International Journal of Academic and Applied Research (IJAAR)
ISSN: 2643-9603
Vol. 4, Issue 6, June – 2020, Pages: 22-31

Double Sequence Space of Fuzzy Real Numbers of Paranormed Type Defined By Double Orlicz Functions

Alaa Fawzi Dabbas* and Ali Hussein Battor

Department of Mathematics, Faculty of Education for Girls University of Kufa, Najaf – Iraq
E-mail: lionwhite812@gmail.com*

ABSTRACT: *Through this paper, we are using dissimilar features of convergent, null and bounded double sequence space of fuzzy real numbers defined by a double Orlicz function, we search some of their features such as completeness, solidness, symmetricity, etc.*

Keywords: Solid space, symmetric space, fuzzy real numbers, completeness.

1. INTRODUCTION

The concept of fuzzyness is widely implementation in many branches of Engineering and Technology. In our study, a double sequence is denoted by $(\mathfrak{X}_{s,r}, \mathfrak{M}_{s,r})$,(a double infinite array of elements $(\mathfrak{X}_{s,r}), (\mathfrak{M}_{s,r})$, where each $(\mathfrak{X}_{s,r}), (\mathfrak{M}_{s,r})$ is a fuzzy real numbers).

The primary studies on double sequences may be found in Bromwich [**Bro 3**]. Thereafter, it was searched by Hardy [**Har 4**], Moricz [**Mor 5**], Basarir and Sonalcan [**Bas** 6], Sarma [**Sar 7**], Tripathy and Sarma [**Tri,Sar 8**] and other scholars. In [**Har 4**], Hardy studied the opinion of regular convergence for double sequence.

The sense of paranormed double sequences was presented by Nakano [**Nak 9**] , Simmons **[Sim 10]** at the first stage, after than, many other writers introduced this topic.

Neamah and Hasan [**Nea1,Has2**] there refers to a double Orlicz function is a function $\Upsilon : [0,\infty) \times [0,\infty) \to [0,\infty) \times [0,\infty)$ such that $\Upsilon(\mathfrak{X},\mathfrak{M}) = (\Upsilon_1(\mathfrak{X}), \Upsilon_2(\mathfrak{M}))$, where $\Upsilon, \Upsilon_1, \Upsilon_2$ are Orlicz functions and

$\Upsilon_1 : [0,\infty) \to [0,\infty)$ and $\Upsilon_2 : [0,\infty) \to [0,\infty)$,

this functions are continuous , non- decreasing ,even, convex and satisfy the following conditions

i) $\Upsilon_1(0) = 0,\quad \Upsilon_2(0) = 0 \Longrightarrow \Upsilon(\mathfrak{X},\mathfrak{M}) = (\Upsilon_1(0), \Upsilon_2(0)) = (0,0)$,

ii) $\Upsilon_1(\mathfrak{X}) > 0, \Upsilon_2(\mathfrak{M}) > 0 \Longrightarrow \Upsilon(\mathfrak{X},\mathfrak{M}) = (\Upsilon_1(\mathfrak{X}), \Upsilon_2(\mathfrak{M})) > (0,0)$,

for $\mathfrak{X} > 0, \mathfrak{M} > 0$, we mean by $(\mathfrak{X},\mathfrak{M}) = (\mathfrak{X}_{s,r}, \mathfrak{M}_{s,r}) > (0,0)$, that $\Upsilon_1(\mathfrak{X}) > 0,\ \Upsilon_2(\mathfrak{M}) > 0$,

iii) $\Upsilon_1(\mathfrak{X}) \to \infty, \Upsilon_2(\mathfrak{M}) \to \infty$ as $\mathfrak{X},\mathfrak{M} \to \infty$, then

$\Upsilon(\mathfrak{X},\mathfrak{M}) = (\Upsilon_1(\mathfrak{X}), \Upsilon_2(\mathfrak{M})) \Longrightarrow (\infty,\infty)$ as $(\mathfrak{X},\mathfrak{M}) \longrightarrow (\infty,\infty)$, we mean by

$\Upsilon(\mathfrak{X},\mathfrak{M}) \longrightarrow (\infty,\infty)$, that $\Upsilon_1(\mathfrak{X}) \to \infty,\quad \Upsilon_2(\mathfrak{M}) \to \infty$.

We refer to the set of all closed and bounded intervals $X = [x_1, x_2]$ on the real line $\mathbb{R}$ by symbol $\mathcal{D}$.

For $X = [x_1, x_2] \in \mathcal{D}$ and $Y = [y_1, y_2] \in \mathcal{D}$ and $Z = [z_1, z_2] \in \mathcal{D}$ and $W = [w_1, w_2] \in \mathcal{D}$, defined

$$d((X,Y),(Z,W)) = \max[(|x_1 - y_1|, |x_2 - y_2|), (|z_1 - w_2|, |z_1 - w_2|)].$$

It is recognized that $(\mathcal{D}, d)$ is a complete metric space **.**

The following information is taken from [**Sar11**]

A fuzzy number $\mathcal{H}$ is a fuzzy set on the real axis, i.e., a mapping $\mathcal{H} : \mathbb{R} \to \not{f}(= [0,1])$ associating each real number v with its membership rank $\mathcal{H}(v)$, satisfies the following conditions :

1) The mapping $\mathcal{H}$ is convex if $\mathcal{H}(v) \geq \mathcal{H}(s) \wedge \mathcal{H}(a) = min\{\mathcal{H}(s), \mathcal{H}(a)\}$, where s $< v < a$.

