G_{2,2}(x_0)}{-h}$

for $2<\beta<3$.

If the fuzzy valued function $f(x)$ is differentiable as in definition 3.1 cases (a1, b1, b3 , c1 , c3 , c5 , c7) it is the Caputo type differentiable in the first form and denoted by $\left({}^{C}D_1^{\beta}f\right)(x_0)$, $\left({}^{C}D_{1,1}^{\beta}f\right)(x_0)$, $\left({}^{C}D_{2,1}^{\beta}f\right)(x_0)$, $\left({}^{C}D_{1,1,1}^{\beta}f\right)(x_0)$, $\left({}^{C}D_{1,2,1}^{\beta}f\right)(x_0)$, $\left({}^{C}D_{2,1,1}^{\beta}f\right)(x_0)$ and $\left({}^{C}D_{2,2,1}^{\beta}f\right)(x_0)$ respectively. If $f(x)$ is differentiable as in definition 3.1 cases (a2, b2, b4 , c2 , c4 , c6 , c8) it is the Caputo type differentiable in the second form and denoted by $\left({}^{C}D_{2}^{\beta} f\right)(x_0)$, $\left({}^{C}D_{1,2}^{\beta} f\right)(x_0)$, $\left({}^{C}D_{2,2}^{\beta} f\right)(x_0)$, $\left({}^{C}D_{1,1,2}^{\beta} f\right)(x_0)$, $\left({}^{C}D_{1,2,2}^{\beta} f\right)(x_0)$, $\left({}^{C}D_{2,1,2}^{\beta} f\right)(x_0)$ and $\left({}^{C}D_{2,2,2}^{\beta} f\right)(x_0)$ respectively.

Theorem 3.2. Let $f(x)\in C^{F}[0,b]\cap L^{F}[0,b]$ be a fuzzy-valued function and $f(x)=\left[\underline{f}(x;r),\overline{f}(x;r)\right]$ for $r\in[0,1]$ and $x_0\in(0,b)$. Suppose that $0<\beta<3$ and m is the number of repetitions of the number 2 among $i_1,i_2,\ldots,i_{\lceil\beta\rceil}$ for $k-1<\beta<k$, $k=1,2,3$. Then

a1. If $f(x)$ is Caputo-type fuzzy fractional differentiable function in the first form, then for $0<\beta<1$

$$\left({}^{C}D_{1}^{\beta} f\right)(x_0)=\left[\left({}^{C}D^{\beta}\underline{f}\right)(x_0;r),\left({}^{C}D^{\beta}\overline{f}\right)(x_0;r)\right]$$

a2. If $f(x)$ is Caputo-type fuzzy fractional differentiable function in the second form, then for $0<\beta<1$

$$\left({}^{C}D_{2}^{\beta} f\right)(x_0)=\left[\left({}^{C}D^{\beta}\overline{f}\right)(x_0;r),\left({}^{C}D^{\beta}\underline{f}\right)(x_0;r)\right]$$

b1. If $\left({}^{C}D_{1}^{\beta} f\right)(x)$ is Caputo-type fuzzy fractional differentiable function in the first form, then for $1<\beta<2$

$$\left({}^{C}D_{1,1}^{\beta} f\right)(x_0)=\left[\left({}^{C}D^{\beta}\underline{f}\right)(x_0;r),\left({}^{C}D^{\beta}\overline{f}\right)(x_0;r)\right]$$

b2. If $\left({}^{C}D_{1}^{\beta} f\right)(x)$ is Caputo-type fuzzy fractional differentiable function in the second form, then for $1<\beta<2$

$$\left({}^{C}D_{1,2}^{\beta} f\right)(x_0)=\left[\left({}^{C}D^{\beta}\overline{f}\right)(x_0;r),\left({}^{C}D^{\beta}\underline{f}\right)(x_0;r)\right]$$

b3. If $\left({}^{C}D_{2}^{\beta} f\right)(x)$ is Caputo-type fuzzy fractional differentiable function in the first form, then for $1<\beta<2$

$$\left({}^{C}D_{2,1}^{\beta} f\right)(x_0)=\left[\left({}^{C}D^{\beta}\overline{f}\right)(x_0;r),\left({}^{C}D^{\beta}\underline{f}\right)(x_0;r)\right]$$

b4. If $\left({}^{C}D_{2}^{\beta} f\right)(x)$ is Caputo-type fuzzy fractional differentiable function in the second form, then for $1<\beta<2$

$$\left({}^{C}D_{2,2}^{\beta} f\right)(x_0)=\left[\left({}^{C}D^{\beta}\underline{f}\right)(x_0;r),\left({}^{C}D^{\beta}\overline{f}\right)(x_0;r)\right]$$

c1. If $\left({}^{C}D_{1,1}^{\beta} f\right)(x)$ is Caputo-type fuzzy fractional differentiable function in the first form, then for $2<\beta<3$

$$\left({}^{C}D_{1,1,1}^{\beta} f\right)(x_0)=\left[\left({}^{C}D^{\beta} \underline{f}\right)(x_0;r),\left({}^{C}D^{\beta} \bar{f}\right)(x_0;r)\right]$$

c2. If $\left({}^{C}D_{1,1}^{\beta} f\right)(x)$ is Caputo-type fuzzy fractional differentiable function in the second form, then for $2<\beta<3$

$$\left({}^{C}D_{1,1,2}^{\beta} f\right)(x_0)=\left[\left({}^{C}D^{\beta} \bar{f}\right)(x_0;r),\left({}^{C}D^{\beta} \underline{f}\right)(x_0;r)\right]$$

