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Novel Transformations for Solving Ordinary Differential Equations with Variable Coefficients

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Abstract: *In this paper we applied a new integral transformation called the Novel transformation. We found general formulas of solutions for the ordinary differential equation with variable coefficients and we applied these formulas in some examples.*

Keywords*:* Transformations, Solving Ordinary Differential Equations, Variable Coefficients

1 Introduction

Integral transformations are with success applied for pretty much 2 centuries in finding several issues in math, mathematical physics, and branch of knowledge. Usually, the origin of the integral transformations together with the Laplace and Fourier transformations is copied back to the celebrated work of P. S. Laplace $(1749 - 1827)$ on applied mathematics within the 1780s. In fact, the Fourier or Laplace transformation methods based on the rigorous mathematical foundation are essentially equivalent to the modern operational calculus. There are many other integral transformations thorough the Hankel transformation, the Hilbert transformation and the Stieltjes transformation which are widely used to solve initial and boundary value problems involving ordinary and partial differential equations and other problems in mathematics, science, and engineering[1,2,3,4,5,6]. The definitions, properties and applications of the new integral Novel transformation (through utilizing the relationship between Novel and Laplace transformations) to ordinary differential equations with variable coefficients are described in this paper. A work has been done on the theory and applications of transformations such as Laplace-Fourier-Melin- Elzaki and Temimi, to name but a few[7,8,9,10,11,12]. The Novel transformation defined for the function $y(t)$ is as follows :

$$\Omega(s) = N_I\big(y(t)\big) = \frac{1}{s}\int_0^\infty e^{-st}\, y(t)\ dt \quad , t > 0 \qquad (1)$$

where $y(t)$, $t > 0$, is a real function, $\frac{e^{-st}}{s}$ is the kernel function and N_I is the operator of Novel transformation. The researchers used the Novel transformation to solve many differential equations, including solving differential equation arising in heat-transfer problem[13,14,15]. In this paper, we applied the Novel transformation to solve the ordinary differential equations with variable coefficients. In the second section we reviewed the important characteristics and theories related to this transformation and its relationship to the Laplace transform. In the third section we found general formulas for solving ordinary differential equations with variable coefficients through theories we have demonstrated. Finally, in the fourth section, we applied these formulas in some examples of ordinary differential equations with variable coefficients

2 Basic definitions and properties of Novel transform.

Definition

The Laplace's transform is defined by:

$$F(s) = L_I\big(y(t)\big) = \int_0^\infty e^{-st}\, y(t)dt \quad ; \quad t > 0 \quad , \qquad (2)$$

where $y(t)$, $t > 0$, is a real function, e^{-st} is the kernel function, and L_I is the operator of Laplace transform. The Laplace and Novel transforms exhibit a duality relationship that expressed as :

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$$\Omega(s) = \frac{1}{s} F(s) \tag{3}$$

The following lemma represent Novel transform for derivative functions and the reader can return for proof to [11] .

Lemma 1

If $\Omega(s) = N_I\big(y(t)\big), t\ > 0, s > 0$, then we have:

$$N_I\big(y^n(t)\big) = s^n N_I\big(y(t)\big) - s^{n-2}y(o) - s^{n-3}\acute{y}(0) - \cdots - y^{(n-2)}(0) - \frac{1}{s}y^{(n-1)} \tag{4}$$

where, $y^n(t)$ is the n-order derivative of $y(t)$.

We will include an appendix at the end of the research for a table to convert Novel to some functions.

3 Formulas of general solution for ordinary differential equation with variable coefficients

The next theorems very useful in study of ordinary differential equations having no constants coefficient.

Theorem 1 :

Let N(y(t)) be Novel transform of the function $y(t)$in A, then Novel transform of

the function $t^n y(t)$ is given

$$N_I(t^n y) = (-1)^n [\frac{d^n}{ds^n} N_I(y) + \frac{n}{s}\frac{d^{n-1}}{ds^{n-1}} N_I(y)] \tag{5}$$

