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The Fuzzy of Maximal Module

Dr. Areej Tawfeeq Hameed*, Dr. Ahmed Hamzah Abed
Department of Mathematics, Faculty of Education for Girls, University of Kufa, Iraq.
*areej.tawfeeq@uokufa.edu.iq, areej238@gmail.com

Abstract— *In this paper, we give the concept of a fuzzy of maximal module as a generalization of maximal module (ordinary), this lead us to study and give some properties of fuzzy of maximal modules. Moreover, we study some types of fuzzy modules and its relationships with fuzzy of maximal modules.*

Keywords—component; Maximal module, fuzzy of maximal module, some types of fuzzy of maximal modules.

1. INTRODUCTION

In 1965, the concept of fuzzy sets was introduced by L.A. Zadeh and in 1993, W.J. Liu introduced the concept of fuzzy modules, fuzzy submodules and since that time many papers were introduced in different mathematical scopes of theoretical and practical applications. In 1998, J.N. Mordeson and D.S. Malik, the concept of fuzzy maximalmodules was introduced and give some properties and theorem of fuzzy maximal modules. The main aim of this paper is to extend and study the notions of (ordinary) maximal submodules and maximal modules into fuzzy maximal submodules and fuzzy maximal modules . This lead us to give some basic properties and theorems. In section one , some basic definitions and results are recalled which will be needed later. In section two, several results about fuzzy maximal submodules of R-module M , are given which are necessary in proving some results in the following sections. Later , we study divinity of fuzzy prime submodule, we give some basic properties of it and we gives the relation between this concept and fuzzy maximalsubmodule. In section three, we introduced fuzzy Jacobson radical of an R-module M , i.e., FJR(M) = ∩ {A | A is a fuzzy maximal submodule of an R-module M}. We give some basic properties of fuzzy Jacobson radical of an R-module M. We give some basic properties of fuzzy maximal submodules with fuzzy Jacobson radical of an R-module M. In section four, we give the external direct sum of fuzzy modules and we study some propositions and theorems about it. In section five, we give the fuzzy factionary of the fuzzy submodule of an R-module M. We study some theorems and properties of it with fuzzy maximal submodules. Throughout this paper (R,+,·) be a commutative ring with unity and M is an R-module.

2. Preliminary

Let X be a nonempty set , (R,+,·) be a commutative ring with unity and M is a R-module. **A fuzzy subset of X** is a function from X into [0,1], [13]. Let A and B be fuzzy subset of X . We write $\mathbf{A \subseteq B}$ if $A(x) \le B(x)$ for all $x \in X$. If $A \subseteq B$ and there exists $x \in X$ such that $A(x) < B(x)$, then we write $\mathbf{A \subset B}$ and we say that A is a proper fuzzy subset of B. Note that **A = B** if and only if $A(x) = B(x)$, for all $x\in X$, [13]. Let $\mathbf{A_X}$ denote the characteristic function of X defined by $A_X(x) = 1$ if $x \in X$ and $\boldsymbol{\lambda_Y}$ denote the characteristic function of Y defined by $\lambda_Y(x) = 1$ if $x \in Y$ and $\lambda_Y(x) = 0$ if $x \notin X \setminus Y$ where Y is anonempty subset , $Y \subseteq X$, [13].
Let A and B be fuzzy subsets of R, **the product A ∘ B** define by: $A \circ B(x) = \sup \{\min \{A(y), B(z)\} \mid x = y \cdot z \} y,z\in R$, for all $x\in R$, [13]. Let A and B be fuzzy subsets of R, **the addition A+B** define by : $A+B(x) = \sup \{\min \{A(y), B(z)\} \mid x = y + z \}$ $y, z \in R$, for all $x \in R$,[13]. Let $f : X \to Y$, A and B are two nonempty fuzzy subsets of nonempty sets X and Y respectively, the fuzzy subset $f(A)$ of Y defined by : $f(A)(y) = \sup A(x)$ if $x \in f^{1}(y) \neq \emptyset$, $y\in Y$ and $f(A)(y) = 0$, otherwise , where $f^{-1}(y) = \{x: f(x) = y\}$. It is called **the image of A under f** and denoted by $f(A)$. The fuzzy subset $f^{1}(B)$ of R defined by: $f^{1}(B)(x) = B(f(x))$, for $x\in X$. (i.e. $f^{1}(B) = (B \circ f)$. Is called **the inverse image of B** and denoted by $f^{1}(B)$. A fuzzy subset A of X is called ***f*-invariant** if $f(x) = f(y)$ implies $A(x) = A(y)$, where $x ,y \in X$. A is called **the sup property** if every set of Im (A), the image of A has a maximal element, [13] . For each $t \in [0,1]$, the set $\mathbf{A_t = \{ x \in X \mid A(x) \ge t \}}$ is called **a level subset of X** and A = B if and only if $A_t = B_t$ and the set $\mathbf{A_* = \{ x \in X \mid A(x) > 0 \}}$ is called **the support of X**, [13]. Let $x \in X$ and $t \in [0,1]$, let x_t denote the fuzzy subset of X defined by $x_t(y) = 0$ if $x \neq y$ and $x_t(y) = t$ if $x = y$ for all $y \in R$. x_t is called **a fuzzy singleton**, [13]. If x_t and y_s are fuzzy singletons, then $x_t + y_s = (x + y)_\lambda$ and $x_t \circ y_s = (x . y)_\lambda$, where $\lambda = \min \{t,s\}$, [13]. Let $\{A_i \mid i \in \Lambda\}$ be a collection of all fuzzy subset of R. Define the fuzzy subset of R (**intersection**) by $(\cap_{i\in\Lambda} A_i)(x) = \inf \{A_i(x) \mid i \in\Lambda\}$ for all $x\in R$,([13],[4]). Define the fuzzy subset of R **(union)** by $(\cup_{i\in\Lambda} A_i)(x) = \sup \{A_i(x) \mid i \in\Lambda\}$ for all $x\in R$,([13] ,[4]). We let ϕ denote $\phi(x) = 0$, for all $x\in R$, **the empty fuzzy subset of R**, [13]. Let A be a nonempty fuzzy subset of R, A is called **a fuzzy subgroup of R** if for all $x, y \in R$, $A(x + y) \ge \min \{A(x), A(y)\}$ and $A(x) = A(-x)$, [13]. A is a nonempty fuzzy subset of R, A is called **a fuzzy ring of R** if and only if for all $x, y \in R$, then $A(x - y) \ge \min \{A(x), A(y)\}$ and $A(x \cdot y) \ge \min \{A(x), A(y)\}$, [13]. A nonempty fuzzy subset A of R is called **a fuzzy ideal of R** if and only if for all $x, y \in R$, then $A(x - y) \ge \min \{A(x), A(y)\}$ and $A(x \cdot y) \ge \max \{A(x), A(y)\}$, [13]. A fuzzy ideal A of R is called **a prime fuzzy ideal of R** if either $A = \lambda_R$ or A is not constant and for any fuzzy ideals B and C of R , if $B \circ C \subseteq A$, then either $B\subseteq A$

or C⊆A, [13]. A nonempty fuzzy subset A of M is called **a fuzzy module of M** if and only if for all x, y ∈ M, then A(x - y) ≥ min {A(x), A(y)} and A(rx) ≥ A(x) and A(0) =, (0 is the zero element of M), [13]. A and B are fuzzy modules of an R-module M, B is called **a fuzzy submodule of A** if and only if B ⊆ A, [13]. Let A be a fuzzy subset of an R-module M. A is a fuzzy submodule of M if and only if A_t is a submodule of M , for all t ∈ [0, 1], [13]. Let A and B be fuzzy submodules of an R-module M**.** Define **AB** by: AB (x) = sup {min { min{A(y_i), B(z_i) }|i=1,.. , n}| y_i, z_i∈M,i=1,.. , n}|x = $\sum_{i=1}^{n} y_i \cdot z_i$, n∈N}, for all x ∈M, [13]. Let B be fuzzy submodule of an R-module M and x_t is fuzzy singleton of R, **the product $x_t \circ$ B** define by: $x_t \circ$ B (a) = sup {x_t(y), B(z) }|a = y ·z} y, z ∈M , for all a∈M ,[13]. Let A be a nonempty fuzzy module of an R-module M . **The Annihilator of A** denoted by (F-Ann A) is defined by {x_t: x∈R, $x_t \circ$ A ⊆0_1},t ∈[0,1] and (F-Ann A)(a) =sup {t: t∈[0,1], $a_t \circ$ A ⊆0_1}, a∈ R; that is F-annA = (0_1 :A),[10,13].