c3. If $\left({}^{C}D_{1,2}^{\beta} f\right)(x)$ is Caputo-type fuzzy fractional differentiable function in the first form, then for $2<\beta<3$

$$\left({}^{C}D_{1,2,1}^{\beta} f\right)(x_0)=\left[\left({}^{C}D^{\beta} \bar{f}\right)(x_0;r),\left({}^{C}D^{\beta} \underline{f}\right)(x_0;r)\right]$$

c4. If $\left({}^{C}D_{1,2}^{\beta} f\right)(x)$ is Caputo-type fuzzy fractional differentiable function in the second form, then for $2<\beta<3$

$$\left({}^{C}D_{1,2,2}^{\beta} f\right)(x_0)=\left[\left({}^{C}D^{\beta} \underline{f}\right)(x_0;r),\left({}^{C}D^{\beta} \bar{f}\right)(x_0;r)\right]$$

c5. If $\left({}^{C}D_{2,1}^{\beta} f\right)(x)$ is Caputo-type fuzzy fractional differentiable function in the first form, then for $2<\beta<3$

$$\left({}^{C}D_{2,1,1}^{\beta} f\right)(x_0)=\left[\left({}^{C}D^{\beta} \bar{f}\right)(x_0;r),\left({}^{C}D^{\beta} \underline{f}\right)(x_0;r)\right]$$

c6. If $\left({}^{C}D_{2,1}^{\beta} f\right)(x)$ is Caputo-type fuzzy fractional differentiable function in the second form, then for $2<\beta<3$

$$\left({}^{C}D_{2,1,2}^{\beta} f\right)(x_0)=\left[\left({}^{C}D^{\beta} \underline{f}\right)(x_0;r),\left({}^{C}D^{\beta} \bar{f}\right)(x_0;r)\right]$$

c7. If $\left({}^{C}D_{2,2}^{\beta} f\right)(x)$ is Caputo-type fuzzy fractional differentiable function in the first form, then for $2<\beta<3$

$$\left({}^{C}D_{2,2,1}^{\beta} f\right)(x_0)=\left[\left({}^{C}D^{\beta} \underline{f}\right)(x_0;r),\left({}^{C}D^{\beta} \bar{f}\right)(x_0;r)\right]$$

c8. If $\left({}^{C}D_{2,2}^{\beta} f\right)(x)$ is Caputo-type fuzzy fractional differentiable function in the second form, then for $2<\beta<3$

$$\left({}^{C}D_{2,2,2}^{\beta} f\right)(x_0)=\left[\left({}^{C}D^{\beta} \bar{f}\right)(x_0;r),\left({}^{C}D^{\beta} \underline{f}\right)(x_0;r)\right]$$

where

$$\left({}^{C}D^{\beta} \underline{f}\right)(x_0;r)=\left[\frac{1}{\Gamma(\lceil\beta\rceil-\beta)}\int_0^x \frac{D^{\lceil\beta\rceil}\underline{f}(t;r)}{(x-t)^{1-\lceil\beta\rceil+\beta}}dt\right]_{x=x_0},$$

$$\left({}^{C}D^{\beta}\bar{f}\right)(x_0;r)=\left[\frac{1}{\Gamma(\lceil\beta\rceil-\beta)}\int_0^x\frac{D^{\lceil\beta\rceil}\bar{f}(t;r)}{(x-t)^{1-\lceil\beta\rceil+\beta}}dt\right]_{x=x_0},$$

$$D^k f(t)=\frac{d^k f(t)}{dt^k}.$$

Proof We prove b3 as follows: Since $\left({}^{C}D^{\beta}_{2,1}f\right)(x)$, $2<\beta<3$ is the Caputo- type fuzzy fractional differentiable function in the first form, then from b3.of definition 3.1, we have :

$$G_{2,1}(x_0+h)\ominus G_{2,1}(x_0)=\left[\underline{G}_{2,1}(x_0+h;r)-\underline{G}_{2,1}(x_0;r),\bar{G}_{2,1}(x_0+h;r)-\bar{G}_{2,1}(x_0;r)\right]$$
$$G_{2,1}(x_0)\ominus G_{2,1}(x_0-h)=\left[\underline{G}_{2,1}(x_0;r)-\underline{G}_{2,1}(x_0-h;r),\bar{G}_{2,1}(x_0;r)-\bar{G}_{2,1}(x_0-h;r)\right]$$

Multiplying both sides by $\frac{1}{h}$, $h>0$, we obtain

$$\frac{G_{2,1}(x_0+h)\ominus G_{2,1}(x_0)}{h}=\left[\frac{\underline{G}_{2,1}(x_0+h;r)-\underline{G}_{2,1}(x_0;r)}{h},\frac{\bar{G}_{2,1}(x_0+h;r)-\bar{G}_{2,1}(x_0;r)}{h}\right]$$
$$\frac{G_{2,1}(x_0)\ominus G_{2,1}(x_0-h)}{h}=\left[\frac{\underline{G}_{2,1}(x_0;r)-\underline{G}_{2,1}(x_0-h;r)}{h},\frac{\bar{G}_{2,1}(x_0;r)-\bar{G}_{2,1}(x_0-h;r)}{h}\right]$$

By taking $h\to 0^+$ on both sides of the above relation , we get

$$\left({}^{C}D^{\beta}f\right)(x_0)=\left[\frac{d}{dx}\underline{G}_{2,1}(x_0;r),\frac{d}{dx}\bar{G}_{2,1}(x_0;r)\right]. \tag{3.6}$$

