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Free Multiplication Gamma Acts and The Gamma ideal η(M)

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Abstract— *In this paper, we define the associated gamma ideal $\eta(M)$,and characterized multiplication gamma acts in terms of the gamma ideals. Basic properties about this concept are discussed. We show that $\eta(M)$ is a globally idempotent multiplication gamma ideal. We clarify the interplay between multiplication gamma act and free gamma act was studied. We introduce the definition of $\mathcal{T}(M)$ and study the relation between it and the gamma ideal $\eta(M)$ stated with examples.*

Keywords— Gamma semigroup , Multiplication gamma act , free gamma act**.**

1. INTRODUCTION.

In 1981 the concept of gamma semigroups was introduced by M.K. Sen [5]. Let S and Γ be nonempty sets. If there is a mapping S×Γ×S → S defined by $(s_1,\alpha, s_2) \mapsto s_1\alpha s_2$, satisfying , $s_1\alpha(s_2\beta s_3) = (s_1\alpha s_2)\beta s_3$ for all s_1, s_2, s_3 in S and α, β in Γ then S is called Γ-semigroup . A Γ-semigroup S, is called commutative if $s\alpha t = t\alpha s$ for all s,t ∈S and α∈Γ. An element a of Γ-semigroup S is said to be identity if $a\alpha s = s = s\alpha a$ for all $s \in$ S and $\alpha \in \Gamma$.A Γ-semigroup with identity is called Γ-monoid. A nonempty subset A is of a Γ-semigroup S called an Γ -ideal or two sided Γ -ideal of S (left and right) if SΓA ⊆ A and AΓS ⊆A [5] . An element s ∈S is called idempotent if $s\alpha s = s$ for all α ∈ Γ,($s\Gamma s$=s). Then, S is said to be a strongly idempotent if for all element in S is an idempotent. A Γ-ideal A of a Γ-semigroup S is said to be globally idempotent (gl-idempotent for short) if AΓA = A. For any subsets A and B of S, then AΓB ={$a\alpha b$: a ∈ A , b ∈ B and α ∈ Γ }. A Γ- ideal P of S is said to be a prime provided that for any two Γ-ideals A, B of S , AΓB ⊆ P , either A ⊆ P or B⊆ P [8]. If S is a gl-idempotent Γ-semigroup then every maximal Γ-ideal of S is a prime Γ-ideal of S **[1]**. Let A be a Γ-ideal of S, then the intersection of all prime Γ-ideals of S containing A is called Γ-radical of A and it's denoted by rad_Γ(A). Let A be ideal of Γ-semigroup S , If $a \in rad_\Gamma$(A) , then $(a\Gamma)^{n-1}a \subseteq$ A for some positive integer n [7]. An Γ-ideal A of a Γ-semigroup S is said to be a maximal Γ-ideal if A is a proper and is not properly contained in any proper Γ-ideal of S. If S is a Γ-monoid then the union of all proper Γ-ideals of S is the unique maximal Γ-ideal of S [8]. In 2016, M.S. Abbas and A. Faris [2] , introduced and studied in the concept of gamma acts which is a generalization of gamma semigroups . A nonempty set M is called left gamma act over S (denoted by S_Γ-act) if there is a mapping S × Γ × M → M defined by (s, α, m) ↦ $s\alpha m$, satisfying $(s_1\alpha s_2)\beta m = s_1\alpha(s_2\beta m)$ for all s_1 , s_2 ∈ S, α, β ∈ Γ and m ∈ M. Similarity, one can define a right gamma acts. If S is a commutative Γ-monoid, then every left S_Γ-act is right S_Γ-act. A non- empty subset N of a left S_Γ-act M is called gamma subact (denoted by S_Γ-subact) if for all s ∈ S, α ∈ Γ and n ∈ N implies that $s\alpha n$ ∈ N. An element θ ∈ M is called a zero of M if $s\alpha$ θ = θ , and if S is a Γ-semigroup with zero then 0αm= θ for all m ∈M ,s ∈ S and α ∈ Γ. Let N be an S_Γ-subact of S_Γ-act M. Then, [N:M] = { s ∈ S | sαm ∈ N for all α ∈ Γ and m ∈ M }. Let M be an S_Γ-act and X a nonempty subset of M [6].Then $[X]_M=\bigcup_{u\in X}$ SΓu where, SΓu = { sαu | s ∈ S and α ∈ Γ},if M = $[X]_M$ then X is said to a generating set of M. Also, if |X| < ∞, then M is finitely generated and M is a cyclic if M = $[\{u\}]_M$ for some u ∈ M**.** Let M be an S_Γ-act. Then ,M is a simple S_Γ-act, if it contain no gamma subact other than M .A Γ-semigroup S is said to be simple if S is S_Γ-act . Let M be an S_Γ-act. A generating set X of M is said to be basis of M, if every element m ∈ M can be uniquely presented in the form m = $s\alpha m_\circ$ for some s ∈ S, α ∈ Γ and $m_\circ$ ∈ X, that is $m = s_1\alpha_1 m_1 = s_2\alpha_2 m_2$ if and only if $s_1 = s_2$, $\alpha_1 = \alpha_2$ and $m_1 = m_2$ [6]. Recently, M. S. Abbas and S. A. Jubrir [3] introduce the concept of multiplication gamma acts and some related concepts. An S_Γ-act M is called a multiplication if every S_Γ-subact N of M is of the form N = AΓM for some Γ-ideal A of S. An S_Γ-act M is multiplication if and only if N=[N:M]ΓM for every S_Γ -subact N of M. For example, cyclic S_Γ -acts are multiplication. A Γ-monoid S, is called multiplication if all its Γ-ideals are multiplication . Let M be a S_Γ-act and s,t ∈ S. Then M is called faithful , if the equality $s\alpha m$ = $t\alpha m$ implies that $s = t$ for all m ∈ M and α∈ Γ [3]. And