2) The mapping $\mathcal{H}$ is normal if there exists $v_0 \in \mathbb{R}$ such that $\mathcal{H}(v_0) = 1$,

International Journal of Academic and Applied Research (IJAAR)
ISSN: 2643-9603
Vol. 4, Issue 6, June – 2020, Pages: 22-31

3) The mapping $\mathcal{H}$ is upper-semi continuous if, in the regular topology of $\mathbb{R}$, $\mathcal{H}^{-1}([0,c+\epsilon))$ is open, for all $\in \not{f}, \epsilon > 0$.

4) The closure of $\{v \in \mathbb{R} : \mathcal{H}(v) > 0\}$, denoted by $[\mathcal{H}]^0$, is compact .

5) The mapping $\mathcal{H}$ is called non-negative if $\mathcal{H}(v) = 0$, for all $v < 0$. The set of all non-negative fuzzy real numbers is denoted by $\mathbb{R}^*(\not{f})$.

For $0 < \propto \leq 1$, The α-level set $[\mathcal{H}]^\propto$, of the fuzzy real number $\mathcal{H}$, defined by

$$[\mathcal{H}]^\propto = \{v \in \mathbb{R} : \mathcal{H}(v) \geq \alpha\}.$$

The set of all upper-semi-continuous, natural , convex fuzzy real numbers is denoted by $\mathbb{R}(\not{f})$ and we say the number belongs to $\mathbb{R}(\not{f})$ throughout the paper, by a fuzzy real number .

Let $\bar{d}: \mathbb{R}^2(\not{f}) \times \mathbb{R}^2(\not{f}) \to \mathbb{R}$ be defined by

$$\bar{d}\big((X,Y),(Z,W)\big) = sup_{0\leq\propto\leq1} d\big(([X]^\propto,[Y]^\propto),([Z]^\propto,[W]^\propto)\big),$$

Then, $\bar{d}$ defines a metric on $\mathbb{R}^2(\not{f})$.

2. DEFINITION AND PRELIMINARES

Let $\mathfrak{X} = (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}})$, $\mathfrak{B} = (\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})$ be double sequence. A double sequences $(\mathfrak{X},\mathfrak{M}) = \big(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\big)$ of fuzzy real numbers is said be a convergent in prin- gsheim's sense to a fuzzy real numbers (ℓ_1,ℓ_2), if for every $\epsilon > 0$, there exists $\mathfrak{s}_0 = \mathfrak{s}_0(\epsilon), \mathfrak{r}_0 = \mathfrak{r}_0(\epsilon) \in \mathbb{N}$ such that $\bar{d}\left(\left(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\ell_1\right),\left(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}},\ell_2\right)\right) < \epsilon$ for all $\mathfrak{s} \geq \mathfrak{s}_0$, $\mathfrak{r} \geq \mathfrak{r}_0$, where $\bar{d}\big(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\ell_1\big) < \epsilon$ and $\bar{d}\big(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}},\ell_2\big) < \epsilon$. A double sequence $(\mathfrak{X},\mathfrak{M}) = \big(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\big)$ is said to be regularly converge if it converges in the pringsheim's sense and the below limits will be occur :

$$\lim_{\mathfrak{s}\to\infty} \bar{d}\big(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\ell_\mathfrak{r}\big) = 0, \qquad (\mathfrak{r} = 1,2,3,\ldots.),$$

$$\lim_{\mathfrak{s}\to\infty} \bar{d}\big(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}},s_\mathfrak{r}\big) = 0, \qquad (\mathfrak{r} = 1,2,3,\ldots.),$$

and

$$\lim_{\mathfrak{r}\to\infty} \bar{d}\big(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{b}_\mathfrak{s}\big) = 0, \qquad (\mathfrak{s} = 1,2,3,\ldots.),$$

$$\lim_{\mathfrak{r}\to\infty} \bar{d}\big(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}},i_\mathfrak{s}\big) = 0, \qquad (\mathfrak{s} = 1,2,3,\ldots.),$$

therefore

$$\lim_{\mathfrak{s}\to\infty} \bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) = (\ell_\mathfrak{r},s_\mathfrak{r}),$$

$$\lim_{\mathfrak{r}\to\infty} \bar{d}\,(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) = (\mathfrak{b}_\mathfrak{s},i_\mathfrak{s}).$$

A double sequence $(\mathfrak{X},\mathfrak{M}) = \big(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}\,,\,\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\big)$ of fuzzy real numbers is said to be bounded, if $sup_{\mathfrak{s},\mathfrak{r}}\,\bar{d}\left(\left(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}\,,\ell_1\right),\left(\,\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\,,\ell_2\right)\right) < \infty$ for $\mathfrak{s},\mathfrak{r} \in \mathbb{N}$ where $sup_{\mathfrak{s},\mathfrak{r}}\,\bar{d}\left(\left(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}\,,\ell_1\right)\right) < \infty$ and $sup_{\mathfrak{s},\mathfrak{r}}\,\bar{d}\left(\left(\,\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\,,\ell_2\right)\right) < \infty$.

Throughout the paper $(l_\infty)_F^{\parallel}, (c)_F^{\parallel}, (c_0)_F^{\parallel}, (c^R)_F^{\parallel}, (c_0^R)_F^{\parallel}$ signify the of all bounded, convergent in pringsheim's sense, null in pringsheim's sense, regularly convergent , regularly null convergent in pringsheim's sense of fuzzy real numbers respectively.

If $\big(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\big) \in E_F^{\parallel}$ where $\big(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}},\mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}\big) \in E_F^{\parallel}$ and $\big|\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\big| \leq \big|\mathfrak{B}_{\mathfrak{s},\mathfrak{r}},\mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}\big|$ where $\big|\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}\big| \leq \big|\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}\big|$ and $\big|\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\big| \leq \big|\mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}\big|$, for all $\mathfrak{s},\mathfrak{r} \in \mathbb{N}$, a double sequence space $E_F^{\parallel}$ is said to be solid.