Now, since $G_2(x_0)$ is equal to the limits defined in a2. of definition 3.1, then we have:

$$G(x_0)\ominus G(x_0+h)=\left[\underline{G}(x_0;r)-\underline{G}(x_0+h;r),\bar{G}(x_0;r)-\bar{G}(x_0+h;r)\right],$$

$$G(x_0-h)\ominus G(x_0)=\left[\underline{G}(x_0-h;r)-\underline{G}(x_0;r),\bar{G}(x_0-h;r)-\bar{G}(x_0;r)\right].$$

Multiplying both sides by $-\frac{1}{h}$, $h>0$, we obtain:

$$\frac{G(x_0)\ominus G(x_0+h)}{-h}=\left[\frac{\bar{G}(x_0+h;r)-\bar{G}(x_0;r)}{h},\frac{\underline{G}(x_0+h;r)-\underline{G}(x_0;r)}{h}\right],$$
$$\frac{G(x_0-h)\ominus G(x_0)}{-h}=\left[\frac{\bar{G}(x_0;r)-\bar{G}(x_0-h;r)}{h},\frac{\underline{G}(x_0;r)-\underline{G}(x_0-h;r)}{h}\right].$$

By taking $h \to 0^+$ on both sides of the above relation , we get:

$$G_2(x_0) = \left[\bar{G}'(x_0;r), \underline{G}'(x_0;r)\right].$$

Then:

$$\underline{G}_2(x_0;r) = \bar{G}'(x_0;r) \;,\; \bar{G}_2(x_0;r) = \underline{G}'(x_0;r) \qquad (3.7)$$

Now, since $G_{2,1}(x_0)$ is equal to the limits defined in b3 of definition 3.1, then we have:

$$G_2(x_0+h) \ominus G_2(x_0) = \left[\underline{G}_2(x_0+h;r) - \underline{G}_2(x_0;r), \bar{G}_2(x_0+h;r) - \bar{G}_2(x_0;r)\right],$$

$$G_2(x_0) \ominus G_2(x_0-h) = \left[\underline{G}_2(x_0;r) - \underline{G}_2(x_0-h;r), \bar{G}_2(x_0;r) - \bar{G}_2(x_0-h;r)\right].$$

Multiplying both sides by $\frac{1}{h}, h>0$, and using relation (3.2) we obtain

$$\frac{G_2(x_0+h) \ominus G_2(x_0)}{h} = \left[\frac{\bar{G}'(x_0+h;r) - \bar{G}'(x_0;r)}{h}, \frac{\underline{G}'(x_0+h;r) - \underline{G}'(x_0;r)}{h}\right]$$

$$\frac{G_2(x_0) \ominus G_2(x_0-h)}{h} = \left[\frac{\bar{G}'(x_0;r) - \bar{G}'(x_0-h;r)}{h}, \frac{\underline{G}'(x_0;r) - \underline{G}'(x_0-h;r)}{h}\right]$$

By taking $h \to 0^+$ on both sides of the above relation ,we get:

$$G_{2,1}(x_0) = \left[\bar{G}''(x_0;r), \underline{G}''(x_0;r)\right]$$

Then $\underline{G}_{2,1}(x_0;r) = \bar{G}''(x_0;r) \;,\; \bar{G}_{2,1}(x_0;r) = \underline{G}''(x_0;r)$ (3.3)

substituting (3.3) in (3.1) yields:

$$\left({}^{C}D^{\beta} f\right)(x_0) = \left[\left(D^3 \bar{G}\right)(x_0;r), \left(D^3 \underline{G}\right)(x_0;r)\right]$$

$$= \left[\left({}^{C}D^{\beta}\bar{f}\right)(x_0;r), \left({}^{C}D^{\beta}\underline{f}\right)(x_0;r)\right]$$

Theorem 3.3. Suppose that $f(x) \in C^{F}[0,\infty) \cap L^{F}[0,\infty)$ and $2<\beta<3$ then :

1. If $\left({}^{C}D_{1,1}^{\beta} f\right)(x)$ is ${}^{C}[i-\beta]$-differentiable fuzzy-valued function, then

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$$L\left[\left({}^{C}D_{1,1,1}^{\beta} f\right)(x)\right]=s^{\beta}L\left[f(x)\right] \ominus s^{\beta-1}f(0) \ominus s^{\beta-2}f'(0) \ominus s^{\beta-3}f''(0)$$

2. If $\left({}^{C}D_{1,1}^{\beta} f\right)(x)$ is ${}^{C}[ii-\beta]$-differentiable fuzzy-valued function, then

$$L\left[\left({}^{C}D_{1,1,2}^{\beta} f\right)(x)\right]=-s^{\beta-1}f(0) \ominus \left(-s^{\beta}\right)L\left[f(x)\right]- s^{\beta-2}f'(0)- s^{\beta-3}f''(0)$$

3. If $\left({}^{C}D_{1,2}^{\beta} f\right)(x)$ is ${}^{C}[i-\beta]$-differentiable fuzzy-valued function, then

$$L\left[\left({}^{C}D_{1,2,1}^{\beta} f\right)(x)\right]=-s^{\beta-1}f(0) \ominus \left(-s^{\beta}\right)L\left[f(x)\right] -s^{\beta-2}f'(0) \ominus s^{\beta-3}f''(0)$$

4. If $\left({}^{C}D_{1,2}^{\beta} f\right)(x)$ is ${}^{C}[ii-\beta]$-differentiable fuzzy-valued function, then

$$L\left[\left({}^{C}D_{1,2,2}^{\beta} f\right)(x)\right]=s^{\beta}L\left[f(x)\right] \ominus s^{\beta-1}f(0) \ominus s^{\beta-2}f'(0) - s^{\beta-3}f''(0)$$