M is a globally faithful(gl-faithful for short) if the equality $s\Gamma m$=t$\Gamma m$for all $m \in$ M implies that $s= t$. Let S be a Γ-monoid and M a faithful S_Γ-act. Then M is a multiplication if and only if : $\bigcap_{i\in I}$ (A_i ΓM)= ($\bigcap_{i\in I} A_i$) ΓM for any nonempty collection of Γ-ideals A_i ,($i \in$ I) of S, with ($\bigcap_{i\in I} A_i$) ≠ ∅ ,and for any S_Γ-subact N of M and Γ-ideal A of S such that N ⊂ AΓM there exists an Γ-ideal B with B ⊂ A and N ⊆ BΓM [4].

Now let M be a S_Γ-act. In this paper we define the associated gamma ideal η(M) = $\bigcup_{m\in M}[S\Gamma m{:}\, M]$ of S has proved useful in studying multiplication gamma acts . Also, we present some properties of the gamma ideal η(M) and put together the several characterizations of multiplication gamma acts interms of the gamma ideal η(M). We show tat η(M) is a globally idempotent multiplication gamma ideal. Moreover, we prove that if M is a faithful multiplication gamma act then η(M)= ⋂{A: A is Γ-ideal of S; M=AΓM}= $\mathcal{T}$(M). Using this gamma ideal η(M) and the result that multiplication gamma acts are free we derive most of our results on multiplication gamma acts. For example, we prove that M is a multiplication gamma act if and only if η(M) = S.

2. The Multiplication Gamma acts and the Gamma ideal η(M).

For S_Γ-act M, we define the Γ-ideal η(M) = $\bigcup_{m\in M}[S\Gamma m{:}\, M]$ to be useful in studying multiplication SΓ-acts as we will see in this section. Using the freeness of SΓ-act some properties are discussed.

2.1 **Definition.** Let M be an S_Γ-act .We define η(M) = $\bigcup_{m\in M}[S\Gamma m{:}\, M]$. We note that η(M) is a Γ- ideal of S. In case M is an Γ-ideal of S , it is clear that M ⊆ η(M).

