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Applications of Mathematical Calculus in Economics

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Abstract: *In Calculus we study Function, Continuity, Differentiability, Integrality and their after, the application of differentiability and inerrability at our school level. In this article we discuss how these concepts of calculus is useful and can be apply in Economics Theory. In fact, this article will be helpful to the beginner, how elementary mathematics is useful in economics and Management.*

Keywords: Calculus, Differentiation, Integration, Basics function of Economics, Cost Function, Revenue Function, Demand Function.

1. INTRODUCTION .

Calculus is a branch of Mathematics which have a wide application in almost all disciplines such as engineering, science, business, financial management, computer science, and information system. Teaching of Calculus using the traditional approach does not help students understand the basic concepts, so the teaching and learning of Calculus should be improved focusing on the conceptual understanding of the subject, as well as the development of problem solving skills. Calculus was developed in the latter half of the 17th century by two mathematicians Gottfried Leibniz's and Isaac Newton. We study calculus in school, which includes function, limit, continuity, derivatives, and integration. The application of calculus has incredible power over the physical worlds by modeling and controlling system. We can model beautifully using calculus such as motion, electricity, heat and light, harmonic, astronomy, radioactive decay, relation rates, birth, and death rates, cost, and revenue and many others. In calculus, we generally study two different branches namely, Differential calculus and Integral calculus. In differential calculus we study the behavior and rate of change, for example distance over a time or investment over a time at interest rate. The integral calculus is reverse process of differential and some time it is called ant derivatives. Some application of integral are, computation involving Area, Volume, Arc length, Pressure, Power series, Fourier series, space, Time and motion. The basic concept of calculus such as constants, variables, functions, limit of functions, continuity of function, differentiability and inerrability of a function will be referred from [2]. The integral formula can be derived as follows (see[1]) :

Let f be a function defined on the closed interval $[a, b]$. Divide this interval into n subintervals by choosing any $(n-1)$ Intermediate points between a and b. Let $x_0 = a$ and $x_n = b$ and $x_1 < x_2 < \cdots < x_{n-1}$ be the intermediate points such that

$x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n$

The points $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ are not necessarily equidistant. .Let $\Delta_1 x$ be the length of the first subinterval so that $\Delta_1 x = x_1 - x_0$ and Let $\Delta_2 x$ be the length of the second subinterval so that $\Delta_2 x = x_2 - x_1$ and so on, thus the length of the lth subinterval is $\Delta_i x$, $\Delta_i x = x_i - x_{i-1}$. A set of all such subintervals of the interval $[a, b]$ is called a partition of the interval $[a, b]$. The partition Δ contains n subintervals. One of these subintervals is longest; however, there may be more than one such subinterval. The length of the longest subinterval of the partition Δ, called the norm of the partition, is denoted by $\|\Delta\|$. Choose a point in each subinterval of the partition Δ : Let ξ_1 be the point chosen in $[x_0 - x_1]$ so that $x_0 < \xi_1 < x_1$. Let ξ_2 be the point chosen in $[x_1 - x_2]$ so that $x_1 < \xi_2 < x_2$. and so forth, so that ξ_1 b is the point chosen in$[x_{i-1} - x_i]$, and $x_{i-1} < \xi_i < x_i$. Form the sum

$f(\xi_1)\Delta_1 x + f(\xi_2)\Delta_2 x + f(\xi_3)\Delta_3 x + \cdots + f(\xi_n)\Delta_n x$, or $\sum_{i=1}^{n} f(\xi_i)\Delta_i x$. Such a sum is called a Riemann sum, named for the mathematician Georg Friedrich Bernhard Riemann (1826-1.866). We are now in a position to define what is meant by a function f being "integrable" on the closed interval $[a, b]$. If $f$is a function defined on the closed interval $[a, b]$, then the definite integral of f from a to b, is given by

$$\int_a^b f(x)\, dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta_i x$$

if the limit exists.