PROPOSITION 2.1[13]: Let A and B be two fuzzy subset of R ,then :
1) A ∘ B⊆A ∩ B, 2) $(A \circ B)_t = A_t . B_t$, t∈[0,1], 3) $(A \cap B)_t = A_t \cap B_t$, t∈[0,1], 4)$(A \cup B)_t = A_t \cup B_t$, t∈[0,1], 5) For each t ∈ [0,1], A = B if and only if, $A_t = B_t$.

PROPOSITION 2.2 : Let A and B be two fuzzy subset of R_1 and R_2 respectively , Let $f : R_1 \to R_2$ be a homomorphism , then :
1) $f(A) \cap f(B) = f(A \cap B)$. 2) $f(A) \circ f(B) = f(A \circ B)$. 3) $f(A_t) = (f(A))_t$. 4) $f^{-1}(B_t) = (f^{-1}(B))_t$.

PROPOSITION 2.3[4]: Let A be a fuzzy module of an R-module M_1 and B be a fuzzy module of an R-module M_2 . Let $f : M_1 \to M_2$ be a homomorphism . Then:1) f(A) is a fuzzy submodule of M_2 if f is an epimorphism . 2) f^{-1} (B) is a fuzzy submodule of M_1 .

PROPOSITION 2.4[4]: Let A and B be two fuzzy modules of an R-module M_1 and C and D be two fuzzy modules of an R-module M_2. Let $f : M_1 \to M_2$ be a homomorphism . Then : 1) $f(A \cap B) = f(A) \cap f(B)$. 2) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

PROPOSITION 2.5[4]: Let {A_i| i ∈Λ} be a collection of fuzzy submodules of an R-module M . Then :
1) $(\cap_{i\in\Lambda} A_i)$ is a fuzzy module of R-module M. 2) $(\cup_{i\in\Lambda} A_i)$ is a fuzzy module of R-module M, where {A_i| i ∈Λ} are chains.

DEFINITION 2.6[18]: Suppose that A and B are two fuzzy submodules of an R-module M , define **(A:B)** by: (A:B) = {r_t : r_t is a fuzzy singleton of R such that $r_t \circ$ B ⊆ A } that mean: (A:B) (r) = sup {t :t ∈[0,1] , $r_t \circ$ B ⊆ A} , r ∈R .
If B = (b_k) , then (A:(b_k))(r) = sup{t :t ∈[0,1], $r_t \circ$ (b_k) ⊆ A , r_t is a fuzzy singleton of R} .

DEFINITION 2.7[18]: Let X be a fuzzy module of an R-module M and A be a fuzzy submodule of X , let I be a fuzzy ideal of R , define (A:I) as (A:I) = {a_t : a_t⊆ M , $a_t \circ$ I ⊆ A}. That mean $(A:_X I)$(b) = sup {t :t ∈[0,1] , I ∘ b_t⊆ A}, b∈M . If I = (r_k), then (A: (r_k)) = { a_t : $a_t \circ$ (r_k) ⊆ A}.

PROPOSITION 2.8[4]: Let A and B be two fuzzy modules of an R-module M, then
1) A + B is a fuzzy module of an R-module M. 2) rA is a fuzzy module of an R-module M where r ∈R.

PROPOSITION 2.9[18]: Let X be a fuzzy module of an R-module M and A and B be two fuzzy submodules of X and let I be a fuzzy ideal of R such that I (0) = 1, then
1) $(A:_X I)$ is a fuzzy module of an R-module M if and only if, $(A:_X I)_t$ is a module of an R-module M, for all t∈[0,1].
2) (A:B) is a fuzzy ideal of R if and only if, $(A:B)_t$ is an ideal of R, for all t∈[0,1] .

PROPOSITION 2.10: Let A and B be two fuzzy modules of an R-module M, then (A:B) is a fuzzy ideal of R.

Proof: Let r_t is a fuzzy singleton of R , r∈R and t ∈ [0,1] . Since (A:B) = {r_t : r_t is a fuzzy singleton of R such that $r_t \circ$ B ⊆ A} that mean: (A:B) (r) = sup {t :t ∈[0,1] , $r_t \circ$ B ⊆ A}, r ∈R.
(A:B)(0) = sup {t :t ∈[0,1] , $0_t \circ$ B ⊆ A} > 0, then (A:B) is a nonempty fuzzy subset of R.
Let x, y ∈ R and t ∈ [0,1] . x_t and y_t are fuzzy singletons of R.
(A:B)(x-y)= sup {t :t ∈[0,1] , $(x\text{-}y)_t \circ$ B ⊆ A} = sup {t :t ∈[0,1] , (x_t -y_t) ∘ B ⊆ A} = sup {t :t ∈[0,1] , ($x_t \circ$ B) – ($y_t \circ$ B) ⊆ A}
= sup{ min{sup {t :t ∈[0,1] , ($x_t \circ$ B) ⊆ A} , sup {t :t ∈[0,1] , ($y_t \circ$ B) ⊆ A}}}.
≥ min{sup {t :t ∈[0,1] , ($x_t \circ$ B) ⊆ A} , sup {t :t ∈[0,1] , ($y_t \circ$ B) ⊆ A}} ≥ min{(A:B)(x), (A:B)(y) }.
(A:B)(x-y) ≥ min{(A:B)(x), (A:B)(y) } … (1) .
(A:B)(x.y) = sup {t :t ∈[0,1] , $(xy)_t \circ$ B ⊆ A} ≥ sup {t :t ∈[0,1] , $x_t \circ$ B ⊆ A} ;
(A:B)(xy) ≥ (A:B)(x) and (A:B)(xy) ≥ (A:B)(y) . Then (A:B)(xy) ≥ max {(A:B)(x) ,(A:B)(xy) } …(2) .
Hence by (1) and (2), (A:B) is a fuzzy ideal of R.

COROLLARY 2.11: Let A and B are fuzzy ideals of R , then (A: B) is a fuzzy ideal of R .

Proof: It is clear .

PROPOSITION 2.12: Let A be a fuzzy submodule of a fuzzy module X, then (A_t: X_t) = $(A: X)_t$for all t ∈ (0,1].