2.2 **Proposition.** Let S be a commutative Γ-semigroup .If M is multiplication S_Γ-act then, M=η(M)ΓM. More generally for any S_Γ-subact N of M ,N= η(M)ΓN.
Proof: Let $m \in$ M , Then SΓm=[SΓm: M]ΓM. Thus, M=$\bigcup_{m\in M}$ SΓm= $\bigcup_{m\in M}$([SΓm: M]ΓM) = ($\bigcup_{m\in M}$[SΓm: M])ΓM=η(M)ΓM. Also, let N be a S_Γ-subact of M .Then there exists Γ-ideal A of S , such that N=AΓM .Hence N=AΓM= AΓ(η(M)ΓM)=(η(M)ΓA)ΓM= η(M)ΓN.

The following example shows that the multiplication gamma act may not free: Let S = ℤ and Γ=ℕ. Then S is S_Γ-act under multiplication of integer numbers. Here ℤ is multiplication ,but it's not free $\mathbb{Z}_\mathbb{N}$-act since {1} is the only generating set of ℤ but 25 ∈ ℤ and 25 = 5.5.1 = 25.1.1, are two distance presentation of 25.

Recall [4] , if M is a S_Γ-act and P a maximal Γ-ideal of S, then we define : T_p(M) = {m ∈M : m= $p\alpha m$ for some $p \in$ P and $\alpha \in$Γ}. Clearly T_p(M) is an S_Γ-subact of M. We say that M is P-cyclic provided there exist $q \in$ S\P such that qΓM ⊆ SΓm , for all $m \in$ M. As generalization of T_p(M) , we can define ,$\overline{T}_p$(M) ={m∈M : m∈$p\Gamma m$,for some $p \in$ P }, it's clear that T_p(M) ⊆ $\overline{T}_p$(M). We proved that, if S is a Γ-monoid ,then M is a multiplication S_Γ-act if and only if for every maximal Γ- ideal P of S, either M = T_p(M) or M is P-cyclic.

Now we introduce the following Theorem:

2.3 **Theorem.** Let S be a commutative Γ-monoid and M a free multiplication S_Γ-act. If A is a Γ-ideal of S, such that M=AΓM , then A∪[**θ**:M]=S.

Proof: Suppose that A ∪ [θ:M]≠S then there a maximal Γ-ideal Q of S, such that A ∪ [θ:M] ⊂ Q. Since M is a multiplication S_Γ-act then either M = T_Q(M) or M is Q-cyclic. First, suppose M = T_Q(M). Since, A ⊆ Q ,then AΓM ⊆ QΓM and hence by hypothesis M=QΓM. Now , let $m \in$ M . Then =$q\beta m_\circ$ for some $q \in Q$, $\beta \in$ Γ and $m_\circ \in$ M.Also, m=a$\alpha m'$ for some $q \in Q$, $\beta \in$ Γ and $m' \in$ M , which is contradiction since M is free. Thus M is Q-cyclic then there exist $q \in$ S\Q such that qΓM ⊆SΓm ,for all $m \in$ M. Let θ∈ M .Then, qΓM ⊆ SΓθ = θ . So, qΓM ⊆ θ .Thus , $q \in$ [θ:M] . It follows that $q \in$ A ∪ [θ:M] ⊂ Q , a contradiction. Therefore, A ∪ [θ:M]=S.

2.4 **Proposition.** Let A be a finitely generated Γ-ideal of commutative Γ-monoid S ,and M a free multiplication S_Γ-act . If A ⊆ η(M) , then AΓM is finitely generated .
Proof: Since A=$\bigcup_{i=1}^{n}$SΓa_i.Then there exists $x_i \in$ M (1≤ i ≤n) such that $a_i \in$ [SΓx_i: M] and hence A ⊆ $\bigcup_{i=1}^{n}$[SΓx_i: M].Then,

$A\Gamma M \subseteq \bigcup_{i=1}^{n}([S\Gamma x_i : M]\Gamma M) = \bigcup_{i=1}^{n} S\Gamma x_i$, ……….. (⋆)

Put B= $\bigcup_{i=1}^{n} S\Gamma x_i$. By Proposition(2.2), B=η(M) ΓB ,and by Theorem (2.3), η(M) ∪ [θ:B]=S. There is element t ∈ η(M) or t ∈[θ:B] such that t=1. Hence, there exist y_1 , y_2, …, y_m ∈ M such that S= η(M) ∪ [θ:B] = $\bigcup_{i=1}^{m}[S\Gamma y_i : M]$ ∪ [θ:B]. Thus ,