Thus in short we have, for a function

$$y = f(x) \tag{1}$$

indefinite integration of the function $f(x)$ with respect to x is given by

$$\int f(x)\,dx + c\ ,c\ is\ arbitary\ constant \tag{2}$$

and definite integration of the function $f(x)$ with respect to x in the interval $[a, b]$ is given by

$$\int_a^b f(x)\,dx \tag{3}$$

For example

$$\int x^2\,dx = \frac{x^3}{3} + c\ ,c\ is\ arbitary\ constant$$

and

$$\int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_1^2 = \left[\frac{2^3}{3} - \frac{1^3}{3}\right] = \frac{1}{3}(8-1) = \frac{7}{3}.$$

If we differentiate equation (1) with respect to x, then we get the first derivative of the function (1) which is denoted by $\frac{dy}{dx}$ or $f^{/}(x)$ which is again a function of x. If we differentiate again $\frac{dy}{dx}$ or $f^{/}(x)$ then we get second order derivative of function (1) and which is denoted by $\frac{d^2y}{dx^2}$or $f^{//}(x)$ and you can take succeeding derivatives as long as the resulting functions are differentiable. In general, we represent the nth derivative of $f(x)$ is denoted by $f^n(x)\ or\ \frac{d^ny}{dx^n}$. The first order and second order derivative play an important role for finding minimum and maximum value of the function at a point for detail method see [1]. This concept is used in economics for determining the maximum profit of a firm.

The role of mathematics in economics has been an ongoing debate for several years. Numerous authors, both economists and non-economists, have addressed the subject and have given pros and cons of the intensive use of mathematical methods in studying social problems. Irrespective of this discussion, the incidence of mathematics being utilized in economics has undoubtedly increased, and nowadays a sophisticated knowledge in mathematics could also be a basic need for any economist willing to travel beyond the undergraduate level. Although there are many arguments both in favor and against the employment of mathematics in economics, this text here merely attempt to provide an objective account of the employment of mathematics in economics. Mathematics provides the economists with a tools often more powerful than the descriptive analysis. Mathematics help to translate verbal arguments be represented into concise quantitative statements or equations. It provides concrete form to economic laws and relationships, and make more practical. The Mathematics helps in systematic understanding of the link and in derivation of certain results which could either be impossible through verbal argument, or would involve complex, and difficult processes. The fashionable mathematical economics began within the 19th century with the utilization of calculus to clarify and explain economic behavior. In today's increasingly complicated international business world, a strong preparation within the basics of both economics and arithmetic are crucial to success. This text is supposed to arrange for a student to travel directly into the business world with skills that are in high demand, in economics or finance. Other occupations include but don't seem to be limited to the following: Economist, management accountant, actuary, examiner, research, analyst, analyst, marketing/sales manager, financial planner, investment manager, realty and investor. We have got looked as in mathematics that, the integral compute the area under a curve. This lets us compute a whole profit, or revenue, or cost, from the related marginal functions. We have got tried sort of applications where this was interpreted as an accumulation over time, including total production of an well and present value of a revenue stream. As an example, considering profit because the world between the worth and revenue curves. In economic, calculus is used to test for determining Marginal revenues and price which helps business manager to maximize their profit and measure the speed of increase in profit that result from each increase in profit that results from each increase in production. As long as marginal revenue exceeds price, the firm increases its profit. In short, application of integration in economic and commerce helps to look out the total cost function and total revenue function from the worth. In study of Business, Economists use the following functions: $C(x)$: cost of producing x items, $R(x)$: revenue from the sale of x items, $P(x)$: profit from the production and sale x items. The equation which give the relation between these function is $P(x) = R(x) - C(x)$ which gives profit. If we take the derivative of this function than economist call it as Marginal function means rate of change and denoted as: $C^{/}(x)$: marginal cost, $R^{/}(x)$: marginal revenue, $P^{/}(x)$: marginal profit. In order to understand the above functions better way we discuss some of relationship between them and application of these functions and their derivative in form of example as follows:

The cost function denoted by (x) , represent the total cost of producing x items. Cost function consists of two parts, fixed

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cost denoted by F and variable cost denoted by $V(x)$. This means that

$$C(x) = F + V(x) \tag{4}$$

It must be noted that, fixed cost includes office expanses, cost of machine, employee wages and many more while variable cost include cost of direct labor, cost of material, cost of transportation and many more. Also $V(x)$ depend upon the number of unit produced (the value of x) where as F is independent of output x.