Proof: Let C = (A: X), then C ∘ X ⊆A, so $(C \circ X)_t$⊆A_t which implies that C_t . X_t⊆A_t by [23,theorem (2.4)]. Hence, C_t⊆(A_t: X_t) ; that is $(A: X)_t$⊆(A_t: X_t) . Let r ∈(A_t: X_t), hence r . y ∈$A_t$for all y∈X_t . So that for all w∈X_t and w ∉A_t , r.w∈A_t. Hence w_t⊄A and $r_t \circ w_t$⊆A. Let r ∈(A_t: X_t), hence r . y ∈$A_t$for all y∈X_t . So that(r . y)$_t$⊆A f or all y∈X_t and $r_t \circ y_t$⊆A for all y_t⊆X . Hence r_t⊆(A: X) , thus r ∈$(A: X)_t$. So that (A_t: X_t)⊆$(A: X)_t$. Therefore, (A_t: X_t) = $(A: X)_t$ for all t ∈ (0, 1].

PROPOSITION 2.13: Let X be a fuzzy module of an R-module M and A be a fuzzy modules of X and I be a fuzzy ideal of R such that I(0) = 1, then $(A:_X I)$ is a fuzzy module of an R-module M.

Proof: Let b_t is a fuzzy singleton of M , b∈M and t ∈ [0,1] . Since $(A :_X I)$ (b) = sup {t :t ∈[0,1] , I ∘ b_t⊆ A}, b∈M.

$(A:_X I)(0) = \sup\{t : t \in [0,1], I \circ 0_t \subseteq A\} > 0$, then $(A:_X I)$ is a nonempty fuzzy subset of R-module M.

Let $x, y \in M$ and $t \in [0,1]$. x_t and y_t are fuzzy singletons of M. $(A:_X I)(x-y) = \sup\{t : t \in [0,1], I \circ (x-y)_t \subseteq A\}$ $= \sup\{t : t \in [0,1], I \circ (x_t - y_t) \subseteq A\} = \sup\{t : t \in [0,1], (I \circ x_t) - (I \circ y_t) \subseteq A\} = \sup\{\min\{\sup\{t : t \in [0,1], (I \circ x_t) \subseteq A\}, \sup\{t : t \in [0,1], (I \circ y_t) \subseteq A\}\}\} \geq \min\{\sup\{t : t \in [0,1], (I \circ x_t) \subseteq A\}, \sup\{t : t \in [0,1], (I \circ y_t) \subseteq A\}\} \geq \min\{(A:_X I)(x), (A:_X I)(y)\}$.
$(A:_X I)(x-y) \geq \min\{(A:_X I)(x), (A:_X I)(y)\}$ … (1).
$(A:_X I)(rx) = \sup\{t : t \in [0,1], I \circ (rx)_t \subseteq A\} = \sup\{t : t \in [0,1], I \circ [r(x_t)] \subseteq A\} \geq \sup\{t : t \in [0,1], I \circ x_t \subseteq A\}$.
$(A:_X I)(rx) \geq (A:_X I)(x)$ …(2). Since $(A:_X I)(0) = 1$ …(3).
Hence by (1), (2) and (3), $(A:_X I)$ is a fuzzy module of R-module M.

PROPOSITION 2.14: Let A be a fuzzy submodule of an R-module M_1 and B be a fuzzy submodule of an R-module M_2. If $f: M_1 \to M_2$ be a R-homomorphism, then :

1) If Im (A) ={1, t} ,then Im (f(A)) = {1, t} , for some $t \in [0, 1)$, if f is f-epimorphism .
2) If Im (B) ={1, t} ,then Im (f^{-1}(B)) = {1, t} , for some $t \in [0, 1)$.

3. Fuzzy Maximal Modules

In this section, we give fuzzy maximal submodules. We give some basic theorems and properties of fuzzy maximal submodules. We give the concept of a fuzzy prime module and we gives the relation between fuzzy prime modules and fuzzy maximal submodules.

DEFINITION 3.1 [14]: A fuzzy submodule A of an R-module M is called **a fuzzy maximal module of M** if for any fuzzy submodule B of M , $A \subseteq B$, then either $A_* = B_*$ or $B = A_M$.

Note that : In our work, since A is fuzzy submodule , then we called A is a fuzzy maximal submodule of M if A is fuzzy maximal module .

EXAMPLES 3.2: Let $X : Z \to [0,1]$ defined by : $X(a) = \begin{cases} 1 & \text{if } a \in 2Z \\ 0 & \text{otherwise} \end{cases}$. Thus X is a fuzzy maximal submodule of Z .

1. Let $Y : Z_8 \to [0,1]$ defined by : $Y(b) = \begin{cases} 1 & \text{if } b \in \{0,2,4,6\} \\ 0 & \text{otherwise} \end{cases}$. Thus Y is a fuzzy maximal submodule of Z_8.

2. Let $X : Z \oplus Z \to [0,1]$ defined by : $X(a) = \begin{cases} 1 & \text{if } a \in 2Z \oplus Z \\ 0 & \text{otherwise} \end{cases}$. Thus X is a fuzzy maximal module of $Z \oplus Z$.

3. Let $X : Z \to [0,1]$ defined by : $X(a) = \begin{cases} 1 & \text{if } a \in 4Z \\ 1/2 & \text{if } a \in 2Z - 4Z \\ 1/3 & \text{otherwise} \end{cases}$. Thus X is not a fuzzy maximal module of Z.

PROPOSITION 3.3 [14]: A fuzzy submodule A of M is a fuzzy maximal submodule of M if and only if, Im A ={1, t} ,for some $t \in [0, 1)$ and A_1 is a maximal submodule of M.

Note that : A_N is a fuzzy maximal submodule of R-module M , then $(A_N)_t$ is a maximal submodule of M, for all $t \in [0, 1]$.

PROPOSITION 3.4 [14]: A fuzzy submodule A of M is a fuzzy maximal submodule of M if and only if Im A ={1, t}, for some $t \in [0, 1)$ and A_* is a maximal submodule of M.

PROPOSITION 3.5: Let N be a submodule of an R-module M and let A_N be the fuzzy moduleofM determined by N . Then N is a maximal submodule of M if and only if A_N is a fuzzy maximal submodule of M .

Proof: Since N is a maximal submodule of M $(N \neq 0)$, then $N = (A_N)_1$and Im (A_N) = {1}, hence A_N is a fuzzy maximal submodule of M by proposition (3.3) .

PROPOSITION 3.6: Let A be a fuzzy maximal submodule of an R-module M_1 and B be a fuzzy maximal submodule of an R-module M_2 . If $f: M_1 \to M_2$ be a R-homomorphism , then :

1) f(A) is a fuzzy maximal submodule of M_2 if f is f-epimorphism .
2) f^{-1}(B) is a fuzzy maximal submodule of M_1 .

Proof:

1) By proposition (2.2) , f(A) is a fuzzy module of M_2 , and by proposition (3.3), A_1 is a maximal module of M_1 , and Im (A) = {1, t}, for some $t \in [0, 1)$, then $f(A_1)$ is a maximal module of M_2 ,and Im $(f(A_1))$ = {1, t}, for some $t \in [0, 1)$ by proposition (2.14) , and by proposition (2.3,(1)) , $f(A_1) = (f(A))_1$ is a maximal module of M_2 and Im $(f(A)_1)$ = {1, t}, for some $t \in [0, 1)$, impels that f(A) is a maximal fuzzy submodule of M_2 by proposition (3.3) .
2) By proposition (2.2) , f^{-1}(B) is a fuzzy module of M_1 , and by proposition (3.3), B_1 is a maximal module of M_2, and Im (B) = {1, t}, for some $t \in [0, 1)$, then $f^{-1}(B_1)$ is a maximal module of M_1, and Im $(f^{-1}(B_1))$ = {1, t}, for some $t \in [0, 1)$ by proposition (2.14) , and by proposition (2.3,(2)) , $f^{-1}(B_1) = (f^{-1}(B))_1$ is a maximal module of M_1, and Im $((f^{-1}(B))_1)$ = {1, t}, for some $t \in [0, 1)$, impels that f^{-1}(B) is a maximal fuzzy submodule of M_1 by proposition (3.3).