AΓM= AΓ(SΓM)=SΓ(AΓM) = ($\bigcup_{i=1}^{m}[S\Gamma y_i : M]$∪[θ:B])Γ(AΓM) = $\bigcup_{i=1}^{m}([S\Gamma y_i : M]\Gamma A\Gamma M)$∪([θ:B]ΓAΓM) = ($\bigcup_{i=1}^{m} A\Gamma[S\Gamma y_i : M]\Gamma M$ ∪[θ:B]ΓAΓM) = $\bigcup_{i=1}^{m} A\Gamma(S\Gamma y_i)$= $\bigcup_{i=1}^{m}(A\Gamma S)\Gamma y_i$= $\bigcup_{i=1}^{m} S\Gamma(A\Gamma y_i)$ as [θ:B]ΓAΓM= θ by (⋆). Therefore, AΓM is finitely generated .

2.5 **Proposition.** Let A be a Γ-ideal of a commutative Γ-monoid S and M a free multiplication S_Γ-act . Then η(M)= S . In particular A ⊆ η(M) .

Proof: By Proposition(2.2), AΓM=η(M) Γ(AΓM) . Hence by Theorem (2.3), η(M) ∪ [θ:AΓM] =S .Thus, η(M)ΓA ∪ [θ:AΓM]ΓA=S. Let $s\alpha a$ ∈ [θ:AΓM]ΓA where s ∈ [θ:AΓM], α ∈ Γ and a ∈ A .Then, $s\Gamma(a\Gamma M)$ ⊆ sΓ(AΓM) ⊆ θ. It follows that $s\Gamma a$ ⊆ [θ :M] . Thus, [θ:AΓM]ΓA ⊆ [θ :M].So , [θ:AΓM]ΓA ⊆ [θ:M] ⊆ η(M) . Also, η(M)ΓA⊆ η(M) Thus , S=η(M)ΓA ∪ [θ:AΓM]ΓA ⊆ η(M) and hence η(M) =S.

2.6 **Proposition.** Let S be a commutative Γ-monoid ,and M a S_Γ-act. Then the followings are equivalent :

i. M is a multiplication.

ii. If M is free, then η(M) = S.

iii. For any S_Γ-subact N of M ,N= η(M)ΓN.

iv. For every maximal Γ-ideal P of S, either M = T_p(M) or M is P-cyclic

Proof: (i) ⟹ (ii) It's clear by Proposition(2.5).

(ii) ⟹ (iii) Let N be a S_Γ-subact of M ,and η(M) = S ,then η(M)ΓN=SΓN= N.

(iii) ⟹ (iv) Let P be a maximal Γ-ideal of S. If η(M) ⊆ P, let m∈M then SΓm is a S_Γ-subact of M . By assumption SΓm = η(M) Γ(SΓm) so there is p∈ η(M) ⊆ P, and α ∈ Γ such that m=$p\alpha m$, thus m∈ T_p(M), Therefore M=T_p(M) .If η(M) ⊈ P. In this case [SΓm: M] ⊈ P for all m∈M . This implies that there is q∈ [SΓm: M] and q ∉ P. It follows that q ∈ S\P such that qΓM ⊆ SΓm ,for all m ∈ M. Hence , M is P-cyclic.

(iv) ⟹ (i) It is clear .

2.7 **Proposition.** Let A be a Γ-ideal of a commutative Γ-monoid S and M a free multiplication S_Γ-act .If A ⊆ η(M) , then A∪[θ:M]= AΓη(M) ∪[θ:M].

Proof: It is sufficient to prove that A ⊆ AΓ η(M)∪[θ:M]. Let a ∈ A . Then aΓM= η(M) Γ(aΓM).By Theorem (2.3), η(M) ∪[θ:aΓM] =S. So, 1∈ η(M) ∪[θ:aΓM] it follows that 1∈ η(M) or 1 ∈ [θ:aΓM] and hence a=$a\alpha$1 ∈ AΓη(M) or 1Γ(aΓM) = (1Γa)ΓM ⊆ θ . So, a=$a\alpha$1 ∈ [θ:M]. Therefore , a ∈ AΓη(M) ∪[θ: M].Hence , A∪[θ:M]= AΓη(M) ∪[θ:M].