If$\big(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\big) \in E_F^{\parallel}$, implies $\big(\mathfrak{X}_{\pi(\mathfrak{s})\pi(\mathfrak{r})},\mathfrak{M}_{\pi(\mathfrak{s})\pi(\mathfrak{r})}\big) \in E_F^{\parallel}$, where π is a permutation of $\mathbb{N}$, a double sequence space $E_F^{\parallel}$ is said to be symmetrical.

Let $K = \{(\mathfrak{s}_i,\mathfrak{r}_i) : i \in \mathcal{N}; \mathfrak{s}_1 < \mathfrak{s}_2 < \mathfrak{s}_3 < \cdots$ and $\mathfrak{r}_1 < \mathfrak{r}_2 < \mathfrak{r}_3 < \cdots\} \subseteq \mathbb{N} \times \mathbb{N}$ and $E_F^{\parallel}$ be a fuzzy double sequence space. A K-step space of $E_F^{\parallel}$ is a double sequence space $\gamma_K^E = \big\{\big(\mathfrak{X}_{\mathfrak{s}_i,\mathfrak{r}_i}\big) \in w_F^{\parallel} : \big(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}\big) \in E_F^{\parallel}\big\}$.

A canonical pre-image of a double sequence $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in E_F^{\|}$ is a double sequence $(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}) \in E_F^{\|}$ defined as follows:

$$(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}) = \begin{cases} (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}), & \text{If } (\mathfrak{s},\mathfrak{r}) \in K, \\ (\bar{0}, \bar{0}), & \text{otherwise.} \end{cases}$$

According to **[Sar 11]** we could say a , a canonical pre-image of a step space γ_K^E is a compilation of canonical pre-images of all elements in γ_K^E.

If it includes the canonical pre-images of all its step spaces, a double sequence space $E_F^{\|}$ is said to be monotonous.

Remark: A double sequence space $E_F^{\|}$ is solid $\Longrightarrow E_F^{\|}$ is monotone [**Mas 1**].

In this paper we defined the following classes of double sequences of fuzzy real numbers**:**

Let $\wp = (\wp_{\mathfrak{s},\mathfrak{r}})$ be a sequence of strictly positive real numbers.

$(l_\infty)_F^{\|}(\Upsilon, \wp)$=

$$\left\{ (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in \mathcal{W}_F^{\|} : \sup_{\mathfrak{s},\mathfrak{r}} \left\{ \left(\Upsilon_1 \left(\frac{\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \bar{0})}{\rho} \right) \right)^{\wp_{\mathfrak{s},\mathfrak{r}}} \vee \left(\Upsilon_2 \left(\frac{\bar{d}(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}, \bar{0})}{\rho} \right) \right)^{\wp_{\mathfrak{s},\mathfrak{r}}} \right\} < \infty \atop \text{for some } \rho > 0, \right\}$$

where

$$(l_\infty)_F^{\|}(\Upsilon_1, \wp) = \left\{ (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}) \in \mathcal{W}_F^{\|} : sup_{k,l} \left(\Upsilon_1 \left(\frac{\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \bar{0})}{\rho} \right) \right)^{\wp_{\mathfrak{s},\mathfrak{r}}} < \infty \atop \text{for some } \rho > 0, \right\},$$

and

$$(l_\infty)_F^{\|}(\Upsilon_2, \wp) = \left\{ (\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in \mathcal{W}_F^{\|} : sup_{\mathfrak{s},\mathfrak{r}} \left(\Upsilon_2 \left(\frac{\bar{d}(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}, \bar{0})}{\rho} \right) \right)^{\wp_{\mathfrak{s},\mathfrak{r}}} < \infty \atop \text{for some } \rho > 0, \right\},$$

$(c)_F^{\|}(\Upsilon, \wp)$=

$$\left\{ (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in \mathcal{W}_F^{\|} : \lim_{\mathfrak{s},\mathfrak{r}} \left\{ \left(\Upsilon_1 \left(\frac{\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \ell_1)}{\rho} \right) \right)^{\wp_{\mathfrak{s},\mathfrak{r}}} \vee \left(\Upsilon_2 \left(\frac{\bar{d}(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}, \ell_2)}{\rho} \right) \right)^{\wp_{\mathfrak{s},\mathfrak{r}}} \right\} = 0 \atop \text{for some } \rho > 0, \right\},$$

where

$$(c)_F^{\|}(\Upsilon_1, \wp) = \left\{ (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}) \in \mathcal{W}_F^{\|} : lim_{\mathfrak{s},\mathfrak{r}} \left(\Upsilon_1 \left(\frac{\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \ell_1)}{\rho} \right) \right)^{\wp_{\mathfrak{s},\mathfrak{r}}} = 0 \atop \text{for some } \rho > 0, \right\},$$

and

International Journal of Academic and Applied Research (IJAAR)
ISSN: 2643-9603
Vol. 4, Issue 6, June – 2020, Pages: 22-31

$$(c)_F^{\|}(\Upsilon_2,\wp) = \left\{ \begin{matrix} (\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in \mathcal{W}_F^{\|}: lim_{\mathfrak{s},\mathfrak{r}} \left(\Upsilon_2 \left(\frac{\bar{d}(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}, \ell_2)}{\rho} \right) \right)^{\wp_{\mathfrak{s},\mathfrak{r}}} = 0 \\ \text{for some } \rho > 0 \end{matrix} \right\}.$$

For $(\ell_1,\ell_2) = (\bar{0},\bar{0})$ we get the class $(c_0)_F^{\|}(\Upsilon,\wp)$.