5.If $\left({}^{C}D_{2,1}^{\beta} f\right)(x)$ is ${}^{C}[i-\beta]$-differentiable fuzzy-valued function, then

$$L\left[\left({}^{C}D_{2,1,1}^{\beta} f\right)(x)\right]=-s^{\beta-1}f(0) \ominus \left(-s^{\beta}\right)L\left[f(x)\right] \ominus s^{\beta-2}f'(0) \ominus s^{\beta-3}f''(0)$$

6. If $\left({}^{C}D_{2,1}^{\beta} f\right)(x)$ is ${}^{C}[ii-\beta]$-differentiable fuzzy-valued function, then

$$L\left[\left({}^{C}D_{2,1,2}^{\beta} f\right)(x)\right]=s^{\beta}L\left[f(x)\right] \ominus s^{\beta-1}f(0) - s^{\beta-2}f'(0) - s^{\beta-3}f''(0)$$

7. If $\left({}^{C}D_{2,2}^{\beta} f\right)(x)$ is ${}^{C}[i-\beta]$-differentiable fuzzy-valued function, then

$$L\left[\left({}^{C}D_{2,2,1}^{\beta} f\right)(x)\right]=s^{\beta}L\left[f(x)\right] \ominus s^{\beta-1}f(0) - s^{\beta-2}f'(0) \ominus s^{\beta-3}f''(0)$$

8. If $\left({}^{C}D_{2,2}^{\beta} f\right)(x)$ is ${}^{C}[ii-\beta]$-differentiable fuzzy-valued function, then

$$L\left[\left({}^{C}D_{2,2,2}^{\beta} f\right)(x)\right]=-s^{\beta-1}f(0) \ominus \left(-s^{\beta}\right)L\left[f(x)\right] \ominus s^{\beta-2}f'(0) - s^{\beta-3}f''(0)$$

Proof We shall prove 1 as follows: Since $\left({}^{C}D_{1,1}^{\beta} f\right)(x)$ is ${}^{C}[i-\beta]-$ differentiable fuzzy-valued function, then by theorem 3.3, we get:

$$\left({}^{C}D_{1,1,1}^{\beta} f\right)(x) = \left[\left({}^{C}D^{\beta}\underline{f}\right)(x;r), \left({}^{C}D^{\beta}\overline{f}\right)(x;r)\right].$$

Therefore, we get:

$$\left({}^{C}\underline{D^{\beta}f}\right)(x;r)=\left({}^{C}D^{\beta}\underline{f}\right)(x;r),\ \left(\overline{{}^{RL}D^{\beta}f}\right)(x;r)=\left({}^{RL}D^{\beta}\overline{f}\right)(x;r). \qquad (3.4)$$

Then, from (3.4), we get:

$$L\left[\left({}^{C}D^{\beta}_{1,1,1}f\right)(x)\right]=L\left[\left({}^{C}\underline{D^{\beta}f}\right)(x;r),\ \left(\overline{{}^{C}D^{\beta}f}\right)(x;r)\right]$$

$$=\left[\ell\left[\left({}^{C}D^{\beta}\underline{f}\right)(x;r)\right],\ell\left[\left({}^{C}D^{\beta}\overline{f}\right)(x;r)\right]\right] \qquad (3.5)$$

By Laplace transform of ordinary Caputo fractional derivative (1.5), equation (3.5) becomes:

$$L\left[\left({}^{C}D^{\beta}_{1,1,1}f\right)(x)\right]=\Big[s^{\beta}\ell\left[\underline{f}(x;r)\right]-s^{\beta-1}\underline{f}(0;r)-s^{\beta-2}\underline{f}'(0;r)-s^{\beta-3}\underline{f}''(0;r),$$
$$s^{\beta}\ell\left[\overline{f}(x;r)\right]-s^{\beta-1}\overline{f}(0;r)-s^{\beta-2}\overline{f}'(0;r)-s^{\beta-2}\overline{f}''(0;r)\Big]. \qquad (3.6)$$

Since $\left({}^{C}D^{\beta}_{1,1}f\right)(x)$ is ${}^{C}\left[(i)-\beta\right]$- differentiable fuzzy – valued function, then by theorem 3.3, we get :

$$\underline{f}'(0;r)=\underline{f'}(0;r),\quad \overline{f}'(0;r)=\overline{f'}(0;r),$$

$$\underline{f}''(0;r)\underline{f''}(0;r),\quad \overline{f}''(0;r)=\overline{f''}(0;r).$$

Then, equation (3.6) becomes:

$$L\left[\left({}^{C}D^{\beta}_{1,1,1}f\right)(x)\right]=\Big[s^{\beta}\ell\left[\underline{f}(x;r)\right]-s^{\beta-1}\underline{f}(0;r)-s^{\beta-2}\underline{f'}(0;r)-s^{\beta-3}\underline{f''}(0;r)$$
$$,s^{\beta}\ell\left[\overline{f}(x;r)\right]-s^{\beta-1}\overline{f}(0;r)-s^{\beta-2}\overline{f'}(0;r)-s^{\beta-2}\overline{f''}(0;r)\Big]$$
$$=s^{\beta}L\left[f(x)\right]\ominus s^{\beta-1}\underline{f}(0)\ominus s^{\beta-2}f'(0)\ominus s^{\beta-3}f''(0).$$

4.Application

Example 4.1. Consider the following FFIVP:

$$\left({}^{C}D^{\beta}y\right)(x)=\sigma;\ \sigma=\left(r^{2},2-r^{2}\right),\ 2<\beta<3, \qquad (4.1)$$

$$y(0)=y'(0)=y''(0)=(r-1,1-r).$$

We note that

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$$\underline{y}(0;r)=\underline{y}'(0;r)=\underline{y}''(0;r)=r-1,$$
$$\overline{y}(0;r)=\overline{y}'(0;r)=\overline{y}''(0;r)=1-r.$$