PROPOSITION 3.7: Let X be a fuzzy module of an R_1-module M and Y be a fuzzy module of an R_2-module M. Let $f: R_1 \to R_2$ be an epimorphism. If 0_1 is a fuzzy maximal module of X, then Y is a fuzzy maximal module, where F-Ann X is a fuzzy maximal module .

Proof: Let $r_t \circ x_k \subseteq 0_1$, for a fuzzy singleton r_t of R_1 and $x_k \subseteq X$. Then $(rx)_\lambda \subseteq 0_1 = f(0_1)$, where $\lambda = \min\{t,k\}$, $t,k, \lambda \in (0,1]$. Since f is epimorphism, then there exists $y \in R_2$ such that $y = f(x)$. Thus $r_t \circ y_k = r_t \circ f(x_k) = r_t \circ f(x)_k = f(rx)_\lambda \subseteq f(0_1)$.
Hence $f^{-1}(f(rx))_\lambda \subseteq f^{-1}(f(0_1))$. But 0_1 is f-invariant , so $f^{-1}(f(0_1)) = 0_1$. Also $(rx)_\lambda \subseteq f^{-1}(f(rx))_\lambda$. Hence $(rx)_\lambda \subseteq 0_1$. But 0_1 is a fuzzy maximal ideal, thus ether $x_k \subseteq 0_1$ or $r_t \subseteq$ F-Ann X. Then $y_k \subseteq f(x_k) \subseteq f(0_1) = 0_1$.
If $r_t \subseteq 0_1$, then $r_t \circ Y \subseteq 0_1$ since $w_x \subseteq Y$, then $w_c = f(z_c) \subseteq Y$ for some $z \in R_1$. So $r_t \circ w_c = f(r_t \circ z_c) \subseteq f(0_1) = 0_1$. Then either $y_k \subseteq 0_1$ or $r_t \subseteq$ F-Ann Y. Since F-Ann X is a fuzzy maximal module, then Im (Y) ={1, t} ,for some $t \in [0, 1)$ by proposition (3.3) and proposition (2.14). Hence Y is a fuzzy maximal module.

REMARKS 3.8: The proposition (3.7) is not true in general, the condition ($0_1$is a fuzzy maximal submodule) is necessary for example: Let $f: Z \to Z/\langle 8\rangle = Z_8$ defined as $f(x) = x$, f is an epimorphism, let $X : Z \to [0,1]$ defined by:

$$X(a) = \begin{cases} 1 & if a \in 2Z \\ 0 & otherwise \end{cases}$$

. Thus X is a fuzzy maximal submodule of Z.

(F-Ann X) = { x_t: $x \in R$, $x_t \circ X \subseteq 0_1$},$t \in [0,1]$ and (F-Ann X) (a) =sup { t: $t \in [0,1]$, $a_t \circ X \subseteq 0_1$}, $a \in R$; that is F-annX = (0_1 :A), F-Ann X is a fuzzy maximal module of Z. And let $Y : Z_8 \to [0,1]$ defined by : $Y(a) = \begin{cases} 1 & if a \in \{0,2,4,6\} \\ 0 & otherwise \end{cases}$.

Thus Y is not fuzzy maximal module of Z_8 . Moreover, 0_1 is not maximal since $f(8) = f(0)$, but $0_1(8) = 0_1 \neq 0_1(0) = 1$.

PROPOSITION 3.9: Let A and B are fuzzy maximal submodules of R-module M, then (A:B) is a fuzzy maximal ideal of R.

Proof: Let A and B be two fuzzy submodules of R-module M, then (A : B) is a fuzzy ideal of R by proposition (2.12) . To prove (A:B) is a fuzzy maximal ideal of R. Let x_t, y_h are fuzzy singletons of R such that $x_t, y_h \subseteq A$, then $x_k \circ y_h = x_k \circ y_\lambda = (xy)_\lambda \subseteq A$ and $(x_k \circ y_h) \circ B = (x_k \circ y_\lambda) \circ B = (xy)_\lambda \circ B \subseteq A$, then $(xy)_\lambda \subseteq (A : B)$, where $\lambda = \min\{h,k\}$.
Since A and B be fuzzy maximal submodules of R-module M, then A_1 and B_1 be maximal modules of R-module M, where $\lambda = \min\{h,k\} \leq 1$. Hence , $(xy) \subseteq (A : B)_1$ is a maximal ideal of R. But $(xy) \subseteq (A : B)_1$ is a maximal ideal of R, then (A: B) is a fuzzy maximal ideal of R.

COROLLARY 3.10: If M = R ,then

1. A is a fuzzy maximal submodule of M if and only if, A is a fuzzy maximal ideal .
2. A is a fuzzy maximal submodule of M, then $|\text{Im}(A)| = 2$.
3. A is a fuzzy submodule of M and A_* is maximal ideal of R and A(0) = 1, then A is a fuzzy maximal submodule of M.

Proof:

(1) By Definition (3.1) of a fuzzy maximal submodule of M and Definition of a fuzzy maximal ideal .
(2) By part (1) and theorem (3.4) in [12] .
(3) By part(1) and theorem (3.7) in [12] .

PROPOSITION 3.11: Let X be a fuzzy module of an R-module M and A be a fuzzy maximal submodules of X and I be a fuzzy ideal of R such that I(0) = 1, then $(A:_X I)$ is a fuzzy maximal submodule of an R-module M.

Proof: Let A be a fuzzy submodule of X and let I be a fuzzy ideal of R such that I (0) = 1, then $(A:_X I)$ is a fuzzy submodule of R-module M by proposition (2.11) and definition of fuzzy submodule . To prove $(A:_X I)$ is a fuzzymaximal submodule of an R-module M.
Since $(A:_X I)(a) = \sup\{t : t \in [0,1], I \circ a_t \subseteq A\}$, $a \in M$. Let x_k be fuzzy singletons of M such that $x_k \subseteq I$, then $x_k \circ a_t = x_\lambda \circ a_\lambda = (xa)_\lambda$, where $\lambda = \min\{t,k\}$ and $(x_k \circ a_t) \circ I = (x_k \circ a_t) \circ I = (xa)_\lambda \circ I \subseteq A$, then $(xa)_\lambda \subseteq (A :_X I)$, where $\lambda = \min\{h,k\}$. Since A be a fuzzy maximal submodule of an R-module M , then A_1 be a maximal submodule of an R-module M , for all $\lambda = \min\{h,k\} \leq 1$. Hence $(xa) \subseteq (A :_X I)_1$ is a maximal submodule of an R-module M ,by proposition (2.9) and [18], Im $((A :_X I))$ ={1, t}, for some $t \in [0, 1)$ by proposition (2.14), then $(A :_X I)$ is a fuzzy maximal submodule of an R-module M by proposition (3.3).

PROPOSITION 3.12: Let X be a fuzzy module of an R-module Mand M be a maximal R-module, then X is a fuzzy maximal submodule if$X_1 \neq 0_1$ and Im(X) ={1, t}, for some $t \in [0,1)$.

Proof: To prove X is a fuzzy maximal submodule of R-module M, that mean X_1 is a maximal module, and Im(X) = {1, t}, for some $t \in [0,1)$. Let I =$X_1 \neq 0_1$, I is a maximal submodule of M by condition implies that X_1 is a maximal submodule of M. And Im(X) = {1, t}, for some $t \in [0,1)$. Hence X is afuzzy maximal submodule of R-module M.