i.e.,$(c_0)_F^{\|}(\Upsilon,\wp)$=

$$\left\{ \begin{matrix} (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in \mathcal{W}_F^{\|}: \lim\limits_{\mathfrak{s},\mathfrak{r}} \left\{ \left(\Upsilon_1 \left(\frac{\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \bar{0})}{\rho} \right) \right)^{\wp_{\mathfrak{s},\mathfrak{r}}} \vee \left(\Upsilon_2 \left(\frac{\bar{d}(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}, \bar{0})}{\rho} \right) \right)^{\wp_{\mathfrak{s},\mathfrak{r}}} \right\} = 0 \\ \text{for some } \rho > 0, \end{matrix} \right\},$$

where

$$(c_0)_F^{\|}(\Upsilon_1,\wp) = \left\{ \begin{matrix} (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}) \in \mathcal{W}_F^{\|}: lim_{\mathfrak{s},\mathfrak{r}} \left(\Upsilon_1 \left(\frac{\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \bar{0})}{\rho} \right) \right)^{\wp_{\mathfrak{s},\mathfrak{r}}} = 0 \\ \text{for some } \rho > 0, \end{matrix} \right\},$$

and

$$(c_0)_F^{\|}(\Upsilon_2,\wp) = \left\{ \begin{matrix} (\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in \mathcal{W}_F^{\|}: lim_{\mathfrak{s},\mathfrak{r}} \left(\Upsilon_2 \left(\frac{\bar{d}(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}, \bar{0})}{\rho} \right) \right)^{\wp_{\mathfrak{s},\mathfrak{r}}} = 0 \\ \text{for some } \rho > 0, \end{matrix} \right\}.$$

Also, a fuzzy double sequences $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in (c^R)_F^{\|}(\Upsilon,\wp)$. If $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in (c)_F^{\|}(\Upsilon,p)$ and the next limits exist **:**

$$\lim_{\mathfrak{s}} \left\{ \Upsilon \left(\frac{\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X}_{\mathfrak{r}})}{\rho} \right) \right\}^{\wp_{\mathfrak{s},\mathfrak{r}}} = 0, as\ \mathfrak{s} \to \infty, \forall\, \mathfrak{r} \in \mathbb{N},$$

$$\lim_{\mathfrak{r}} \left\{ \Upsilon \left(\frac{\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, s_{\mathfrak{s}})}{\rho} \right) \right\}^{\wp_{\mathfrak{s},\mathfrak{r}}} = 0, as\ \mathfrak{r} \to \infty, \forall\, \mathfrak{s} \in \mathbb{N},$$

$$\lim_{\mathfrak{s}} \left\{ \Upsilon \left(\frac{\bar{d}(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{P}_{\mathfrak{r}})}{\rho} \right) \right\}^{\wp_{\mathfrak{s},\mathfrak{r}}} = 0, as\ \mathfrak{s} \to \infty, \forall\, \mathfrak{r} \in \mathbb{N},$$

$$\lim_{\mathfrak{r}} \left\{ \Upsilon \left(\frac{\bar{d}(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}, r_{\mathfrak{s}})}{\rho} \right) \right\}^{\wp_{\mathfrak{s},\mathfrak{r}}} = 0, as\ \mathfrak{r} \to \infty\ \forall\, \mathfrak{s} \in \mathbb{N},$$

therefore

$$\lim_{\mathfrak{s}} \left\{ \Upsilon \left(\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \right) \right\}^{\wp_{\mathfrak{s},\mathfrak{r}}} = (0,0),\ \ as\ \mathfrak{s} \to \infty, \text{for each } \mathfrak{r} \in \mathbb{N},$$

$$\lim_{\mathfrak{r}} \left\{ \Upsilon \left(\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \right) \right\}^{\wp_{\mathfrak{s},\mathfrak{r}}} = (0,0),\ \ as\ \mathfrak{r} \to \infty, \text{for each } \mathfrak{s} \in \mathbb{N}.$$

International Journal of Academic and Applied Research (IJAAR)
ISSN: 2643-9603
Vol. 4, Issue 6, June – 2020, Pages: 22-31

A double sequence $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in (c_0^R)_F^{\|}(\Upsilon)$, if

$$\mathfrak{X} = \mathfrak{X}_{\mathfrak{r}} = s_{\mathfrak{s}} = \bar{0}, \quad \forall\ \mathfrak{s},\mathfrak{r} \in \mathbb{N}$$

and

$$\mathfrak{B} = \mathfrak{P}_{\mathfrak{r}} = r_{\mathfrak{s}} = \bar{0}, \quad \forall\ \mathfrak{s},\mathfrak{r} \in \mathbb{N}.$$

We define

$$(m)_F^{\|}(\Upsilon, \wp) = (c)_F^{\|}(\Upsilon, \wp) \cap (l_\infty)_F^{\|}(\Upsilon, \wp),$$

$$(m_0)_F^{\|}(\Upsilon, \wp) = (c_0)_F^{\|}(\Upsilon, \wp) \cap (l_\infty)_F^{\|}(\Upsilon, \wp).$$

3. MAIN RESULTS

Theorem 3.1: Let $(\wp_{\mathfrak{s},\mathfrak{r}})$ be bounded. Then the classes of double sequences $(l_\infty)_F^{\|}(\Upsilon, \wp)$, $(c^R)_F^{\|}(\Upsilon, \wp)$, $(c_0^R)_F^{\|}(\Upsilon, \wp)$ are complete metric spaces with respect to the distance defined by