2.8 **Corollary.** Let S be a commutative Γ-monoid and M a faithful free multiplication S_Γ-act . Then

i. η(M) is a multiplication Γ-ideal of S.

ii. η(M) is an gl-idempotent Γ-ideal of S.

Proof: (i) Let A be a Γ-ideal of S, such that A ⊆ η(M) .Then Proposition(2.7), A∪[θ:M] = AΓη(M) ∪[θ:M] and hence AΓM∪[θ:M]ΓM = AΓ η(M)ΓM∪ [θ:M]ΓM .So, AΓM=AΓη(M) ΓM as [θ:M]ΓM=θ and hence , A=AΓη(M) .

(ii) By Proposition(2.7), as A= η(M) we have

η(M)∪[θ:M] = η(M)Γη(M) ∪[θ:M]

(η(M)∪[θ:M])ΓM = (η(M)Γη(M)∪[θ:M])ΓM

η(M)ΓM ∪ [θ:M]ΓM = [η(M)Γη(M)]ΓM ∪ [θ:M]ΓM .

Thus, η(M)ΓM= [η(M)Γη(M)]ΓM. By faithfulness of M, η(M) = η(M) Γη(M) .

Now, we introduce the concept of $\mathcal{T}$**(M)** and we give some properties about it. Also, we study the relation between it and η(M) .

2.9 **Definition.** Let M be an S_Γ-act and $\mathcal{T}$={A: A is Γ-ideal of S; M=AΓM} .Define $\mathcal{T}(M)=\bigcap_{A\in\mathcal{T}} A$ .If M is a faithful multiplication S_Γ-act, then M= $\mathcal{T}$(M)ΓM, since $\mathcal{T}(M)\Gamma M = (\bigcap_{A\in\mathcal{T}} A)\Gamma M = \bigcap_{A\in\mathcal{T}}(A\Gamma M) = M$.

In following some properties of $\mathcal{T}$(M) are given.

2.10 **Proposition.** Let S be a commutative Γ-monoid ,and M be a faithful multiplication S_Γ-act. Then:

i. $m \in \mathcal{T}$(M)Γm for each $m\in$ M.
ii. $\mathcal{T}$(M) is a gl-idempotent Γ-ideal of S.
iii. M is a multiplication $\mathcal{T}(M)_\Gamma$-act .
iv. M ≠ QΓM for each proper Γ-ideal Q of $\mathcal{T}$(M).

Proof:(i) Let $m \in$ M. Then SΓm = BΓM for some Γ-ideal B of S. Thus ,SΓm = BΓM = BΓ($\mathcal{T}$(M)ΓM) =$\mathcal{T}$(M)Γ(BΓM)= $\mathcal{T}$(M)Γ(SΓm) ⊆ $\mathcal{T}$(M)Γm ,and hence $m \in \mathcal{T}$(M)Γm.

(**ii**) M= $\mathcal{T}$(M)ΓM = $\mathcal{T}$(M)Γ($\mathcal{T}$(M)ΓM)= ($\mathcal{T}$(M)Γ$\mathcal{T}$(M))ΓM which implies , $\mathcal{T}$(M) = $\mathcal{T}$(M)Γ$\mathcal{T}$(M) by the definition of $\mathcal{T}$(M).

(**iii**) By (**i**) Define the mapping:

$\mathcal{T}$(M) × Γ × M → M by (a,α, m) ⟶ $a\alpha m = m$ Clearly ,M is $\mathcal{T}(M)_\Gamma$-act .Let N be a $\mathcal{T}(M)_\Gamma$-subact of M. Then, by (i) N= $\mathcal{T}$(M)ΓN .Now, we show that N is an S_Γ-subact of M. Let $s\alpha n$ ∈SΓN where $s \in S$, $\alpha \in \Gamma$ and $n \in$ N. Thus, $s\alpha n$ ∈SΓN= SΓ($\mathcal{T}$(M)ΓN)= (SΓ$\mathcal{T}$(M))ΓN= SΓ($\mathcal{T}$(M)ΓN)=N .
Hence, N is S_Γ-subact of M. Thus, there exists an Γ-ideal C of S such that N = CΓM and hence N=CΓM = CΓ($\mathcal{T}$(M)ΓM)= (CΓ$\mathcal{T}$(M))ΓM. But CΓ$\mathcal{T}$(M) is an Γ-ideal of $\mathcal{T}$(M). It follows that M is a multiplication $\mathcal{T}(M)_\Gamma$-act.