Now, we study givethe concept of a fuzzy prime module and we gives the relation between them .

DEFINITION 3.13 [10]: Let X be a fuzzy module of an R-module M and A be a fuzzy submodule of X. A is called **a fuzzy prime submodule of an R-module M** whenever $r_t \circ a_k \subseteq A$, for fuzzy singleton r_t of R and $a_k \subseteq X$ we have either $r_t \subseteq (A :_R X)$ or $a_k \subseteq A$, where

$(A :_R X) = \{ r_t : r_t \circ X \subseteq A , r_t$ fuzzy singleton of $R \}$.

PROPOSITION 3. 14[10] : Let X be a fuzzy module of an R-module M and A be a fuzzy submodule of X. A is a prime fuzzy submodule of an R-module M and let P = (A:X), then :

1) A is a fuzzy prime submodule of an R-module M.
2) For each fuzzy submodule B of X and for any fuzzy ideal I of R , if $I \circ B \subseteq A$, then either $B \subseteq A$ or $I \subseteq (A: X)$.
3) If A is a fuzzy prime submodule of an R-module M, then (A: X) is a fuzzy prime ideal of R .
4) A fuzzy submodule A of M is a fuzzy prime submodule of M if and only if Im A ={1, t}, for some $t \in [0,1)$ and A_1 is a prime submodule of M .

PROPOSITION 3. 15 :

Let X be a fuzzy module of an R-module M and A be a fuzzy submodule of X. If A is a fuzzy maximal submodule of an R-module M , then A is a fuzzy prime submodule of of an R-module M .

Proof: Let A be a fuzzy maximal submodule of R-module M , then A_1 be maximal module of R-module M , and Im (A) ={1, t} for some $t \in [0,1)$by proposition (3.3). By [21] , A_1 be prime module of R-module M. Hence A is a fuzzy prime submodule of an R-module M by proposition (3.14(4)) .

REMARKS 3.16: The Proposition (3.15) is not true in general, since $0_1$is a prime fuzzy submodule by [10] , but is not fuzzy maximal submodule .

PROPOSITION 3. 17 : Let X be a fuzzy module of an R-module M. If X is fuzzy Boolean, then every fuzzy prime submodule of an R-module M is a fuzzy maximal submodule of of an R-module M.

Proof: Since X is fuzzy Boolean of R-module M , then R is a Boolean ringby definition of fuzzy Boolean. Since every prime of a Boolean ring R is maximal, implies that every fuzzy prime of a Boolean ring R is fuzzy maximal by proposition (2.14,(4)) and proposition (3.3).

REMARK 3.18: Let an element $\alpha \neq 1, \alpha \in [0, 1]$ is dual atom if there is no $\beta \in [0, 1]$ such that $\alpha < \beta < 1$. Clearly α is a dual atom if and only if α is a maximal element of [0,1).

PROPOSITION 3.19: Let A be the fuzzy submodule of an R-module M. A is a fuzzy maximal submodule of an R-module M if and only if, there exists a maximal submodule A_* of M and α is a dual atom in [0, 1] such that $A(x) = \begin{cases} 1 & if x \in A_* \\ 0 & otherwise \end{cases}$.

Proof: Suppose A is a fuzzy maximal submodule of an R-module M, then A(0)= 1. For any α in [0,1], $(A\cup\alpha)(x) = A(x) \cup \alpha$, forall x in M. Then $A \cup \alpha$ is a fuzzy submodule since [0,1] is distributive and $A \subseteq A\cup\alpha$ and hence either $A \cup\alpha$ is constant or $A = A \cup\alpha$. If A(x) < 1 and A(y) < 1, for $x ,y \in M$, then $(A \cup A)(0) = A(0) \cup A(x) = 1$ and $(A \cup A(x))(x) = A(x)$ implies that $A \cup A(x)$ is not constant implies that $A(y) = A(y) \cup A(x)$. Hence $A(y) \geq A(x)$. By symmetry we get $A(x) \geq A(y)$.
Thus A assume exactly one value say α other than 1. Nowα is a dual atom for if $\alpha < \beta \in [0, 1]$, then B defined by :

$B(x) = \begin{cases} 1 & if A(x) = 1 \\ \beta & otherwise \end{cases}$. B is the fuzzy submodule of an R-module M and $A \subseteq B$, then B is constant implies that β = 1 .

Now $A_* = \{x \in M \mid A(x) = 1\}$ is a maximal submodule of M . Hence $B(x) = A(x) = \begin{cases} 1 & if x \in A_* \\ 0 & otherwise \end{cases}$.

Conversely, let A_* be a maximal submodule of M and α is a dual atom in [0, 1]. If A is defined as above ,then A is a not constant fuzzy submodule by [4] . If B is a not constant fuzzy submodule of an R-module M such that $A \subseteq B$, then $A_* \subseteq \{x \in M \mid B(x) = 1\} \neq M$ implies that $A_* = \{x \in M \mid B(x) = 1\}$and for any $x \notin A_*$, $\alpha = A(x) \leq B(x) < 1$ implies that A(x) = B(x). Hence A is a fuzzy maximal submodule of an R-module M .

COROLLARY 3.20 : If M is a module in which each maximal submodule is prime , then each fuzzy maximal submodule is fuzzy prime submodule .

PROPOSITION 3.21 : Let A and B be two fuzzy maximal submodules of R-module M , then (A + B) is a fuzzy maximal submodule of R-module M .

Proof: Let A and B be two fuzzy submodules of R-module M , then (A + B) is a fuzzy submodule of R-module M by proposition (2.9) . To prove (A+ B) is a fuzzy maximal submodule of R-module M. Let x_k and y_h are fuzzy singletons of R such that $x_k \subseteq A$ and $y_h \subseteq B$, then $x_k + y_h = x_k + y_\lambda = (x + y)_\lambda \subseteq (A + B)$, where $\lambda = \min \{h ,k \} \leq 1$.

Hence , $(x + y) \subseteq (A + B)_1$ is a maximal module of R-module M , and Im (A+ B) = {1, t} , for some$t \in [0,1)$. Then (A + B) is a fuzzy maximal submodule of R-module M.

4. Fuzzy Jacobson Radical :

In this section, we introduced fuzzy Jacobson radical of an R-module M as the fuzzy Jacobson radical of R, that mean FJR(M) = ∩ {A | A is a fuzzy maximal ideal of R}. We give some basic properties of fuzzy Jacobson radical of an R-module M. We give some basic properties of fuzzy maximal submodules with fuzzy Jacobson radical of an R-module M .

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DEFINITION 4.1: The intersection of all the fuzzy maximal submodule of an R-module M is called **a fuzzy Jacobson radical of M**; i.e. , FJR(M) = ∩ {A | A is a fuzzy maximal submodule of R-module M}.

PROPOSITION 4.2: Let M be finitely generated and let A be fuzzy submoduleof an R-module M. If B is a fuzzy ideal of R such that BA = A , then there exists $t \in [0, 1]$ such that $B_t M = M$.

Proof: Let t = inf { A(x) | $x \in M$ }. By [7] , there exists generators $b_1, b_2, \dots, b_n$ of M such that $A(b_1) = A(b_2) = \dots = A(b_n) = t$, Let $x \in M$ and $\varepsilon > 0$, then $t - \varepsilon < A(x) = (BA)(x) = \sup\{\min\{\min\{B(y_i), A(z_i)\} | i=1,.., n\} | y_i, z_i \in M, i=1,.., n\} | x = \sum_{i=1}^{n} y_i \cdot z_i, n \in N\}$, for all $x \in M$.