$G((\mathfrak{X}, \mathfrak{M}), (\mathfrak{B}, \mathfrak{Q}))=$

$$inf\left\{\rho^{\frac{\wp_{\mathfrak{s},\mathfrak{r}}}{\mathcal{J}}} > 0: sup_{\mathfrak{s},\mathfrak{r}}\left\{\left(\Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}})}{\rho}\right)\right) \vee \left(\Upsilon_2\left(\frac{\bar{d}(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}})}{\rho}\right)\right)\right\} \le 1\right\},$$

$$\mathcal{J} = \max(1, 2^{T-1})$$

where

$$G(\mathfrak{X}, \mathfrak{M}) = inf\left\{\rho^{\frac{\wp_{\mathfrak{s},\mathfrak{r}}}{\mathcal{J}}} > 0: sup_{\mathfrak{s},\mathfrak{r}}\left(\Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}})}{\rho}\right)\right) \le 1\right\}$$

$$G(\mathfrak{B}, \mathfrak{Q}) = inf\left\{\rho^{\frac{\wp_{\mathfrak{s},\mathfrak{r}}}{\mathcal{J}}} > 0: sup_{\mathfrak{s},\mathfrak{r}}\left(\Upsilon_2\left(\frac{\bar{d}(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}})}{\rho}\right)\right) \le 1\right\}$$

Proof : Let us consider the case $(l_\infty)_F^{\|}(\Upsilon, \wp)$ and the other cases can be established next similar techniques..

Let $(\mathfrak{X}^i), (\mathfrak{M}^i)$ be any Cauchy sequences in $(l_\infty)_F^{\|}(\Upsilon_1, \wp)$, $(l_\infty)_F^{\|}(\Upsilon_2, \wp)$ respectively, hence $(\mathfrak{X}^i, \mathfrak{M}^i) = (\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}})$ be a double Cauchy sequence

Let $\epsilon > 0$, $\mathfrak{X}_0, r > 0$ be fixed. Then for each $\frac{\epsilon}{r\mathfrak{X}_0} > 0$, there exists a positive integer N such that $G_{\Upsilon_1}(\mathfrak{X}^i, \mathfrak{X}^j) < \frac{\epsilon}{r\mathfrak{X}_0}$ and $G_{\Upsilon_2}(\mathfrak{M}^i, \mathfrak{M}^j) < \frac{\epsilon}{r\mathfrak{X}_0}$, for $i, j \ge N$, and consequently,

$$G_\Upsilon\left((\mathfrak{X}^i, \mathfrak{X}^j), (\mathfrak{M}^i, \mathfrak{M}^j)\right) = \left(G_{\Upsilon_1}(\mathfrak{X}^i, \mathfrak{X}^j), G_{\Upsilon_2}(\mathfrak{M}^i, \mathfrak{M}^j)\right) < \frac{\epsilon}{r\mathfrak{X}_0},$$

for all $i, j \ge N$.

By definition of G, we obtain

$$inf\left\{\rho > 0: sup_{\mathfrak{s},\mathfrak{r}}\left\{\Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})}{\rho}\right) \vee \Upsilon_2\left(\frac{\bar{d}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})}{\rho}\right)\right\} \le 1\right\}$$

Thus,

International Journal of Academic and Applied Research (IJAAR)
ISSN: 2643-9603
Vol. 4, Issue 6, June – 2020, Pages: 22-31

$$sup_{\mathfrak{s},\mathfrak{r}}\left\{\Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})}{\rho}\right)\vee\Upsilon_2\left(\frac{\bar{d}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})}{\rho}\right)\right\}\leq 1$$

for all $i,j\ \geq N$.

$$\Rightarrow sup_{\mathfrak{s},\mathfrak{r}}\left\{\Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})}{G_{\Upsilon_1}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})}\right)\vee\Upsilon_2\left(\frac{\bar{d}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})}{G_{\Upsilon_2}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})}\right)\right\}\leq 1$$

for each $i,j\ \geq N$,

Since $p_{\mathfrak{s},\mathfrak{r}}$ bounded it follows that

$$\left\{\Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})}{G_{\Upsilon_1}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})}\right)\vee\Upsilon_2\left(\frac{\bar{d}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})}{G_{\Upsilon_2}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})}\right)\right\}\leq 1$$

for each $\mathfrak{s},\mathfrak{r}\geq 1$ and for all $i,j\ \geq N$.

Hence one can find $r>0$ with $\Upsilon_1\left(\frac{r\mathfrak{X}_0}{2}\right)\geq 1$ and $\Upsilon_2\left(\frac{r\mathfrak{X}_0}{2}\right)\geq 1$, such that

$$\Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})}{G_{\Upsilon_1}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})}\right)\leq\ \Upsilon_1\left(\frac{r\mathfrak{X}_0}{2}\right)\text{and }\Upsilon_2\left(\frac{\bar{d}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})}{G_{\Upsilon_2}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})}\right)\leq\Upsilon_2\left(\frac{r\mathfrak{X}_0}{2}\right)$$

Hence, $\Upsilon\left(\frac{r\mathfrak{X}_0}{2},\frac{r\mathfrak{X}_0}{2}\right)=\left(\Upsilon_1\left(\frac{r\mathfrak{X}_0}{2}\right),\Upsilon_2\left(\frac{r\mathfrak{X}_0}{2}\right)\right)\geq(1,1)$, therefore,

$$\left\{\Upsilon_1\left(\frac{\bar{d}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})}{G_{\Upsilon_1}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})}\right),\Upsilon_2\left(\frac{\bar{d}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})}{G_{\Upsilon_2}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})}\right)\right\}\leq\left(\Upsilon_1\left(\frac{r\mathfrak{X}_0}{2}\right),\Upsilon_2\left(\frac{r\mathfrak{X}_0}{2}\right)\right).$$

This implies that.