By taking fuzzy Laplace transform for both sides of equation (4.1) we get:

$$L\left[\left({}^{C}D^{\beta}y\right)(x)\right]=L[\sigma]. \tag{4.2}$$

Now, we have $2^3=8$ cases as follows:

Case 1. if $\left({}^{C}D_{1,1}^{\beta}y\right)(x)$ is ${}^{RL}[(i)-\beta]$-differentiable, then equation (4.2) becomes:

$$s^{\beta}L[y(x)]\ominus s^{\beta-1}y(0)\ominus s^{\beta-2}y'(0)\ominus s^{\beta-3}y''(0)=L[\sigma].$$

Then, we get :

$$s^{\beta}\ell\left[\underline{y}(x;r)\right]-s^{\beta-1}\underline{y}(0;r)-s^{\beta-2}\underline{y}'(0;r)-s^{\beta-3}\underline{y}''(0;r)=\frac{r^2}{s},$$

$$s^{\beta}\ell\left[\overline{y}(x;r)\right]-s^{\beta-1}\overline{y}(0;r)-s^{\beta-2}\overline{y}'(0;r)-s^{\beta-3}\overline{y}''(0;r)=\frac{2-r^2}{s}.$$

Therefore we have :

$$s^{\beta}\ell\left[\underline{y}(x;r)\right]=\frac{r^2}{s}+(r-1)\left(s^{\beta-1}+s^{\beta-2}+s^{\beta-3}\right),$$

$$s^{\beta}\ell\left[\overline{y}(x;r)\right]=\frac{2-r^2}{s}+(1-r)\left(s^{\beta-1}+s^{\beta-2}+s^{\beta-3}\right).$$

Consequently, applying inverse of Laplace transform on the both sides we have:

$$\underline{y}(x;r)=r^2\ell^{-1}\left[\frac{1}{s^{\beta+1}}\right]+(r-1)\left(\ell^{-1}\left[\frac{1}{s}\right]+\ell^{-1}\left[\frac{1}{s^2}\right]+\ell^{-1}\left[\frac{1}{s^3}\right]\right),$$

$$\overline{y}(x;r)=\left(2-r^2\right)\ell^{-1}\left[\frac{1}{s^{\beta+1}}\right]+(1-r)\left(\ell^{-1}\left[\frac{1}{s}\right]+\ell^{-1}\left[\frac{1}{s^2}\right]+\ell^{-1}\left[\frac{1}{s^3}\right]\right).$$

Finally, we determine the solution of FFIVP (4.1) as follows :

$$\underline{y}(x;r)=r^2\frac{x^{\beta}}{\Gamma(\beta+1)}+(r-1)\left(1+x+\frac{x^2}{2}\right),$$

$$\overline{y}(x;r)=(2-r^2)\frac{x^\beta}{\Gamma(\beta+1)}+(1-r)\left(1+x+\frac{x^2}{2}\right).$$

Case 2 If $\left({}^C D_{1,1}^{\beta} y\right)(x)$ is ${}^C\left[(ii)-\beta\right]$-differentiable, then equation (4.2) becomes:

$$-s^{\beta-1}y(0)\ominus(-s^\beta)L\left[y(x)\right]-s^{\beta-2}y'(0)-s^{\beta-3}y''(0)=L[\sigma].$$

Then, we get :

$$-s^{\beta-1}\underline{y}(0;r)+s^\beta\ell\left[\underline{y}(x;r)\right]-s^{\beta-2}\underline{y}'(0;r)-s^{\beta-3}\underline{y}''(0)=\frac{2-r^2}{s},$$

$$-s^{\beta-1}\overline{y}(0;r)+s^\beta\ell\left[\overline{y}(x;r)\right]-s^{\beta-2}\overline{y}'(0;r)-s^{\beta-3}\overline{y}''(0)=\frac{r^2}{s}.$$

Therefore, we have:

$$s^\beta\ell\left[\underline{y}(x;r)\right]=\frac{2-r^2}{s}+(r-1)\left(s^{\beta-1}+s^{\beta-2}+s^{\beta-3}\right),$$

$$s^\beta\ell\left[\overline{y}(x;r)\right]=\frac{r^2}{s}+(1-r)\left(s^{\beta-1}+s^{\beta-2}+s^{\beta-3}\right).$$

Consequently, applying inverse of Laplace transform on the both sides we have:

$$\underline{y}(x;r)=(2-r^2)\ell^{-1}\left[\frac{1}{s^{\beta+1}}\right]+(r-1)\left(\ell^{-1}\left[\frac{1}{s}\right]+\ell^{-1}\left[\frac{1}{s^2}\right]+\ell^{-1}\left[\frac{1}{s^3}\right]\right),$$

$$\overline{y}(x;r)=r^2\ell^{-1}\left[\frac{1}{s^{\beta+1}}\right]+(1-r)\left(\ell^{-1}\left[\frac{1}{s}\right]+\ell^{-1}\left[\frac{1}{s^2}\right]+\ell^{-1}\left[\frac{1}{s^3}\right]\right).$$

Finally, we determine the solution of FFIVP (4.1) as follows :

$$\underline{y}(x;r)=(2-r^2)\frac{x^\beta}{\Gamma(\beta+1)}+(r-1)\left(1+x+\frac{x^2}{2}\right),$$

$$\overline{y}(x;r)=r^2\frac{x^\beta}{\Gamma(\beta+1)}+(1-r)\left(1+x+\frac{x^2}{2}\right).$$