Therefore, there exists a representation $x = y_1 \cdot z_1 + y_2 \cdot z_2 + \dots + y_n \cdot z_n$, where $y_i, z_i \in M, i=1,.., n$ such that $t - \varepsilon < \{\min\{B(y_i), A(z_i)\} | i=1,.., n\}$ implies that $t - \varepsilon < B(y_i)$ and $t - \varepsilon < A(z_i)$ for all i=1,.. , n. Thus $y_i \in B_{t-\varepsilon}$ and $z_i \in A_{t-\varepsilon}$ for all i=1,.. , n , hence $x = \sum_{i=1}^{n} y_i \cdot z_i \in B_{t-\varepsilon} A_{t-\varepsilon}$ for all ε . $x \in B_t A_t$ implies that $M \subseteq B_t A_t$, then $M = B_t A_t$ … (1) .

$z_i \in A_{t-\varepsilon 1}$ for all $\varepsilon > 0$, that mean is $z_i \in A_{t1}$ for all i , then $M = A_t$ … (2) .

From (1) and (2), there exists $t \in [0, 1]$ such that $B_t M = M$.

THEOREM 4.3 : Let M be generated by $b_1, b_2, \dots, b_n$ and let A fuzzy submodule of an R-module such that $A(b_i) \neq 0$ for all i . If BA = A , where B is a fuzzy ideal of R and if $B \subseteq FJR(R)$, then M = <0> (A is constant) .

Proof: Let t = inf { A(x) | $x \in M$ }. Since $A(b_i) \neq 0$ for all i , then $t \neq 0$. And since $B \subseteq FJR(R)$ it follows that B is contained in every fuzzy maximal submodule of an R-module M . Let J be any maximal submodule of M . Define a fuzzy subset D of R-module M by : $D(x) = \begin{cases} 1 & if x \in J \\ \alpha & if x \notin J, \alpha \in [0,1) \end{cases}$. D is a fuzzy maximal submodule of an R-module M , therefore $B \subseteq D$ and $B(x) \leq D(x)$,for all $x \in M$. Let $y \in B_t$, then $B(y) \geq t > \alpha$, then $D(y) > \alpha$, implies that D(y) = 1 and $y \in J$. Hence $B_t \subseteq J(M)$. Now , proposition (4.2) and Nakayama's lemma , M= <0> and hence A is constant ,

PROPOSITION 4.4 [19]: Let N be a submodule of a finitely generated module M and let I be an ideal of R such that $I \subseteq J(R)$. If IM + N = M , then N = M .

THEOREM 4.5: Let A and B be fuzzy submodules of an R-module such that $A(x) \geq B(x)$ for all $x \in M$. Let C be a fuzzy ideal of R . If M is generated by $b_1, b_2, \dots, b_n$, $C \subseteq FJR(R)$ and $A(bi) \neq 0$, for some I , then there exists $t \in [0, 1]$ such that $A_t = B_t$.

Proof: If $C \subseteq FJR(R)$, then $A_t \subseteq J(R)$ for some $t \in [0, 1]$ (as the proof of theorem (4.3)) and by proposition(4.4) , hence $M = B_t$. But $M = A_t$, then $A_t = B_t$.

PROPOSITION 4.6: Let A be fuzzy submoduleof an R-module M such that $A \subseteq FJR(M)$, then e +a is invertible where $a \in A_1$.

Proof: Suppose e + a is not invertible, then there exists a maximal submodule J of M containing e + a >

Define a fuzzy submodule D of an R-module M by : $D(x) = \begin{cases} 1 & if x \in J \\ \alpha & if x \notin J, \alpha \in [0,1) \end{cases}$. D is a fuzzy maximal submodule of an R-module M . By hypothesis $t = A(a) \leq D(a)$ implies that $D(a) > \alpha$. Hence $a \in M$ when $e \in M$, this is contradiction .

Hence e +a is invertible where $a \in A_1$.

Now, we definition the fuzzy coset of fuzzy submodule asthe fuzzy coset of fuzzy ideal in[6].

DEFINITION 4.7: Let A be fuzzy submodule of an R-module M and for any $x \in R$, the fuzzy subset A_x^* of M defined by : $A_x^*(r) = A(r-x)$ for all $r \in R$, is termed as the **fuzzy coset determined by x and A .**

REMARK 4.8: The binary operations of addition and multiplication as [6] are $A_x^* + A_y^* = A_{x+y}^*$ and $A_x^* A_y^* = A_{xy}^*$ for all $x,y \in R$. And A_e^* is the identity element and A_0^* is the zero element .

LEMMA 4.9:

1- Let A be fuzzy submoduleof an R-module M where R is a commutative ring and A(x) < A(y), for some $x, y \in R$, then A(x-y) = A(x) = A(y-x) .
2- Let A be fuzzy submoduleof an R-module M, then A(x) = A(0) if and only if, $A_x^* = A_0^*$ where $x \in R$.

Proof: The part (1) is obvious .

To proof (2) , let A(x) = A(0) , so that $A(r) \leq A(x)$ for all $r \in R$.

If A(r) < A(x) ,then A(r-x) = A(r) by part (1) . If A(r) = A(x) ,then $r, x \in At$ where t =A(0) = 1. Hence A(r-x) = A(0) = A(r) implies that A(r-x) = A(r) for all $r \in R$. Consequently, $A_x^* = A_0^*$ where $x \in R$.

The converse is obvious .

PROPOSITION 4.10 [19]: Let $a \in R$, then $a \in JR(R)$ if and only if e-ra is invertible for all $r \in R$, where JR(R) is the Jacobson radical of M .

DEFINITION 4.11[8]: Let A be fuzzy submodule of an R-module M. A is called **fuzzy semiprime** if $A(x^n) = A(x)$, for all $x \in M$, for all $n \in N$.

THEOREM 4.12 : Intersection of fuzzy maximal submodule of R-module M (FJR(M)) is fuzzy semiprime submodule of R-module M.

Proof: By definition (4.11) and definition (4.1).

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5. External Direct Sum of Maximal Fuzzy Modules

In this section, we give the external direct sum of fuzzy modules and we shall study sum properties about it .

DEFINITION 5.1[10]: Let X be a fuzzy module of M_1 and Y be a fuzzy module of M_2 . Let T : $M_1 \oplus M_2 \to [0,1]$ definite by $T(a,b) = \min\{X(a),Y(b)\}$ for all $(a,b) \in M_1 \oplus M_2$ .T is called **a fuzzy external direct sum of denoted by $X \oplus Y$.**

PROPOSITION 5.2 [10]: If X and Y are fuzzy modules of a fuzzy module M_1 and M_2 respectively , then $T = X \oplus Y$ is a fuzzy module of $M_1 \oplus M_2$.

PROPOSITION 5.3: If X and Y are fuzzy submodules of a fuzzy module M_1 and M_2 respectively. Let A be a fuzzy submodule of X such that $A \subseteq X$ and B be a fuzzy submodule of Y such that $B \subseteq Y$, then $A \oplus B$ is a fuzzy submodule of $M_1 \oplus M_2$ such that $(A \oplus B) \subseteq (X \oplus Y) = T$, where $(A \oplus B)(a,b) = \min\{A(a),B(b)\}$, for all $(a,b) \in M_1 \oplus M_2$.