Proof: We would like to verify by mathematical induction. If n = 1

$$N_I(y) = \int_0^\infty \frac{1}{s}\ y(t)e^{-st}\ \mathrm{dt}\ = \mathbf{\Omega}(\mathrm{s}),$$

by deriving the both sides, we get:

$$\frac{d}{ds}[\int_0^\infty \frac{1}{s}\ y(t)e^{-st}\ \mathrm{dt}\ ] = \frac{d}{ds}\Omega(s)$$

$$\frac{1}{s}\int_0^\infty (-t)\, y(t)e^{-st}\ \mathrm{dt}\ - \frac{1}{s^2}\int_0^\infty y(t)\, e^{-st}\ \mathrm{dt}\ = \frac{d}{ds}\Omega(s)$$

$$\frac{-1}{s}\int_0^\infty t\, y(t)e^{-st}\ \mathrm{dt}\ - \frac{1}{s}[\frac{1}{s}\int_0^\infty y(t)\, e^{-st}\ \mathrm{dt}\ ] = \frac{d}{ds}\Omega(s)$$

$$-N_I(ty) - \frac{1}{s}N_I(y) = \frac{d}{ds}N_I(y)\text{ , hence:}$$

$$N_I(ty) = (-1)[\frac{d}{ds}N_I(y) + \frac{1}{s}N_I(y)] \tag{6}$$

Next, we assume that the equality holds for n = k :

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ISSN: 2643-9670
Vol. 4, Issue 6, June – 2020, Pages: 43-51

$N_I(t^k y) = (-1)^n[\frac{d^k}{ds^k} N_I(y) + \frac{k}{s}\frac{d^{k-1}}{ds^{k-1}} N_I(y)]$ (7)

Let us we show that;

$N_I(t^{k+1} y) = (-1)^{k+1}[\frac{d^{k+1}}{ds^{k+1}} N_I(y) + \frac{k+1}{s}\frac{d^k}{ds^k} N_I(y)]$

$N_I(t^{k+1} y) = N_I(t.t^k y),$

from (6), we have:

$N_I(t^{k+1} y) = (-1)[\frac{d}{ds} N_I(t^k y)] + \frac{1}{s} N_I(t^k y)$ (8)

Substituting (7) into (8), and by simple calculation, we get :

$N_I(t^{k+1} y) = (-1)^{k+1}\left[\frac{d^{k+1}}{ds^{k+1}} N_I(y) + \frac{k+1}{s}\frac{d^k}{ds^k} N_I(y)\right]$

The validity of the equality for all natural number n follows by mathematical induction. It is clear that theorem 1 be rewritten by:

$N_I(t^n y) = (-1)^n[\frac{d^n}{ds^n} N_I(y) + \frac{n}{s}\frac{d^{n-1}}{ds^{n-1}} N_I(y)]$

Theorem 2

If Novel transform of the function $y(t)$given by$N_I\big(y(t)\big) = \Omega(s)$, then**:**

$N_I(ty^n) = -s^n \frac{d}{ds} N_I(y) - (n+1)s^{n-1} N_I(y) + (n-1)s^{n-3} y(0) + (n-2)s^{n-4}\acute{y}(0) + \cdots +$
$y^{(n-1)}(0) + \frac{1}{s} y^{(n-2)}(0)$ (9)

Proof: We would like to verify by mathematical induction. If n = 1

$N_I(t\acute{y}) = \int_0^\infty \frac{1}{s}\ t\ \acute{y}(t) e^{-st}\ dt,$ (10)

let $\acute{y}(t) = g(t)$………(*), substituting this equation in (14), we have;

$N_I(tg(t)) = \frac{1}{s}\int_0^\infty\ t\ \ g(t)e^{-st}\ dt$,

Hence from theorem (1), we have:

$N_I(t\acute{y}) = -\frac{d}{ds} N_I(\acute{y}) - \frac{1}{s} N_I(\acute{y}),$ (11)

apply lemma(1),we get:

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$$N_I(t\acute{y}) = -\frac{d}{ds}[sN_I(y) - \frac{1}{s}y(0)] - \frac{1}{s}\left[sN_I(y) - \frac{1}{s}y(0)\right]$$

$$N_I(t\acute{y}) = -s\frac{d}{ds}N_I(y) - 2N_I(y) \qquad (12)$$

In the general case. Next, we suppose that,

$$N_I(ty^k) = -s^k\frac{d}{ds}N_I(y) - (k+1)s^{k-1}N_I(y) + (k-1)s^{k-3}y(0) + (k-2)s^{k-4}\acute{y}(0) + \cdots + y^{(k-1)}(0) + \frac{1}{s}y^{(k-2)}(0)$$

And show that $N_I(ty^n)$ can be expressed by;

$$N_I(ty^{k+1}) = -s^{k+1}\frac{d}{ds}N_I(y) - (k+2)s^{n-1}N_I(y) + (k)s^{k-2}y(0) + (k-1)s^{k-3}\acute{y}(0) + \cdots + y^{(k)}(0) + \frac{1}{s}y^{(k-1)}(0)\ ,$$