Case 3. If $\left({}^C D_{1,2}^{\beta} y\right)(x)$ is ${}^C\left[(i)-\beta\right]$-differentiable, then equation (4.2) becomes:

$$-s^{\beta-1}y(0)\ominus(-s^\beta)L\left[y(x)\right]-s^{\beta-2}y'(0)\ominus s^{\beta-3}y''(0)=L[\sigma].$$

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Then, we get :

$$-s^{\beta-1}\underline{y}(0;r)+s^{\beta}\ell\left[\underline{y}(x;r)\right]-s^{\beta-2}\underline{y}'(0;r)-s^{\beta-3}\overline{y}''(0)=\frac{2-r^2}{s},$$

$$-s^{\beta-1}\overline{y}(0;r)+s^{\beta}\ell\left[\overline{y}(x;r)\right]-s^{\beta-2}\overline{y}'(0;r)-s^{\beta-3}\underline{y}''(0)=\frac{r^2}{s}.$$

Therefore, we have:

$$s^{\beta}\ell\left[\underline{y}(x;r)\right]=\frac{2-r^2}{s}+(r-1)\left(s^{\beta-1}+s^{\beta-2}-s^{\beta-3}\right),$$

$$s^{\beta}\ell\left[\overline{y}(x;r)\right]=\frac{r^2}{s}+(1-r)\left(s^{\beta-1}+s^{\beta-2}-s^{\beta-3}\right).$$

Consequently, applying inverse of Laplace transform on the both sides we have:

$$\underline{y}(x;r)=\left(2-r^2\right)\ell^{-1}\left[\frac{1}{s^{\beta+1}}\right]+(r-1)\left(\ell^{-1}\left[\frac{1}{s}\right]+\ell^{-1}\left[\frac{1}{s^2}\right]-\ell^{-1}\left[\frac{1}{s^3}\right]\right),$$

$$\overline{y}(x;r)=r^2\ell^{-1}\left[\frac{1}{s^{\beta+1}}\right]+(1-r)\left(\ell^{-1}\left[\frac{1}{s}\right]+\ell^{-1}\left[\frac{1}{s^2}\right]-\ell^{-1}\left[\frac{1}{s^3}\right]\right).$$

Finally, we determine the solution of FFIVP (4.1) as follows :

$$\underline{y}(x;r)=\left(2-r^2\right)\frac{x^{\beta}}{\Gamma(\beta+1)}+(r-1)\left(1+x-\frac{x^2}{2}\right),$$

$$\overline{y}(x;r)=r^2\frac{x^{\beta}}{\Gamma(\beta+1)}+(1-r)\left(1+x-\frac{x^2}{2}\right).$$

Case 4. If $\left({}^{C}D_{1,2}^{\beta}y\right)(x)$ is ${}^{C}\left[(ii)-\beta\right]$-differentiable, then equation (4.2) becomes:

$$s^{\beta}L\left[y(x)\right]\ominus s^{\beta-1}y(0)\ominus s^{\beta-2}y'(0)-s^{\beta-3}y''(0)=L[\sigma].$$

Then, we get :

$$s^{\beta}\ell\left[\underline{y}(x;r)\right]-s^{\beta-1}\underline{y}(0;r)-s^{\beta-2}\underline{y}'(0;r)-s^{\beta-3}\overline{y}''(0;r)=\frac{r^2}{s},$$

$$s^{\beta}\ell\left[\overline{y}(x;r)\right]-s^{\beta-1}\overline{y}(0;r)-s^{\beta-2}\overline{y}'(0;r)-s^{\beta-3}\underline{y}''(0;r)=\frac{2-r^2}{s}.$$

Therefore we have :

$$s^{\beta}\ell\left[\underline{y}(x;r)\right]=\frac{r^2}{s}+(r-1)\left(s^{\beta-1}+s^{\beta-2}-s^{\beta-3}\right),$$

$$s^{\beta}\ell\left[\overline{y}(x;r)\right]=\frac{2-r^2}{s}+(1-r)\left(s^{\beta-1}+s^{\beta-2}-s^{\beta-3}\right).$$

Consequently, applying inverse of Laplace transform on the both sides we have:

$$\underline{y}(x;r)=r^2\ell^{-1}\left[\frac{1}{s^{\beta+1}}\right]+(r-1)\left(\ell^{-1}\left[\frac{1}{s}\right]+\ell^{-1}\left[\frac{1}{s^2}\right]-\ell^{-1}\left[\frac{1}{s^3}\right]\right),$$

$$\overline{y}(x;r)=\left(2-r^2\right)\ell^{-1}\left[\frac{1}{s^{\beta+1}}\right]+(1-r)\left(\ell^{-1}\left[\frac{1}{s}\right]+\ell^{-1}\left[\frac{1}{s^2}\right]-\ell^{-1}\left[\frac{1}{s^3}\right]\right).$$

Finally, we determine the solution of FFIVP (4.1) as follows :

$$\underline{y}(x;r)=r^2\frac{x^{\beta}}{\Gamma(\beta+1)}+(r-1)\left(1+x-\frac{x^2}{2}\right),$$

$$\overline{y}(x;r)=\left(2-r^2\right)\frac{x^{\beta}}{\Gamma(\beta+1)}+(1-r)\left(1+x-\frac{x^2}{2}\right).$$