Proof: Since A and B are fuzzy submodules of X and Y respectively . To prove $(A \oplus B)$ is a fuzzy submodule of $X \oplus Y$.
$(A \oplus B)(0,0) = \min\{A(0), B(0)\} = 1$, then $(A \oplus B)$ is a nonempty fuzzy subset of $M_1 \oplus M_2$ …(1) .

Let $(a_1, b_1), (a_2, b_2) \in M_1 \oplus M_2$,

$$\begin{aligned}(A\oplus B)[(a_1,b_1)-(a_2,b_2)] &= (A\oplus B)(a_1-a_2, b_1-b_2)\\ &= \min\{A(a_1-a_2), B(b_1-b_2)\}\\ &\geq \min\{\min\{A(a_1),A(a_2)\}, \min\{B(b_1),B(b_2)\}\}\\ &= \min\{\min\{A(a_1),B(b_1)\}, \min\{A(a_2),B(b_2)\}\}, \text{ by [13] .}\\ &= \min\{(A\oplus B)(a_1,b_1), (A\oplus B)(a_2,b_2)\} \quad \ldots (2) .\end{aligned}$$

$$\begin{aligned}(A\oplus B)[r(a_1,b_1)] &= (A\oplus B)(ra_1, rb_1)\\ &= \min\{A(ra_1), B(rb_1)\}\\ &\geq \min\{A(a_1), B(b_1)\}\\ &= \min\{(A\oplus B)(a_1,b_1)\} \quad \ldots (3) .\end{aligned}$$

From (1) , (2) and (3) , then $(A \oplus B)$ is a fuzzy submodule of $M_1 \oplus M_2$.

$$\begin{aligned}\text{But } (A\oplus B)(a_1,b_1) &= \min\{A(a_1), B(b_1)\}\\ &\geq \min\{X(a_1), Y(b_1)\}\\ &= X\oplus Y(a_1,b_1) .\\ &= T(a_1,b_1) .\end{aligned}$$

Hence $(A \oplus B)$ is a fuzzy submodule of $(X \oplus Y) = T$.

PROPOSITION 5.4: If X and Y are fuzzy submodules of a fuzzy module M_1 and M_2 respectively. Let D is a fuzzy submodule of $M_1 \oplus M_2$ such that $D \subseteq (X \oplus Y) = T$, then there exists A be a fuzzy submodule of X such that $A \subseteq X$ and B be a fuzzy submodule of Y such that $B \subseteq Y$and $D = (A\oplus B)$.

Proof: Let D be a fuzzy submodule of $M_1 \oplus M_2$, then D_t is an submodule of $M_1 \oplus M_2$, for all $t \in [0, A(0,0)]$. Thus there exists I and J of M_1 and M_2 respectively such that $D_t = I \oplus J$.

Let A: $M_1 \to [0, 1]$ such that : $A(a) = \begin{cases} A(a,0) & if\, a \in I \\ 0 & otherwise \end{cases}$ and B: $M_2 \to [0, 1]$ such that : $B(b) = \begin{cases} B(0,b) & if\, b \in J \\ 0 & otherwise \end{cases}$.

It follows that A and B are fuzzy submodules ofM_1 and M_2 respectively such that $A \subseteq X$ and $B \subseteq Y$.
To prove $D = A \oplus B$, it is enough to prove that $D_t = (A \oplus B)_t = A_t \oplus B_t$, by lemma (2.4.1.10) in [13] .
It is clear that $A(0) = B(0) = D(0, 0)$.
Note that : $A_t = \{a \in M_1 \mid A(a) \geq t\}$, $t \in [0, A(0)]$.
Let $a \in A_t$, $t > 0$. Therefore, $A(a) \geq t > 0$ which implies $A(a)= D(a, 0)$. Thus $a \in I$, so that $A_t \subseteq I$ …. (1).
Let $a \in I$, $A(a) = D(a, 0)$, But $a \in I$ implies that $(a, 0) \in I \oplus J = A_t$. Thus $D(a, 0) \geq t > 0$. Hence, $A(a) \geq t$, so $a \in A_t$ and $I \subseteq A_t$, $t > 0$ …. (2) .
Form (1) and (2), we get that $I = A_t$, $t > 0$.
Similarly, $J = B_t$, $t > 0$. Thus$I \oplus J = A_t \oplus B_t = D_t$, for all $t \in [0, A(0,0)]$. Hence by remark (1.3,11) in [13], $D = (A\oplus B)$.

PROPOSITION 5.5: If A is a fuzzy submodules of an R-module M_1 and B is a fuzzy submodules of an R-module M_2 . Then $(A \oplus B)_t = A_t \oplus B_t$, for all $t \in [0,1]$.

Proof: Let $(a, b) \in M_1 \oplus M_2$ such that $(a, b) \in (A \oplus B)_t$ implies that $(A \oplus B)(a, b) \geq t$, thus $\min\{A(a), B(b)\} \geq t$, so $A(a) \geq t$ and $B(b) \geq t$. Therefore, $a \in A_t$ and $b \in B_t$. Hence $(a, b) \in A_t \oplus B_t$, for all $t \in [0,1]$.

Conversely , let $(a, b) \in M_1 \oplus M_2$ such that $(a, b) \in A_t \oplus B_t$ implies that $a \in A_t$ and $b \in B_t$, thus $A(a) \geq t$ and $B(b) \geq t$, so $\min\{A(a), B(b)\} \geq t$. Thus $(A \oplus B)(a, b) \geq t$ and $(a, b) \in (A \oplus B)_t$, for all $t \in [0,1]$. Hence $(A \oplus B)_t = A_t \oplus B_t$, for all $t \in [0,1]$.

PROPOSITION 5.6: If A and B are fuzzy submodules of an R-module M , then for each $t \in [0,1]$, A = B if and only if, $A_t = B_t$, for all $t \in [0,1]$.

Proof: It is obvious as proposition (2.3(5)) .

PROPOSITION 5.7: If X and Y are fuzzy modules of a fuzzy module M_1 and M_2 respectively . Let A be a fuzzy submodule of X such that $A \subseteq X$ and B be a fuzzy submodule of Y such that $B \subseteq Y$. If Im $(A \oplus B)= \{1,t\}$ for some $t \in [0,1)$if and only if Im (A)= {1,t} for some $t \in [0,1)$ and Im (B)= {1,t} for some $t \in [0,1)$.

Proof: Since A and B are fuzzy submodules ofM_1 and M_2 respectively such that $A \subseteq X$ and $B \subseteq Y$. Thenthere exists $(A \oplus B)$ be a fuzzy submodule of $M_1 \oplus M_2$implies that $(A \oplus B)_t$ is an submodule of $M_1 \oplus M_2$, for all $t \in [0, A\oplus B (0,0)]$. Thus there exists I and J of M_1 and M_2 respectively such that $(A \oplus B)_t = I \oplus J$,

Let A: $M_1 \rightarrow [0, 1]$ such that : $A(a) = \begin{cases} A(a,0) & if a \in I \\ t & otherwise \end{cases}$, for some $t \in [0,1)$.

and B: $M_2 \rightarrow [0, 1]$ such that : $B(b) = \begin{cases} B(0,b) & if b \in J \\ t & otherwise \end{cases}$, for some $t \in [0,1)$.

A and B are fuzzy submodules ofM_1 and M_2 respectively such that $A \subseteq X$ and $B \subseteq Y$ by proposition (5.4) .

Since A is a fuzzy submodule, then A(0) = 1and $0 \in I$, thus $A(a) = \begin{cases} 1 & if a \in I \\ t & otherwise \end{cases}$.

Hence Im (A)= {1,t} for some $t \in [0,1)$ and similarly Im (B)= {1,t}for some $t \in [0,1)$.