If we differentiate (4) with respect to x then we get rate of change of cost C per unit change in output level of x unit which called as Marginal Cost (MC) . Thus we write it as

$$MC = \frac{dC}{dx} \tag{5}$$

Since, we know that integration is reverse process of differentiation or also called as antiderivatives, that is if

$\frac{d}{dx} f(x) = F(x)$, then

$\int F(x)dx = f(x) + c, c\ is\ arbitary\ constant.$

Therefore, if we integrate equation (5) on both side with respect to x, we get the total cost of x
units as follow:

$$C(x) = \int (MC)\, dx + c\ , c\ is\ abitrary\ constant \tag{6}$$

The Revenue Function ($R(x)$) give the total money obtained (or total turnover) by selling x units of product. If x units are sold at \$$p$ per unit, then

$$\boldsymbol{R(x) = p \times x} \tag{7}$$

If we differentiate equation (7) with respect to x, we get the rate of change in revenue per unit change in out , which is called as Marginal Revenue(MR) and given as

$$MR = \frac{dR}{dx} \tag{8}$$

Taking integration of equation (8) both side with respect to x, we get total revenue of x units sold and given by

$$R(x) = \int (MR)\, dx + c\ , c\ is\ abitrary\ constant \tag{9}$$

The difference between total revenue $R(x)$ and the total cost $C(x)$ is known as profit function denoted by $P(x)$ which is given by

$$P(x) = R(x) - C(x) \tag{10}$$

If the equation (10) is differentiated with respect to x, then we get the rate of change in profit per unit change in output which is known as the Marginal Profit (MP) which is given by

$$MP = \frac{dP}{dx} \tag{11}$$

Suppose the demand for a product is given as a function of price by the expression $x = f(p)$. This demand function gives the quantity demanded of the product for a price. We define the elasticity of demand for a good with demand equation $x = f(p)$, as the function

$$E(p) = \frac{-pf^{/}(p)}{f(p)} \tag{12}$$

Economists use the following terms to describe demand.
If demand is elastic if $E(p) > 1$.
If demand is inelastic if $E(p) < 1$.
If demand is unitary if $E(p) = 1$.
The notion of elasticity of demand governs how responsive the demand for a product is in regard to changes in the price. We will discuss some of application of above functions as follows:

2. APPLICATIONS:

We will discuss some of application of above functions as follows:

Application 1 (Marginal Cost Analysis): The total cost in thousands of rupees for a daily production of an item is $C(x) = 50 + 20x - x^2$, then by (5), we have

$$Marginal\ cost = C^1(x) = \frac{d}{dx}(50 + 20x - x^2) = 20 - 2x$$

and the Marginal cost when 4 units are produced is given by

$$Marginal\ cost = C^1(4) = 20 - 2(4) = 20 - 8$$
$$= 12\ thousand.$$

Application2 (Total Revenue): The Marginal revenue of a function is $MR = 7 - 2x - x^2$. The total revenue is given by

$$Total\ Revenue = \int (7 - 2x - 3x^2)dx = 7x - \frac{2x^2}{2} - \frac{3x^3}{3}$$
$$= 7x - x^2 - x^3.$$

Application 3 (Tangent line to Cost function curve) : A company sells notebook for \$3 each and the cost associated with these notebook are given by $C(x) = 1.25x + 0.01x^2 + 50$. The Marginal cost is $C^{/}(x) = 1.25 + 0.02x$ and Revenue function is $R(x) = 3x$ with marginal revenue $P^{/}(x) = R^{/}(x) - C^{/}(x) = 1.75 - 0.02x$. Thus, if we take $x = 2$, then $C^{/}(2)$ will represent the slope of the tangent line to the cost function $C(x)$ at a point $x = 2$.