(**iv**) Let Q be a proper Γ-ideal of $\mathcal{T}$(M) such that M = QΓM. Then M = QΓ($\mathcal{T}$(M)ΓM) = (QΓ$\mathcal{T}$(M))ΓM and QΓ$\mathcal{T}$(M) is an Γ-ideal of S. By the definition of $\mathcal{T}$(M) we have $\mathcal{T}$(M) ⊆ QΓ$\mathcal{T}$(M) ⊆Q ⊆ $\mathcal{T}$(M), that is $\mathcal{T}$(M)=Q.

2.11 **Theorem.** Let S be a commutative Γ-monoid and M a faithful multiplication S_Γ-act . Then $\mathcal{T}$(M) = η(M) .
Proof: Since M is faithful multiplication then, M= $\mathcal{T}$(M)ΓM .By Proposition (2.2) , M= η(M)ΓM. It follows that $\mathcal{T}$(M)ΓM= η(M)ΓM ,and hence $\mathcal{T}$(M) = η(M).

2.12 **Corollary .** Let S be a commutative Γ-monoid and M a faithful multiplication S_Γ-act . Then, η (η(M)) = η(M). **Proof:** By Corollary(2.8), η(M) is a faithful multiplication Γ-ideal of S and η(M) = η(M)Γη(M) .So, by Theorem(2.11) , η(η (M))= $\mathcal{T}$(η (M)) ⊆ η(M) and since η(M) ⊆ η (η(M)). Therefore, η (η(M))= η(M) .

Now , we inteoduce the following Lemma and Theorem which are need in or work .

2.13 **Lemma.** Let A be a Γ-ideal of Γ-semigroup S .Then for any collection of Γ-ideals {B_i: $i\in$ I} of S, we have

i. $[A : \bigcup_{i\in I} B_i] = \bigcap_{i\in I}[A : B_i]$
ii. $[\bigcap_{i\in I} B_i : A] = \bigcap_{i\in I}[B_i : A]$.

Proof. Clear.

2.14 **Theorem**. Let S be gl-idempotent Γ-monoid and A , B are Γ-ideals of S with M is a gl- faithful multiplication S_Γ-act. Then AΓM ⊆ BΓM if and only if M=[B:A]ΓM.

Proof:(⟸)Assume that M=[A:B]ΓM. Then AΓM=AΓ([A:B]ΓM)=(AΓ[A:B])ΓM=([A:B]ΓA)ΓM ⊆ BΓM.
(⟹)Suppose that AΓM ⊆ BΓM. First we show that [B:SΓa]ΓM=M for all $a\in$ A . If [B: SΓa]=S ,then it's clear that [B :SΓa]ΓM=M. Thus, let [B : SΓa]≠ S. Suppose M≠PΓM for some prime Γ-ideal P of S ,such that [B :SΓa] ⊆ P. Then there exists $x\in$ M with $x \notin$ PΓM. Since M is multiplication, then SΓx=[SΓx: M] ΓM . Which implies that [SΓx: M] ⊄P. Thus there is $c\in$ S\P such that cΓM ⊆ SΓx. Now, cΓ(aΓM) ⊆ cΓ(AΓM) ⊆ cΓ(BΓM)=BΓ(cΓM) ⊆ BΓ(SΓx)=BΓx, thus (cαa)βx = bγx for some b∈B and $\alpha, \gamma, \beta \in \Gamma$ .Since M is gl-faithful ,then (cαa)=b and hence $c\alpha a$ ∈ B .It follows cα(1$\alpha\circ a$)∈ B. So, we have cΓ(SΓa) ⊆ B .Thus $c \in$ [B : SΓa] ⊆ P. Hence, $c \in$ P which is a contradiction .Therefore, M=PΓM for all prime Γ-ideal P of S, with [B : SΓa] ⊆ P. Since M is a gl-faithful then , M= ⋂(PΓM)= (⋂P)ΓM . Hence, M = rad_Γ([B: SΓa])ΓM. Now, let $m \in$ M then,