Case 5. If $\left({}^{C}D_{2,1}^{\beta}y\right)(x)$ is ${}^{C}\left[(i)-\beta\right]$-differentiable, then equation (4.2) becomes:

$$-s^{\beta-1}y(0)\ominus\left(-s^{\beta}\right)L\left[y(x)\right]\ominus s^{\beta-2}y'(0)\ominus s^{\beta-3}y''(0)=L[\sigma].$$

Then, we get :

$$-s^{\beta-1}\underline{y}(0;r)+s^{\beta}\ell\left[\underline{y}(x;r)\right]-s^{\beta-2}\overline{y'}(0;r)-s^{\beta-3}\overline{y''}(0)=\frac{2-r^2}{s},$$

$$-s^{\beta-1}\overline{y}(0;r)+s^{\beta}\ell\left[\overline{y}(x;r)\right]-s^{\beta-2}\underline{y'}(0;r)-s^{\beta-3}\underline{y''}(0)=\frac{r^2}{s}.$$

Therefore, we have:

$$s^{\beta}\ell\left[\underline{y}(x;r)\right]=\frac{2-r^2}{s}+(r-1)\left(s^{\beta-1}-s^{\beta-2}-s^{\beta-3}\right),$$
$$s^{\beta}\ell\left[\overline{y}(x;r)\right]=\frac{r^2}{s}+(1-r)\left(s^{\beta-1}-s^{\beta-2}-s^{\beta-3}\right).$$

Consequently, applying inverse of Laplace transform on the both sides we have:

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$$\underline{y}(x;r)=\left(2-r^2\right)\ell^{-1}\left[\frac{1}{s^{\beta+1}}\right]+(r-1)\left(\ell^{-1}\left[\frac{1}{s}\right]-\ell^{-1}\left[\frac{1}{s^2}\right]-\ell^{-1}\left[\frac{1}{s^3}\right]\right),$$

$$\overline{y}(x;r)=r^2\ell^{-1}\left[\frac{1}{s^{\beta+1}}\right]+(1-r)\left(\ell^{-1}\left[\frac{1}{s}\right]-\ell^{-1}\left[\frac{1}{s^2}\right]-\ell^{-1}\left[\frac{1}{s^3}\right]\right).$$

Finally, we determine the solution of FFIVP (4.1) as follows :

$$\underline{y}(x;r)=\left(2-r^2\right)\frac{x^\beta}{\Gamma(\beta+1)}+(r-1)\left(1-x-\frac{x^2}{2}\right),$$

$$\overline{y}(x;r)=r^2\frac{x^\beta}{\Gamma(\beta+1)}+(1-r)\left(1-x-\frac{x^2}{2}\right).$$

Case 6. If $\left({}^{C}D^{\beta}_{2,1}y\right)(x)$ is ${}^{C}\left[(ii)-\beta\right]$-differentiable, then equation (4.2) becomes:

$$s^\beta L\left[y(x)\right]\ominus s^{\beta-1}y(0)-s^{\beta-2}y'(0)-s^{\beta-3}y''(0)=L[\sigma],$$

Then, we get :

$$s^\beta\ell\left[\underline{y}(x;r)\right]-s^{\beta-1}\underline{y}(0;r)-s^{\beta-2}\overline{y'}(0;r)-s^{\beta-3}\overline{y''}(0;r)=\frac{r^2}{s},$$

$$s^\beta\ell\left[\overline{y}(x;r)\right]-s^{\beta-1}\overline{y}(0;r)-s^{\beta-2}\underline{y'}(0;r)-s^{\beta-3}\underline{y''}(0;r)=\frac{2-r^2}{s}.$$

Therefore we have :

$$s^\beta\ell\left[\underline{y}(x;r)\right]=\frac{r^2}{s}+(r-1)\left(s^{\beta-1}-s^{\beta-2}-s^{\beta-3}\right),$$

$$s^\beta\ell\left[\overline{y}(x;r)\right]=\frac{2-r^2}{s}+(1-r)\left(s^{\beta-1}-s^{\beta-2}-s^{\beta-3}\right).$$

Consequently, applying inverse of Laplace transform on the both sides we have:

$$\underline{y}(x;r)=r^2\ell^{-1}\left[\frac{1}{s^{\beta+1}}\right]+(r-1)\left(\ell^{-1}\left[\frac{1}{s}\right]-\ell^{-1}\left[\frac{1}{s^2}\right]-\ell^{-1}\left[\frac{1}{s^3}\right]\right)$$

$$\overline{y}(x;r)=\left(2-r^2\right)\ell^{-1}\left[\frac{1}{s^{\beta+1}}\right]+(1-r)\left(\ell^{-1}\left[\frac{1}{s}\right]-\ell^{-1}\left[\frac{1}{s^2}\right]-\ell^{-1}\left[\frac{1}{s^3}\right]\right)$$

Finally, we determine the solution of FFIVP (4.1) as following :

$$\underline{y}(x;r)=r^2\frac{x^\beta}{\Gamma(\beta+1)}+(r-1)\left(1-x-\frac{x^2}{2}\right),$$

$$\overline{y}(x;r)=(2-r^2)\frac{x^\beta}{\Gamma(\beta+1)}+(1-r)\left(1-x-\frac{x^2}{2}\right).$$

Case 7 If $\left({}^{C}D_{2,2}^{\beta}y\right)(x)$ is ${}^{C}\left[(i)-\beta\right]$-differentiable, then equation (4.2) becomes:

$$s^\beta L\left[y(x)\right]\ominus s^{\beta-1}y(0)-s^{\beta-2}y'(0)\ominus s^{\beta-3}y''(0)=L[\sigma].$$