Application 4 (Economic scale): The average cost per unit is defined as cost divided by numbers of units produced which is denoted by $\bar{C}(x)$ and given as $\bar{C}(x) = \frac{C(x)}{x}$. The cost function for the company is $C(x) = 500 + 35x$, the average cost function is given by $C(x) = \frac{500}{x} + 35$ and therefore marginal cost will be $\bar{C}^{/} = -\frac{500}{x^2}$. Since the derivative of the average cost is native for all , we see that the average cost is decreasing for all x. This implies that the per unit cost is decreasing as the production quantity rises , which refer in economic circle as economic of scale.

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Application 5 (Elastic demand) : Let the demand equation for a product be given by

$$x = f(p) = \sqrt{400 - 5p} \text{. Then}$$

$$E(p) = \frac{-pf^{/}(p)}{f(p)} = \frac{5p}{2(400 - 4p)}$$

If the price of the product is set at 40, then the elasticity of demand is $\frac{1}{2} < 1$, this classifies the demand as inelastic. If the price increases past 40, we can expect the revenue to increase. This means that the price increases in the product are not enough to depress sales. Thus, increasing the price of the product will generate additional revenue. In contrast, a price of 60 will result in an elasticity of demand of $2 > 1$. This classifies the demand as elastic. Price increases past 60 will result in a decrease in revenue. That is, price increases will drive off customers in sufficient quantity to depress revenue.

The first and second order derivatives play an important role for finding minimum and maximum value of the function at a point for detail method see [1]. This concept is used in economics for determining the maximum profit of a firm. We need the following theorems without proof before stating the problem.

Theorem 1: Let $f(x)$ be continuous and differentiable, then.

1. if $f^{/}(x) > 0$ for all x in an interval (a, b), then $f(x)$ is increasing on (a, b) .
2. if $f^{/}(x) < 0$ for all x in an interval (a, b), then $f(x)$ is decreasing on (a, b).

We define $x = c$, where c is a point in the domain of f(x), to be a critical value of $y = f(x)$ if $f^{/}(x) = 0$ or if $f^{/}(x)$ fails to exist if the tangent line is vertical. Critical values are very important in the remainder of our studies.

Theorem 2: If $f(x)$ is continuous on the interval (a, b), then any local maximum or minimum must occur at a critical value of $f(x)$.

It is important to realize that this theorem does not say the function must have a local maximum or minimum at a critical value. It says "If there are any local extrema (a term meaning either local maxima or minima), they must occur at critical values." The existence of critical values does not guarantee a local extrema.

Theorem 3 (The First Derivative Test) : Let c be a critical value of $f(x)$. If $f^{/}(x) < 0$ for $x < c$ and $f^{/}(x) > 0$ for $> c$, then $(c, f(c))$ is a local minimum. If $f^{/}(x) > 0$ for $x < c$ and $f^{/}(x) < 0$ for $x > c$, then $(c, f(c))$ is a local maximum.

Theorem 4 (The Second Derivative Test) : Suppose $f(x)$ is differentiable and $f^{/}(c) = 0$. If $f^{//}(c) > 0$, then $f(x)$ has a local minimum at $(c, f(c))$. If $f^{//}(c) < 0$, then $f(x)$ has a local Maximum at $(c, f(c))$.

Theorem 5 : If $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ attains an absolute maximum and an absolute minimum on $[a, b]$. These absolute extrema of $f(x)$ will occur at a critical point or at an end point. To find absolute extrema, we find all critical points, then evaluate $f(x)$ at the endpoints of the interval and at the critical values. The largest value of $f(x)$ will be the absolute maximum, and the smallest will be the absolute minimum.

To find absolute extrema, we find all critical points, then evaluate $f(x)$ at the endpoints of the interval and at the critical values. The largest value of $f(x)$ will be the absolute maximum, and the smallest will be the absolute minimum.