Conversely, since A and B are fuzzy submodules of X and Y respectively , then $(A \oplus B)$ is a fuzzy submodule of $X \oplus Y$ by proposition (5.3) . $(A\oplus B)(0,0) = \min\{A(0), B(0)\}= 1$. Let $(a_1 , b_1) \in M_1 \oplus M_2$, $(A\oplus B)(a_1 , b_1) = \min\{A(a_1), B(b_1)\}= t$, for some $t \in [0,1)$ .Hence Im $(A \oplus B)= \{1,t\}$ for some $t \in [0,1)$.

PROPOSITION 5.8: If X and Y are fuzzy modules of a fuzzy module M_1 and M_2 respectively . Let A be a fuzzy submodule of X such that $A \subseteq X$ and B be a fuzzy submodule of Y such that $B \subseteq Y$. $A \oplus B$ is a fuzzy maximal submodule of $X \oplus Y$if and only if A and B are fuzzy maximal submodules.

Proof: Since $(A \oplus B)$ is a fuzzy submodule of $X \oplus Y$, then A and B are fuzzy submodules of X and Y respectively by proposition (5.4). To prove A and B are fuzzy maximalsubmodules of X and Y respectively, since $(A \oplus B)$ is a fuzzy maximal submodule of $X \oplus Y$, implies that $(A \oplus B)_1$ is a maximal submodule of $M_1 \oplus M_2$, and Im $(A \oplus B)= \{1,t\}$ for some $t \in [0,1)$, by proposition (3.3). But $(A \oplus B)_1 = A_1\oplus B_1$ by proposition (5.5), then $A_1\oplus B_1$ is a maximal submodule of $M_1 \oplus M_2$. And Im (A)= {1,t} for some $t \in [0,1)$ and Im (B)= {1,t} for some $t \in [0,1)$ by proposition (5.7) . Hence A and B are maximal fuzzy modules of X and Y respectively by proposition (3.3) .

Conversely, SinceA and B are fuzzy submodules of X and Y respectively , then $(A \oplus B)$ is a fuzzy submodule of $X \oplus Y$ by proposition (5.3). To prove $(A \oplus B)$ is a fuzzy maximal submodule of $X \oplus Y$, sinceA and B are fuzzy maximal submodules of X and Y respectively , implies that $A_1$and $B_1$are maximal submodules of $M_1$and M_2 respectively, and Im (A)= {1,t} for some $t \in [0,1)$ and Im (B)= {1,t} for some $t \in [0,1)$by proposition (3.3) .

But $A_1\oplus B_1$ is a maximal submodule of $M_1 \oplus M_2$ and $(A \oplus B)_1 = A_1\oplus B_1$ by proposition (5.5) , then $(A \oplus B)_1$ is a maximal submodule of $M_1 \oplus M_2$. And Im $(A \oplus B)= \{1,t\}$ for some $t \in [0,1)$, by proposition (5.7). Hence $A \oplus B$ is a maximal fuzzy module of $X \oplus Y$, by proposition (3.3) .

6. Factorization of Fuzzy Submodules:

In this section, we give the fuzzy fractionary of the fuzzy submodule of an R-module M. We study some theorems and properties of it with fuzzy maximal submodules.

PROPOSITION 6.1: Let $B_1, B_2, \dots, B_n$ and A be fuzzy submodules of R-modules M such that $A = B_1, B_2, \dots, B_n$ and $B_1(0) = B_2(0) = \dots B_n(0)$, then $M(0) = B_i(0)$, i = 1, .. , n .If $B_1, B_2, \dots, B_n$ are finite valued , then $A_t = (B_1)_t, (B_2)_t, \dots, (B_n)_t$,for all $t \in [0, 1]$.

Proof: $A(0) = \sup\{ \min \{\min \{B_i(x_{ij}) \mid i= 1, \dots , n\} \mid 0 = \sum_{j=1}^{n} x_{1j}, \dots , x_{nj} ,\ x_{nj}\in M, i= 1 , \dots , n, j= 1, \dots , m; m\in N\}\}$
$= \min \{ B_i(0) \mid i= 1 , \dots , n\}= B_i(0)$, i= 1, … , n.

Let $t \in [0, A(0)]$, then $x \in A_t$if and only if $\sup\{ \min \{\min \{B_i(x_{ij}) \mid i= 1, \dots , n\} \mid x = \sum_{j=1}^{n} x_{1j}, \dots , x_{nj} ,\ x_{nj}\in M, i= 1 , \dots , n, j= 1, \dots , m; m\in N\} \geq t$ since for some $m \in N$, $x_{ij}\in B_i(t)$, i= 1,… , n; j=1 , .. , m such that $x = \sum_{j=1}^{n} x_{1j}, \dots , x_{nj}$ if and only if, $x \in (B_1)_t, (B_2)_t, \dots, (B_n)_t$.

The implication becomes an equivalence since the Bi are finite valued. Thus$A_t = (B_1)_t, (B_2)_t, \dots, (B_n)_t$,for all $t \in [0, A(0)]$.

PROPOSITION 6.2[14]: Let A, B and C be fuzzy submodules of an R-module M such that A is nonzero and (A:B) = (A: C), then B=C .

PROPOSITION 6.3[14]: Let A and B be two fuzzy submodules of an R-module M such that $B \subseteq A$, then there exists a fuzzy submodule C of R-module M such that B = (A: C).

PROPOSITION 6.4[14]: Let A, B and C be finite valued fuzzy submodules of an R-module M and for all $t \in [0, 1]$, $A_t \neq <0>$. If (A:B) = (A: C), then B=C.

PROPOSITION 6.5: Let A be fuzzy submodules of R-modules M such that Im (A) = $\{t_0, t_1, \dots , t_n\}$, where $t_0 < t_1 < \dots < t_n$, then there exists submodules B and C ofan R-module M such that A = (B: C), C is fuzzy maximal submodule of an R-module M with t_0 = Im (C) , Im (B) = $\{t_0, t_1, \dots , t_n\}$, if A_{t1} is not a fuzzy maximal submodule of an R-module M and Im (B) = $\{, t_1, \dots , t_n\}$, if A_{t1} is a fuzzy maximal submodule of an R-module M.

Proof: Let $A(0) = \sup\{ \min \{\min \{B_i(x_{ij}) \mid {}_{i}= 1, \dots , n\} \mid 0 = \sum_{j=1}^{n} x_{1j}, \dots , x_{nj} ,\ x_{nj} \in M, i= 1 , \dots , n, j= 1, \dots , m; m \in N\}\}$ $= \min \{ B_i(0) \mid i= 1 , \dots , n\}= B_i(0) = 1 , i= 1, \dots , n.$
Let $t \in [0, 1]$, then $x \in A_t$ if and only if $\sup\{ \min \{\min \{B_i(x_{ij}) \mid {}_{i}= 1, \dots , n\} \mid x = \sum_{j=1}^{n} x_{1j}, \dots , x_{nj} ,\ x_{nj} \in M, i= 1 , \dots , n, j= 1, \dots , m; m \in N\} \geq t$ since for some $m \in N$, $x_{ij} \in B_i(t)$, $i= 1,\dots , n; j=1 , .. , m$ such that $x = \sum_{j=1}^{n} x_{1j}, \dots , x_{nj}$ if and only if, $x \in (B_1)_t, (B_2)_t, \dots, (B_n)_t$. The implication becomes an equivalence since the Bi are finite valued. Thus$A_t = (B_1)_t, (B_2)_t, \dots, (B_n)_t$,for all $t \in [0, 1]$.

7. References

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