$$S\Gamma m = [S\Gamma m:M]\Gamma M = [S\Gamma m:M]\Gamma(rad_\Gamma([B : S\Gamma a])\Gamma M)$$
$$= rad_\Gamma([B : S\Gamma a])\Gamma([S\Gamma m:M]\,\Gamma M)$$
$$= rad_\Gamma([B : S\Gamma a])\Gamma(S\Gamma m) .$$

Hence, $m = s\alpha m$ for some, s ∈ rad_Γ([B : SΓa]) and $\alpha \in \Gamma$. Thus there exist a positive integer such that $(s\Gamma)^{n-1}s \subseteq$ [B : SΓa] and hence $(s\alpha)^{n-1}s \in$ [B : SΓa] for all $\alpha \in \Gamma$. It follows, $= s\alpha m = \cdots = ((s\alpha)^{n-1}s)\alpha m \in$ [B: SΓa]ΓM . Therefore , M=[B:SΓa]ΓM. Finally, by Lemma(2.13), [B :A]ΓM=[B : $\bigcup_{a\in A} S\Gamma a$]ΓM = $(\bigcap_{a\in A}[B : S\Gamma a])\Gamma M = \bigcap_{a\in A}([B : S\Gamma a]\Gamma M) = M$.

2.15 **Theorem.** Let S be a gl-idempotent Γ-monoid and M be a gl-faithful multiplication S_Γ-act. If A and B are Γ-ideals of S, which contained in η(M) then AΓM = BΓM if and only if A=B.

Proof: (⟸) Clear .

(⟹) Let AΓM=BΓM .Then by Theorem(2.14), M=[B:A]ΓM = M=[A:B]ΓM . Thus, by Theorem(2.11), η(M) = $\mathcal{T}$(M) ⊆ [B:A]. Since A ⊆ η(M) then by Proposition(2.7), A= AΓη(M) ⊆ AΓ[B:A]⊆ B . Similarly , B ⊆ A . Hence, A=B.

2.16 **Corollary.** Let M be a faithful multiplication S_Γ-act. If A and B are Γ-ideals of S, which contained in η(M) ,then

i. AΓM = BΓM if and only if A ∩ η(M) = B ∩ η(M) .
ii. A ∩ η(M) =AΓη(M) .

Proof:(**i**) (⟹)Suppose that AΓM = BΓM . Since M is a faithful multiplication then (A ∩ η(M))ΓM = AΓM ∩ η(M)ΓM=BΓM ∩ η(M)ΓM = (B ∩ η(M))ΓM . Therefore, by Theorem(2.15), A ∩ η(M) = B ∩ η(M).

(⟸) Let A ∩ η(M) = B ∩ η(M) .Since A⊆ η(M) thus,

AΓM=AΓM ∩ η(M) ΓM=(A ∩ η(M))ΓM
= (B ∩ η(M))ΓM = BΓM ∩ η(M)ΓM= BΓM.

(**ii**) (A ∩ η(M))ΓM = AΓM ∩ η(M)ΓM=AΓM=AΓ(η(M) ΓM)= (AΓη(M))ΓM. Thus, by Theorem (2.15) , A ∩ η(M)=AΓη(M) .

2.17 **Corollary.** Let M be a gl-faithful multiplication S_Γ-act, and A be a Γ-ideal of S contained in η(M). Then

i. [AΓM:M]=[A: η(M)] .
ii. A ∩ η(M) =[A: η(M)] ∩ η(M) .

Proof: (**i**) Let s∈ [A : η(M)]. Then sΓη(M) ⊆ A, which implies that sΓ(η(M)ΓM) ⊆ AΓM ,and hence sΓM ⊆ AΓM. So, s ∈ [AΓM:M]. Hence, [A : η(M)]⊆ [AΓM:M] .Conversely, since M is a multiplication, AΓM=[AΓM:M]ΓM .By Theorem (2.15), [AΓM:M]=A . Hence, [AΓM:M]Γη(M) =AΓη(M) ⊆ A. Therefore, [AΓM:M]=[A: η(M)].