From (11), we have:

$$N_I(ty^{k+1}) = -\frac{d}{ds}N_I(y^{k+1}) - \frac{1}{s}N_I(y^{k+1}),$$

using lemma (1) and by simple calculation, we get :

$$N_I(ty^{k+1}) = -s^{k+1}\frac{d}{ds}N_I(y) - (k+2)\ s^kN_I(y) - (k)s^{k-2}y(0) - (k-1)s^{k-3}\acute{y}(0) - \cdots + \frac{1}{s}y^{(k-1)}(0)]$$

for all natural number n. Thus, if the equality holds for k, it holds for k +1. Therefore, by mathematical induction, the equality is true for all natural number n.

$$N_I(ty^n) = -s^n\frac{d}{ds}N_I(y) - (n+1)s^{n-1}N_I(y) + (n-1)s^{n-3}y(0) + (n-2)s^{n-4}\acute{y}(0) + \cdots + y^{(n-1)}(0) + \frac{1}{s}y^{(n-2)}(0)$$

Theorem 3

If Novel transform of the function $y(t)$given by $N_I\big(y(t)\big) = \Omega(s)$, then**:**

$$N_I(t^2y^n) = s^n\frac{d^2}{ds^2}N_I(y) + 2(n+1)s^{n-1}\frac{d}{ds}N_I(y) + n(n+1)s^{n-2}N_I(y) - (n-1)(n-2)s^{n-4}y(0) + (n-2)(n-3)s^{n-5}\acute{y}(0) + \cdots + y^{(n-2)}(0) + \frac{2}{s}y^{(n-3)}(0) \quad (13)$$

Proof:

We would like to establish the validity of the statement by the mathematical induction. For n = 1,

From theorem (1), we get:

International Journal of Academic Multidisciplinary Research (IJAMR)
ISSN: 2643-9670
Vol. 4, Issue 6, June – 2020, Pages: 43-51

$N_I(t^2\acute{y}) = \frac{d^2}{ds^2}N_I(\acute{y}) + \frac{2}{s}\frac{d}{ds}N_I(\acute{y}),$

by using lemma (1) and by simple calculation, we get :

$N_I(t^2\acute{y}) = \frac{d^2}{ds^2}[sN_I(y) - \frac{1}{s}y(0)] + \frac{2}{s}\frac{d}{ds}[sN_I(y) - \frac{1}{s}y(0)]$

$$N_I(t^2\acute{y}) = s\frac{d^2}{ds^2}N_I(y) + 4\frac{d}{ds}N_I(y) + \frac{2}{s}N_I(y) \quad (14)$$

holds. Next, we suppose that

$N_I(t^2y^k) = s^k\frac{d^2}{ds^2}N_I(y) + 2(k+1)s^{k-1}\frac{d}{ds}N_I(y) + k(k+1)s^{k-2}N_I(y) - (k-1)(k-2)s^{k-4}y(0) + (k-2)(k-3)s^{k-5}\acute{y}(0) + \cdots + y^{(k-2)}(0) + \frac{2}{s}y^{(k-3)}(0)$

Let us show that:

$N_I(t^2y^{k+1}) = s^{k+1}\frac{d^2}{ds^2}N_I(y) + 2(k+2)s^k\frac{d}{ds}N_I(y) + (k+1)(k+2)s^{k-1}N_I(y) - (k)(k-1)s^{k-3}y(0) + (k-1)(k-2)s^{k-4}\acute{y}(0) + \cdots + y^{(k-1)}(0) + \frac{2}{s}y^{(k-2)}(0)$

By theorem (2), we have:

$N_I(t^2y^{k+1}) = \frac{d^2}{ds^2}N_I(y^{k+1}) + \frac{2}{s}\frac{d}{ds}N_I(y^{k+1}),$

Using lemma (1), and by simple calculation, yield:

$N_I(t^2y^{k+1}) = s^{k+1}\frac{d^2}{ds^2}N_I(y) + 2(k+2)s^k\frac{d}{ds}N_I(y) + (k+1)(k+2)s^{k-1}N_I(y) - (k)(k-1)s^{k-3}y(0) + (k-1)(k-2)s^{k-4}\acute{y}(0) + \cdots + y^{(k-1)}(0) + \frac{2}{s}y^{(k-2)}(0)$

Thus, if the equality holds for k, it holds for k+1. Therefore, by mathematical induction, the equality is true for all natural number n.