$\bar{d}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})\leq\frac{r\mathfrak{X}_0}{2}\cdot G_{\Upsilon_1}((\mathfrak{X}^i,\mathfrak{X}^j))$, for all $i,j\geq n_0$.

$$\bar{d}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})\leq\frac{r\mathfrak{X}_0}{2}\cdot\frac{\epsilon}{r\mathfrak{X}_0}=\frac{\epsilon}{2}\qquad\text{for all } i,j\ \geq n_0.$$

$$\Rightarrow\bar{d}(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}})\leq\frac{\epsilon}{2}\qquad\text{for all } i,j\ \geq n_0.$$

and

$$\bar{d}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})\leq\frac{r\mathfrak{X}_0}{2}\cdot G_{\Upsilon_2}((\mathfrak{M}^i,\mathfrak{M}^j)),\ \text{for all } i,j\ \geq n_0.$$

$$\bar{d}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})\leq\frac{r\mathfrak{X}_0}{2}\cdot\frac{\epsilon}{r\mathfrak{X}_0}=\frac{\epsilon}{2}\qquad\text{for all } i,j\geq n_0$$

$$\Rightarrow\bar{d}(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})\leq\frac{\epsilon}{2}\qquad\text{for all } i,j\ \geq n_0.\ \text{then}$$

$$\bar{d}\left((\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}}),(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}})\right)\leq\frac{r\mathfrak{X}_0}{2}.\frac{\epsilon}{r\mathfrak{X}_0}=\frac{\epsilon}{2}\quad\text{for all } i,j\ \geq n_0.$$

Hence $(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}})$ is a double Cauchy sequence in $R^2(f)$.

Thus ,

For each $(0<\epsilon<1)$, there exists a positive integer N such that $\bar{d}\left((\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{X}),(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}},\mathfrak{M})\right)<\epsilon$ for all $i,j\ \geq N$, where $\bar{d}(\mathfrak{X}^i,\mathfrak{X})<\epsilon$ and $\bar{d}\ (\mathfrak{M}^i,\mathfrak{M})<\epsilon$ for all $i,j\ \geq N$.

Taking $j \to \infty$ and fixing i, so by using the continuity of $\Upsilon = (\Upsilon_1, \Upsilon_2)$, we get

$$sup_{\mathfrak{s},\mathfrak{r}} \left\{ \Upsilon_1 \left(\frac{\bar{d}\left(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \lim_{j\to\infty} \mathfrak{X}^j_{\mathfrak{s},\mathfrak{r}}\right)}{\rho} \right) \vee \Upsilon_2 \left(\frac{\bar{d}\left(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \lim_{j\to\infty} \mathfrak{M}^j_{\mathfrak{s},\mathfrak{r}}\right)}{\rho} \right) \right\} \le 1$$

Thus,

$$sup_{\mathfrak{s},\mathfrak{r}} \left\{ \Upsilon_1 \left(\frac{\bar{d}\left(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X}\right)}{\rho} \right) \vee \Upsilon_2 \left(\frac{\bar{d}\left(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}\right)}{\rho} \right) \right\} \le 1,$$

On taking the infimum of such ρ' s, we get,

$$inf \left\{ \rho > 0 : sup_{\mathfrak{s},\mathfrak{r}} \left\{ \Upsilon_1 \left(\frac{\bar{d}\left(\mathfrak{X}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{X}\right)}{\rho} \right) \vee \Upsilon_2 \left(\frac{\bar{d}\left(\mathfrak{M}^i_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}\right)}{\rho} \right) \right\} \le 1 \right\} \le \epsilon$$

for all $i \ge N$ and $j \to \infty$.

Since $(\mathfrak{X}^i, \mathfrak{M}^i) \in (l_\infty)^{\Vert}_F(\Upsilon, \wp)$ and Υ is continuous, it follows that $(\mathfrak{X}, \mathfrak{M}) \in (l_\infty)^{\Vert}_F(\Upsilon, \wp)$.

Theorem 3. 2 : The space of double sequences $(l_\infty)^{\Vert}_F(\Upsilon, \wp)$ is symmetric but the space of double sequences $(c)^{\Vert}_F(\Upsilon, \wp)$, $(c_0)^{\Vert}_F(\Upsilon, \wp)$, $(c^R_0)^{\Vert}_F(\Upsilon, \wp)$, $(c^R)^{\Vert}_F(\Upsilon, \wp)$, are not symmetric.

Proof: Noticeably a space of double sequences $(l_\infty)^{\Vert}_F(\Upsilon, \wp)$is symmetric. However, other spaces of double sequences, could be indicated by the following example .

Example 3.1: Let's say the double sequences $(c)^{\Vert}_F(\Upsilon, \wp)$.

if $\Upsilon(\mathfrak{X}, \mathfrak{M}) = (\mathfrak{X}, \mathfrak{M})$, and $\wp_{1\mathfrak{r}} = 2$ for all $\mathfrak{r} \in \mathbb{N}$ and $\wp_{\mathfrak{s},\mathfrak{r}} = 3$, otherwise.