Then, we get :

$$s^\beta\ell\left[\underline{y}(x;r)\right]-s^{\beta-1}\underline{y}(0;r)-s^{\beta-2}\overline{y}'(0;r)-s^{\beta-3}\underline{y}''(0;r)=\frac{r^2}{s},$$

$$s^\beta\ell\left[\overline{y}(x;r)\right]-s^{\beta-1}\overline{y}(0;r)-s^{\beta-2}\underline{y}'(0;r)-s^{\beta-3}\overline{y}''(0;r)=\frac{2-r^2}{s}.$$

Therefore we have :

$$s^\beta\ell\left[\underline{y}(x;r)\right]=\frac{r^2}{s}+(r-1)\left(s^{\beta-1}-s^{\beta-2}+s^{\beta-3}\right),$$

$$s^\beta\ell\left[\overline{y}(x;r)\right]=\frac{2-r^2}{s}+(1-r)\left(s^{\beta-1}-s^{\beta-2}+s^{\beta-3}\right).$$

Consequently, applying inverse of Laplace transform on the both sides we have:

$$\underline{y}(x;r)=r^2\ell^{-1}\left[\frac{1}{s^{\beta+1}}\right]+(r-1)\left(\ell^{-1}\left[\frac{1}{s}\right]-\ell^{-1}\left[\frac{1}{s^2}\right]+\ell^{-1}\left[\frac{1}{s^3}\right]\right)$$

$$\overline{y}(x;r)=(2-r^2)\ell^{-1}\left[\frac{1}{s^{\beta+1}}\right]+(1-r)\left(\ell^{-1}\left[\frac{1}{s}\right]-\ell^{-1}\left[\frac{1}{s^2}\right]+\ell^{-1}\left[\frac{1}{s^3}\right]\right)$$

Finally, we determine the solution of FFIVP (4.1) as follows :

$$\underline{y}(x;r)=r^2\frac{x^\beta}{\Gamma(\beta+1)}+(r-1)\left(1-x+\frac{x^2}{2}\right),$$

$$\overline{y}(x;r)=(2-r^2)\frac{x^\beta}{\Gamma(\beta+1)}+(1-r)\left(1-x+\frac{x^2}{2}\right).$$

Case 8. If $\left({}^{C}D_{2,2}^{\beta}y\right)(x)$ is ${}^{C}\left[(ii)-\beta\right]$-differentiable, then equation (4.2) becomes:

$$-s^{\beta-1}y(0)\ominus\left(-s^{\beta}\right)L\left[y(x)\right]\ominus s^{\beta-2}y'(0)-s^{\beta-3}y''(0)=L[\sigma].$$

Then, we get :

$$-s^{\beta-1}\underline{y}(0;r)+s^{\beta}\ell\left[\underline{y}(x;r)\right]-s^{\beta-2}\overline{y'}(0;r)-s^{\beta-3}\underline{y''}(0)=\frac{2-r^2}{s},$$

$$-s^{\beta-1}\overline{y}(0;r)+s^{\beta}\ell\left[\overline{y}(x;r)\right]-s^{\beta-2}\underline{y'}(0;r)-s^{\beta-3}\overline{y''}(0)=\frac{r^2}{s}.$$

Therefore, we have:

$$s^{\beta}\ell\left[\underline{y}(x;r)\right]=\frac{2-r^2}{s}+(r-1)\left(s^{\beta-1}-s^{\beta-2}+s^{\beta-3}\right),$$
$$s^{\beta}\ell\left[\overline{y}(x;r)\right]=\frac{r^2}{s}+(1-r)\left(s^{\beta-1}-s^{\beta-2}+s^{\beta-3}\right).$$

Consequently, applying inverse of Laplace transform on the both sides we have:

$$\underline{y}(x;r)=\left(2-r^2\right)\ell^{-1}\left[\frac{1}{s^{\beta+1}}\right]+(r-1)\left(\ell^{-1}\left[\frac{1}{s}\right]-\ell^{-1}\left[\frac{1}{s^2}\right]+\ell^{-1}\left[\frac{1}{s^3}\right]\right),$$

$$\overline{y}(x;r)=r^2\ell^{-1}\left[\frac{1}{s^{\beta+1}}\right]+(1-r)\left(\ell^{-1}\left[\frac{1}{s}\right]-\ell^{-1}\left[\frac{1}{s^2}\right]+\ell^{-1}\left[\frac{1}{s^3}\right]\right).$$

Finally, we determine the solution of FFIVP (4.1) as follows :

$$\underline{y}(x;r)=\left(2-r^2\right)\frac{x^{\beta}}{\Gamma(\beta+1)}+(r-1)\left(1-x+\frac{x^2}{2}\right),$$

$$\overline{y}(x;r)=r^2\frac{x^{\beta}}{\Gamma(\beta+1)}+(1-r)\left(1-x+\frac{x^2}{2}\right).$$

5. Conclusions

In this paper, definition of fuzzy Caputo fractional derivatives about the order $2<\beta<3$ for fuzzy-valued function f is introduced and, fuzzy Laplace transforms for fuzzy Caputo fractional derivatives of the order $2<\beta<3$ are found under H-differentiability. FFIVP of the order $2<\beta<3$ is solved .

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تحويلات لابلاس الضبابية لمشتقة كابوتو الكسورية الضبابية

من الرتبة $2<\beta<3$

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المستخلص

الهدف الرئيسي من هذا البحث هو إيجاد صيغ المشتقات الكسورية الضبابية من النوع كابوتو من الرتبة $0<\beta<3$ لدالة القيمة الضبابية f , وكذلك إيجاد صيغ تحويلات لابلاس الضبابية للمشتقات الكسورية الضبابية من النوع كابوتو من الرتبة $2<\beta<3$ من باستخدام فرق -H.