$N_I(t^2y^n) = s^n\frac{d^2}{ds^2}N_I(y) + 2(n+1)s^{n-1}\frac{d}{ds}N_I(y) + n(n+1)s^{n-2}N_I(y) - (n-1)(n-2)s^{n-4}y(0) + (n-2)(n-3)s^{n-5}\acute{y}(0) + \cdots + y^{(n-2)}(0) + \frac{2}{s}y^{(n-3)}(0)$

4 Applications

Now we apply the above theorem to find Novel transform for some ordinary differential equations with variable coefficients:

Example (1) : To solve the differential equation

International Journal of Academic Multidisciplinary Research (IJAMR)
ISSN: 2643-9670
Vol. 4, Issue 6, June – 2020, Pages: 43-51

$$t^2 y'' + 4t y' + 2y = 12t^2 \qquad (15)$$

with the initial conditions $y(0) = y'(0) = 0$

By using Novel transform into equation (15) and the theorem 2 and theorem 3, we have:

$$s^2 N_I''(y) + 6s N_I'(y) + 6N_I(y) - 4s N_I'(y) - 8N_I(y) + 2N_I(y) = \frac{24}{s^4}$$

$$s^2 N_I''(y) + 2s N_I'(y) = \frac{24}{s^4} \quad , \qquad (20)$$

after simple calculation, we get :

$N_I(y) = \frac{2}{s^4} + \frac{c_1}{s} + c_2$, where c_1 , c_2 are constants .

By making up the initial conditions, we get:

$N_I(y) = \frac{2}{s^4}$,

by using the inverse Novel transform we obtain the Solution in the form of

$y(t) = t^2$

Example (2): Consider the non-constant coefficient differential equation in the form of

$$t^2 y''' + 6t y'' + 6y' = 60t^2 \qquad (21)$$

with the initial conditions $y(0) = y'(0) = 0$,

by using Novel transform and apply theorem (3), we have:

$$s^3 N_I''(y) - 8s^2 N_I'(y) + 12s N_I(y) - \frac{2}{s} y(0) - 6s^2 N_I'(y) - 18s N_I(y) + \frac{6}{s} y(0) + 6s N_I(y) - \frac{6}{s} y(0) = \frac{120}{s^4} ,$$

substituting the condition into equation, we get:

$$s^3 N_I''(y) + 2s^2 N_I'(y) = \frac{120}{s^4} \ . \qquad (22)$$

and by simple calculation, we get :

$N_I(y) = \frac{6}{s^5} + c_1 s + c_2$,where $c_1$and c_2 are constants. (23)

After substituting the initial conditions, we find that $c_1 = c_2 = 0$, and from that, we get :

$N_I(y) = \frac{6}{s^5}$,

by taking the inverse Novel transform we obtain the Solution in the form of:

$y(t) = t^3$

Example (3): To solve the differential equation:

$$\acute{\acute{y}} + 3t\acute{y} - 6y = 2, \tag{24}$$

with the initial conditions $y(0) = \acute{y}(0) = 0,$

by using Novel transform and apply theorem (2), we have:

$s^2 N_I(y) - y(0) - \frac{1}{s}\acute{y}(0) - 3s\acute{N}_I(y) - 6N_I(y) - 6N_I(y) = \frac{2}{s^2},$

after substituting the initial conditions, we get :

$(s^2 - 12)N_I(y) - 3s\acute{N}_I(y) = \frac{2}{s^2},$

we find the integral factor and then we multiply both sides of the equation by then we integrate both sides, yields:

$s^4 e^{\frac{-s^2}{6}} N_I(y) = 2e^{\frac{-s^2}{6}} + c_1$,such that c_1 is constant

After substituting the initial conditions, we find that $c_1 = 0$, and doing simple calculation, we get:

$N_I(y) = \frac{2}{s^4},$

by taking the inverse Novel transform we obtain the solution in the form of $y(t) = t^2$

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Appendix

Table 1. The Novel integral transformation of some functions:

	$y(t)$	$\Omega(s) = sL[y(t)]$
1	1	$\frac{1}{s^2}$
2	t^n	$\frac{n!}{s^{n+1}}$
3	e^{at}	$\frac{a}{s(s-a)}$
4	$\sin at$	$\frac{a}{s(s^2+a^2)}$
5	$\cos at$	$\frac{1}{s^2+a^2}$

6	$\sinh at$	$\frac{a}{s(s^2-a^2)}$
7	$\cosh at$	$\frac{1}{s^2-a^2}$