Suppose the double sequence $\left(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\right)$ be defined by

$(\mathfrak{X}_{1\mathfrak{r}}, \mathfrak{M}_{1\mathfrak{r}}) = (\bar{2}, \bar{2})$ for all $\mathfrak{r} \in \mathbb{N}$

where

$$\mathfrak{X}_{1\mathfrak{r}} = \bar{2} \quad \text{for all } \mathfrak{r} \in \mathbb{N},$$

and

$$\mathfrak{M}_{1\mathfrak{r}} = \bar{2} \quad \text{for all } \mathfrak{r} \in \mathbb{N}.$$

For $\mathfrak{s} > 1$,

$$\left(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}\right)(v) = \begin{cases} v+2, & \text{for } -2 \le v \le -1, \\ -v, & \text{for } -1 \le v \le 0, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\left(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\right)(v) = \begin{cases} v+2, & \text{for } -2 \le v \le -1, \\ -v, & \text{for } -1 \le v \le 0, \\ 0, & \text{otherwise.} \end{cases}$$

consequently, $\left(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}\right)$ can be defined as

International Journal of Academic and Applied Research (IJAAR)
ISSN: 2643-9603
Vol. 4, Issue 6, June – 2020, Pages: 22-31

$$(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}})(v) = \begin{Bmatrix} (v+2, v+2), & \text{for } -2 \le v \le -1, \\ (-v, -v), & \text{for } -1 \le v \le 0, \\ (0,0), & \text{otherwise.} \end{Bmatrix}$$

Let $(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}})$ be a rearrangement of $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}})$, $(\mathfrak{Q}_{\mathfrak{s},\mathfrak{r}})$ be a rearrangement of $(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})$

defined as

$$\mathfrak{B}_{\mathfrak{s}\mathfrak{s}} = \bar{2}$$

and

$$\mathfrak{Q}_{\mathfrak{s}\mathfrak{s}} = \bar{2}$$

Then $(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}})$ be a rearrangement of $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}})$ defined by

$$(\mathfrak{B}_{\mathfrak{s},\mathfrak{s}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{s}}) = (\bar{2}, \bar{2}).$$

For $\mathfrak{s} \neq \mathfrak{r}$,

$$(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}})(v) = \begin{Bmatrix} v+2, & \text{for } -2 \le v \le -1, \\ -v, & \text{for } -1 \le v \le 0, \\ 0, & \text{otherwise.} \end{Bmatrix}$$

and

$$(\mathfrak{Q}_{\mathfrak{s},\mathfrak{r}})(t) = \begin{Bmatrix} v+2, & \text{for } -2 \le v \le -1, \\ -v, & \text{for } -1 \le v \le 0, \\ 0, & \text{otherwise.} \end{Bmatrix}$$

Therefore, $(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}})$ can be defined as

$$(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}})(v) = \begin{Bmatrix} (v+2\,, v+2), & \text{for } -2 \le v \le -1, \\ (-v\,, -v), & \text{for } -1 \le v \le -1, \\ (0\,, 0), & \text{otherwise.} \end{Bmatrix}$$

Thus $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in (c)_F^{\Vert}(\Upsilon, \wp)$ but $(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}) \notin (c)_F^{\Vert}(\Upsilon, \wp)$.

Hence $(c)_F^{\Vert}(\Upsilon, \wp)$ is not symmetric. In same sense, it can be indicated that other spaces of double sequences are not symmetric too.

Theorem 3.3 : The spaces $(l_\infty)_F^{\Vert}(\Upsilon, \wp)$, $(c_0)_F^{\Vert}(\Upsilon, \wp)$,$(c_0^R)_F^{\Vert}(\Upsilon, \wp)$ are solid.

Proof : Consider the space of double sequences $(l_\infty)_F^{\Vert}(\Upsilon, \wp)$. Let $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}})$

$\in (l_\infty)_F^{\Vert}(\Upsilon, \wp)$and $(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}})$ be such that.

$$\bar{d}(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \bar{0}) \le \bar{d}(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \bar{0})$$

and

$$\bar{d}(\mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}, \bar{0}) \le \bar{d}(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}, \bar{0})$$

and consequently

$$\bar{d}\big((\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}), (\bar{0}, \bar{0})\big) \le \bar{d}\big((\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}), (\bar{0}, \bar{0})\big)$$

The result follows from the inequality

So,

$$\left(\bar{d}\left((\mathfrak{B}_{\mathfrak{s},\mathfrak{r}},\mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}),(\bar{0},\bar{0})\right)\right)^{\wp_{\mathfrak{s},\mathfrak{r}}} \leq \left(\bar{d}\left((\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}),(\bar{0},\bar{0})\right)\right)^{\wp_{\mathfrak{s},\mathfrak{r}}}$$

as $\Upsilon = (\Upsilon_1, \Upsilon_2)$ is increasing , we have

$$sup_{\mathfrak{s},\mathfrak{r}}\left\{\left(\Upsilon_1\left(\frac{\bar{d}\left((\mathfrak{B}_{\mathfrak{s},\mathfrak{r}},\mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}),(\bar{0},\bar{0})\right)}{\rho}\right)\right)^{\wp_{\mathfrak{s},\mathfrak{r}}}\right\} \leq sup_{\mathfrak{s},\mathfrak{r}}\left\{\left(\Upsilon_2\left(\frac{\bar{d}\left((\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}),(\bar{0},\bar{0})\right)}{\rho}\right)\right)^{\wp_{\mathfrak{s},\mathfrak{r}}}\right\}.$$

Hence, the spaces of double sequences $(l_\infty)_F^{\parallel}(\Upsilon, \wp)$ is solid. In same way, we could recognize other spaces are solid too by following same sense .

Proposition 3.4: The spaces of double sequences $(c)_F^{\parallel}(\Upsilon, \wp)$,$(c^R)_F^{\parallel}(\Upsilon, \wp)$ and $(m)_F^{\parallel}(\Upsilon, \wp)$ are not monotone and hence not solid.

Proof: The following following Example will lead to such result.