Application 6 : A company sells shoes to dealers at $20 per pair if fewer than 50 pairs are ordered. If 50 or more pairs are ordered (up to 600), the price per pair is reduced 2 cents times the number ordered. What size order produces maximum revenue for the company?

Revenue price number sold = (price)(number sold) , and we let x be order size.

Revenue= $R(x) = 20x \; if \; x \in [0{,}500]$ and taking the quantity discount into consideration, we get

$R(x) = (20 - 0.02x)(x) \; if \; x \in [51{,}600]$

For $x \in [0{,}500]$ we have $R(x) = 20x$ and it is obvious that revenue is a maximum when $x = 50$ producing a revenue of $(20)(50) = \$1000$. For $x \in [51{,}600]$ we must use calculus to locate the maximum for the revenue function, since

$R(x) = (20 - 0.02x)(x) = 20x - 0.02x^2$

We have

$R^{/}(x) = 20 - 0.04x \; and \; R^{//}(x) = -0.04.$

The critical number for revenue is $\frac{20}{0.04} = 500$, and since $R^{//}(x) < 0 \; for \; all \; x$, we know that $R(x)$ has a maximum at $x = 500$.

$R(500) = (20 - 0.02 \times 500)(500) =$
$5000 \; and \; R(500) > R(50).$

So we know that the order size producing the most income for the company is a 500 pair order.

Now we see the application of integration in economics. The supply function or supply curve gives the quantity of an item that producers will supply at any given price. The demand function or demand curve gives the quantity that consumers will demand at any given price. We will denote the price per unit by p and the quantity supplied or demanded at that price by q. As is the convention in economics, we will always write p as a function of q. Thus the supply curve will be denoted by the formula

$$p = S(q) \tag{13}$$

and represented by a graph where the x and y axes correspond to q and p values respectively. Similarly, we will use

$$p = D(q) \tag{14}$$

to denote the demand curve.

As you might expect, the supply function S is increasing – the higher the price, the more the producers will supply. The demand function D is decreasing – the higher the price, the less the consumers will buy. The point of intersection (q_c, p_c) of the supply and demand curves is called the market equilibrium point. The numbers q_c and p_c are termed equilibrium quantity and equilibrium price respectively. The economic significance of the market equilibrium is the following: consider the case of bread. As long as $p < p_c$the demand for bread exceeds its supply, pushing up the price until it reaches equilibrium price p_c . At this point, the quantity supplied is equal to the quantity demanded which is the equilibrium quantity. Conversely, if the price exceeds equilibrium, the supply of bread exceeds demand, bringing the price down.

In an ideal free market both consumers and producers gain by buying and selling at the equilibrium price. It is easy to understand this in principle, but the goal of this section is to compute exactly how much the consumers gain by buying at the equilibrium price rather than at a higher price. We first compute the total amount spent by the consumers if everyone buys at the equilibrium price p_c. In that case q_c units are supplied and bought, and the total amount spent is the number of units bought times the price per unit, i.e.,

total amount spent at equilibrium price = $p_c \times q_c$ (15).

Next, let us compute the total amount that would be spent if every consumer paid the maximum price that each is willing to pay. Divide the interval $[0, q_c]$ into n subintervals, each of length $\Delta x = \frac{q_c}{n}$, with endpoints

$$x_0 = 0, x_1 = \frac{q_c}{n}, x_2 = \frac{2q_c}{n}, \dots, x_n = \frac{nq_c}{n} = q_c.$$

Consider the first interval $[0, x_1]$, i.e., suppose that only x_1 units had been available. Then the selling price per unit could have been set at $D(x_1)$ dollars and x_1 units sold. Of course, at this price it would have been impossible to sell any more. The total expenditure from buying these first x_1 units of commodity is therefore

(price per unit) × (number of units) = $D(x_1)\Delta x$ dollars.