(**ii**) By Corollary(2.16)(ii) [A: η(M)] ∩ η(M)=[A: η(M)]Γη(M)
=[A∩ η(M): η(M)]Γη(M)=A ∩ η(M).

As η(M) is a multiplication Γ-ideal by Corollary(2.8).

Recall [4], if S is a Γ-monoid and M a faithful S_Γ-act, then, M is a multiplication if and only if AΓM is a multiplication S_Γ-act for all multiplication Γ-ideal A of S.

2.18 **Theorem.** Let A be a Γ-ideal of commutative Γ-monoid S and N a S_Γ-subact of a faithful multiplication S_Γ-act M .Then ,

i. N is a multiplication if and only if [K:M]Γη(M) = [K:N]Γ[N:M]Γη(M) for each S_Γ-subact K of N.
ii. AΓη(M) =[AΓM:M]Γ η(M) .
iii. AΓM is a multiplication S_Γ-subact if and only if AΓη(M) is a multiplication Γ-ideal of S.

Proof: (**i**) (⟹) Let N be a multiplication and K an S_Γ-subact of N. Then, K=[K:N]ΓN=[K:N]Γ([N:M]ΓM). Also, K=[K:M]ΓM. Thus, ([K:N]Γ[N:M])ΓM =[K:M]ΓM. Since M is a faithful ,then ([K:N]Γ[N:M])Γη(M)=[K:M]Γη(M) .

(⟸) Suppose [K:M]Γη(M) = [K:N]Γ[N:M]Γη(M) for each S_Γ-subact K of N. Let L be a S_Γ-subact of N. Then L is a S_Γ-subact of M.

L=[L:M]ΓM =([L:M]Γη(M))ΓM =([L:N]Γ[N:M]Γ η(M))ΓM = [L:N]Γ[N:M]ΓM =[L:N]ΓN.

Thus, N is a multiplication S_Γ-act.

(**ii**) AΓ(η(M)ΓM)=AΓM=[AΓM:M]ΓM=([AΓM:M]Γη(M))ΓM. By faithfulness of M, AΓη(M)=[AΓM:M]Γη(M).

(**iii**) (⟸) Suppose AΓη(M) is a multiplication Γ-ideal of S. Then ,AΓM= AΓ(η(M)ΓM)= (AΓη(M))ΓM is a multiplication S_Γ-act .

(⟹) Assume that AΓM is a multiplication. Let B be a Γ-ideal of S such that B ⊆ AΓη(M). Since BΓM = BΓ(η(M)ΓM) then B= BΓη(M). Since AΓM is a multiplication and BΓM ⊆ (AΓη(M))ΓM =AΓM. Then , by (i) and (ii) ,we get:

B=BΓη(M) =[BΓM:M]Γη(M) =[BΓM: AΓM]Γ[AΓM:M]Γη(M) =[BΓM: AΓM]Γ(AΓη(M)) .

Therefore, AΓη(M) is a multiplication Γ-ideal of S.

2.19 **Corollary.** Let M be a faithful multiplication S_Γ-act and N an S_Γ-subact of M. Then N is a multiplication if and only if [N: M]Γη(M) is a multiplication Γ-ideal of S.

Proof: Clear.

2.20 **Corollary.** Let M be a faithful free multiplication S_Γ-act and N an S_Γ-subact of M. If N is multiplication then for any S_Γ-subact K of M, [K:M] =[K:N]Γ[N:M].

Proof: Let N be a multiplication S_Γ-subact. Then by Theorem(2.18), (i) [K:M]Γη(M)=[K:N]Γ[N:M]Γη(M). By Proposition (2.5), [K:M] ⊆ η(M). Thus, [K:M]=[K:M]Γη(M),by (Proposition(2.7)). Since [K:N]Γ[N:M]⊆[K:M] ⊆ η(M) .Then, [K:N]Γ[N:M]=[K:N]Γ[N:M]Γη(M)=[K:M]Γη(M) =[K:M] .

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