Example 3.2: Suppose a double sequence space $(c)_F^{\parallel}(\Upsilon, \wp)$ and Suppose $\Upsilon(\mathfrak{X},\mathfrak{M}) = (\mathfrak{X},\mathfrak{M})$. Let $j = \{(\mathfrak{s},\mathfrak{r}): \mathfrak{s} + \mathfrak{r} \text{ is even}\} \subseteq \mathbb{N}\times\mathbb{N}$ and let

$$\wp_{\mathfrak{s},\mathfrak{r}} = \begin{cases} 3, & \text{for } \mathfrak{s}+\mathfrak{r} \text{ even}, \\ 2, & \text{otherwise}. \end{cases}$$

and let $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})$ be defined as :

For all $\mathfrak{s},\mathfrak{r} \in \mathbb{N}$,

$(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})(v)=$

$$\left\{\begin{array}{ll} (v+3, v+3), & \text{for } -3 \leq v \leq -2, \\ (\mathfrak{s}v(3\mathfrak{s}-1)^{-1} + 3\mathfrak{s}(3\mathfrak{s}-1)^{-1}), \mathfrak{s}v(3\mathfrak{s}-1)^{-1} + 3\mathfrak{s}(3\mathfrak{s}-1)^{-1}), & \text{for } -2 \leq v \leq -1+\mathfrak{s}^{-1}, \\ (0,0), & \text{otherwise}. \end{array}\right\}$$

where

$$(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}})(v) = \left\{\begin{array}{ll} (v+3), & \text{for } -3 \leq v \leq -2, \\ \mathfrak{s}v(3\mathfrak{s}-1)^{-1} + 3\mathfrak{s}(3\mathfrak{s}-1)^{-1}, & \text{for } -2 \leq v \leq -1+\mathfrak{s}^{-1}, \\ 0, & \text{otherwise}. \end{array}\right\}$$

and

$$(\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})(v) = \left\{\begin{array}{ll} (v+3), & \text{for } -3 \leq v \leq -2\,, \\ \mathfrak{s}v(3\mathfrak{s}-1)^{-1} + 3\mathfrak{s}(3\mathfrak{s}-1)^{-1}, & \text{for } -2 \leq v \leq -1+\mathfrak{s}^{-1}, \\ 0, & \text{otherwise}. \end{array}\right\}.$$

Then $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) \in (c)_F^{\parallel}(\Upsilon, \wp)$.

Let $(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}},\mathfrak{Q}_{\mathfrak{s},\mathfrak{r}})$ be the canonical pre-image of $(\mathfrak{X}_{\mathfrak{s},\mathfrak{r}},\mathfrak{M}_{\mathfrak{s},\mathfrak{r}})_J$ of the sub sequence j of $\mathbb{N}\times\mathbb{N}$ .Then

$$(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}})(v) = \left\{\begin{array}{ll} (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}) & if\ (\mathfrak{s},\mathfrak{r}) \in j, \\ \bar{0}, & \text{otherwise}. \end{array}\right\}$$

and

$$(\mathfrak{Q}_{\mathfrak{s},\mathfrak{r}})(v) = \left\{\begin{array}{ll} (\mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) & if\ (\mathfrak{s},\mathfrak{r}) \in j, \\ \bar{0}, & \text{otherwise}. \end{array}\right\}.$$

and consequently

$$(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}})(v) = \begin{cases} (\mathfrak{X}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{M}_{\mathfrak{s},\mathfrak{r}}) & if\,(\mathfrak{s},\mathfrak{r}) \in j, \\ (\bar{0}, \bar{0}) & \text{otherwise.} \end{cases}$$

Thus, $(\mathfrak{B}_{\mathfrak{s},\mathfrak{r}}, \mathfrak{Q}_{\mathfrak{s},\mathfrak{r}}) \notin (c)_F^{\|}(\Upsilon, \wp)$. Hence, $(c)_F^{\|}(\Upsilon, \wp)$ does not regard as a monotone. In the same way, It can be indicated that other spaces of double sequences are not monotone too.

Hence, the spaces $(c)_F^{\|}(\Upsilon, \wp)$, $(c^R)_F^{\|}(\Upsilon, \wp)$ and $(m)_F^{\|}(\Upsilon, \wp)$ are not solid.

REFERENCES

[1] Azeez .M., (2017), **''On Statistically convergent double sequence spaces defined by Orlicz functions''**, Master Thesis, University of Kofa.

[2] Hayder. Z. ,(2017), **'' On Statistical convergent of generalized difference double sequence spaces which defined by Orlicz functions'',** Master Thesis, University of Kofa.

[3] Bromwich. T. J. Ia. **''An Introduction to the Theory of Infinite Series'',** MacMillan and Co. Ltd. New York, (1965).

[4] Hardy, G. H. **''On the convergence of certain multiple series'',** Proc. Camb. Phil. Soc, 19 (1917),

[5] Moricz, F **'' Extension of the spaces c and c_0 from single to double sequences'',** Acta. Math. Hungerica. 57(1-2), (1991), 129-136.

[6] Basarir, M. and Sonalcan , O. **''On some double sequence spaces'',** J. Indian Acad . Math, (1999), 21(2) 193-200.

[7] Sarma B. **'' double sequences spaces defined over an n normed sequence space '',** Afrika Matematika, 24(4),683-689.

[8] Tripathy B. C. Sarma B. **''Statisitically convergent double sequences ''** Tamkang J. Math . 34(3) (2003), 231-237.

[9] Nakano, H. **''Modular Sequence Spaces'',** proc. Japan Acad. 27 (1951), 508-512. Fuzzy Sets and Systems,99 (1998), 353-356. .

[10] Simons, S. **''The Sequence Spaces l(p_v) and $m(p_v)$'',**Proc. London Math. Soc .,(3) 15 (1965), 422-436 .

[11] Sarma. B. ,(2018), **'' double sequence spaces of fuzzy numbers of paranormed under an Orlicz function'',** Science Direct.