After selling the first x_1 units, suppose that more units become available, so that now a total of x_2 units have been produced. Setting the price at $D(x_2)$, the remaining $x_2 - x_1 = \Delta x$ units can be sold, yielding $D(x_2)\Delta x$ dollars. Note that each group of buyers paid as much for the commodity as it was worth to them. Continuing this process of price discrimination, the total amount of money paid by consumers willing to pay at least p_c is approximately equal to

$$D(x_1)\Delta x + D(x_2)\Delta x + \cdots + D(x_n)\Delta x \tag{16}$$

The sum in (16) is a Riemann sum for the integral $\int_0^{q_c} D(q)dq$, and the sum gets closer and closer to the integral as n gets larger. At the same time, the sum giver better and better approximations to the total expenditure. The conclusion is therefore

Total amount of maximum price is given by

$$\int_0^{q_c} D(q)dq \tag{17}$$

The quantity in (17) is the area under the demand curve from $q = 0$ to $q = q_c$. As the figure shows, it is greater than $p_c \times q_c$, which is the area of the rectangle either sides $[0, q_c]$ and $[0, p_c]$, and which according to the formula (15) represents the total amount spent by consumers at the equilibrium price. The difference between these two areas (17) - (15) represents the total that consumers save by buying at equilibrium price. This is called the consumer surplus for this product and given by

$$Consumer\ surplus = \int_0^{q_c} D(q)\,dq - p_c q_c \tag{18}$$

A similar analysis (which you should try out) shows that the producers also gain by trading at the equilibrium price. Their gain called producer surplus is given by

$$Producer\ surplus = p_c q_c - \int_0^{q_c} S(q)\,dq \tag{19}$$

We will, now discuss some application of the above integral formulas, which we put in terms of examples:

Application 7 (Maximizing Revenue): The demand equation for a certain product is $p = 6 - \frac{1}{2}q$ dollars. The level of production that results in maximum revenue is given as follows:

The revenue function $R(q) = q \times p = q \times \left(6 - \frac{1}{2}q\right) = 6q - \frac{1}{2}q^2$.

The Marginal revenue is given by $R^1(q) = 6 - q$. Also $R^1(q) = 6 - q = 0 \Rightarrow q = 6$, which the critical point. Now $R^{11} = -1 < 0$, hence we get maximum value at $q = 6$ and the value is $R(6) = 6 \times 6 - \frac{1}{2}6^2 = 18$. Thus, the rate of production resulting in maximum revenue is $q = 6$, which results in total revenue of 18 dollars.

Application 8 (Use of Integral in Economics): Consumer surplus is the gain made by consumers when they purchase an item at the competitive market price rather than the (highest) price that they would have been willing to pay for it. For example: If you would be willing to pay $60 for a ticket to see the World Cup final, but you can buy a ticket for $50, in this case, your consumer surplus is £10. Analogously, producer surplus is the gain made by producers when they sell an item at the market price rather than the (lowest) price that they would also have accepted for it. For Example, If a firm would sell a good at $4, but the market price is $7, the producer surplus is $3. The demand function for a commodity is $p = 28 - 4q$ and supply function on the market is $= 3q$, then $(p_c, q_c) = (12,4)$ with

$$Consumer\ surplus = \int_0^{q_c} D(q)\,dq - p_c q_c$$

$$= \int_0^{4} (28 - 4q)\,dq - 12 \times 4 = 32$$

and

$$Producer\ surplus = p_c q_c - \int_0^{q_c} S(q)\,dq$$

$$= 12 \times 4 - \int_0^{4} 3q\,dq = 24$$

3. Conclusion:

We conclude that Integral Calculus in mathematics play a very important role in supply and demand theory. In fact, the price of an item determines the supply and the demand for the item. As price increases demand for the item usually falls. Conversely, as the price increases, the quantity producers are willing to supply will increase. On the other hand first and second derivatives are useful in optimization problems which determine the maximum profits. In short, we say that, the application of calculus to business and economics, such as maximum and minimum problems, marginal analysis, and cost analysis can be computed more easily.

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