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$\widetilde{\mathrm{T}}$-Transformation for n^{th}- Order Ordinary Differential Equations

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Abstract: *Fuzzy Tarig transform ($\tilde{T}$-Transformation) method be used in this paper to estimate the exact solutions of a fuzzy differential equations with generalized differentiability of Hukuhara. Moreover, this fuzzy integral is more suitable and accurate for solving fuzzy n^{th} order differential equations due to that fuzzy transform such as Laplace Sumudu and Tarig reduce the ordinary differential equation to an algebraic systems. To explain this approach, some important concepts and theorems are discussed in this work associated with some examples.*

Keywords: Fuzzy differential equation; fuzzy n^{th}-order differential equations; fuzzy Tarig transformation.

1. Inroduction

Fuzzy set theory is a important method for modeling. Especially, the linear equations with constant coefficients are worth studying due to they fit as mathematical models for famous physical problems and another fields such as real problems in the golden mean [5], particle systems [8], quantum mechanics and gravity [9], synchronize hyperchaotic systems [17], unstable systems [10,15,166], medicine [1,3], difficulties in engineering [11]. Actually, fuzzy transformation was proposed as a pilot fuzzy approximation approach to apply in various application fields such as numerical solution of ordinary differential equations. In recent years, many researchers in the theoretical and functional fields did several works (see [4,6 ,13,14,17]).

2.Basic Concepts

The basic definitions of a fuzzy number are given as follows:

Definition 1. [2]

A fuzzy number is a fuzzy set like : $R \to [0, 1]$ which satisfies:

1. π is an upper semi-continuous function,
2. $\pi(\lambda) = 0$ outside some interval [a,d],
3. A real numbers x, y, z, w such as $x \leq y \leq z \leq w$ and

3.1 $\pi(\lambda)$ a function is monotonic increasing on $[x, y]$,
3.2 $\pi(\lambda)$ a function is monotonic decreasing on $[z, w]$,
3.3 $\pi(\lambda) = 1$ for all $\lambda \in [y, z]$.

4. π is upper semi-continuous,
5. π is fuzzy convex,
6. π is normal,
7. supp(A) is the support of the π , and its closure cl(supp(A)) is compact.

Definition 2. [14,7] The metric structure is given by the distance from Hausdorff to satisfy the following properties, that R_F is denoted the class of fuzzy subsets of real axis:

$\Pi : R_F \times R_F \to R_+ \cup 0$, $\Pi(\rho(r), v(r)) = \mathrm{Max}\{\sup|\underline{\rho} - \underline{v}|, \sup|\overline{\rho} - \overline{v}|\}$, (R_F,Π) is a complete metric space and following properties are well known:

$\Pi(\rho + \omega, v + \omega) = \Pi(\rho, v)$, $\forall \rho, v, \omega \in R_F$.
$\Pi(k\rho, kv) = |k|\Pi(\rho, v)$, $\forall \rho, v \in R_F$, $\forall k \in R$.
$\Pi(\rho+v,\omega+e) \leq \Pi(\rho,\omega)+\Pi(v, e)$, $\forall \rho, v,\omega, e \in R_F$.

Definition 3. [2]

Let $m, n \in R_F$. If there exists $z \in R_F$ such that $m = n + z$ then z is called the H-differential of m, n and it is denoted by $m \ominus n$.

Definition 4. [2]

Suppose $\Omega(m)$ be a fuzzy valued function on [a, b]. Suppose that $\Omega(m, n)$ and $\Omega(m, n)$ are improper Rieman-integrable on [a, b] then we say that $\Omega(x)$ is improper on [a, b], furthermore,

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$\underline{(\int_a^b \Omega(m,n)dm)} = (\int_a^b \underline{\Omega(m,n)}dm)$, $\overline{(\int_a^b \Omega(m,n)dm)}$=$(\int_a^b \overline{\Omega(m,n)}dm)$.

3. Generalization of $\tilde{T}$-Transformation

Theorem 1. [2]

Let f(t) be a fuzzy valued function on [a,∞) represented by $(\underline{\Gamma}(t,\propto),\overline{\Gamma}(t,\propto))$. For any fixed $\propto\in[0,1]$, assume $\underline{\Gamma}(t,\propto)$ and $\overline{\Gamma}(t,\propto)$ are Riemann-integrable on [a,b] for every b≥ a, and there are two positive functions $\underline{M}(\propto)$ and $\overline{M}(\propto)$ such that $\int_a^b|\underline{\Gamma}(t,\propto)|dt\le\underline{M}(\propto)$ and $\int_a^b|\overline{\Gamma}(t,\propto)|dt\le\overline{M}(\propto)$ for every b≥ a. Then Γ(t) is improper fuzzy Riemann-integable on [a,∞) and the improper fuzzy Riemann-integral is fuzzy number. Furthermore, we have : $\int_a^\infty\Gamma(t)dt=\left(\int_a^\infty\underline{\Gamma}(t,\propto)dt,\int_a^\infty\overline{\Gamma}(t,\propto)dt\right)$.

Proposition 1. [2]

If each of Γ(t) and Φ(t) are fuzzy valued functions and fuzzy Riemann-integable on [a,∞) then Γ(t)+Φ(t) is fuzzy Riemann-integable on [a,∞). Moreover, we have: $\int_1(\Gamma(t)+\Phi(t))dt$=$\int_1\Gamma(t)dt+\int_1\Phi(t)dt$.

Theorem 2. [12]

Suppose that $\mu(\tau),\mu^{`}(\tau),\dots,\mu^{(n-1)}(\tau)$ are differentiable fuzzy valued functions such that $\mu^{i_1}(\tau),\mu^{i_2}(\tau),\dots,\mu^{i_m}(\tau)$ are (ii)-differentiable functions for $0\le i_1\le i_2\le\cdots\le i_m\le n-1$, $0\le m\le n$, $\mu^{(p)}(\tau)$ is (i)-differentiable for $p\ne i_j, j = 1,2,\dots,m$ and $\mu(\tau)$ is denoted by $\mu(\tau) = [\underline{\mu}(\tau),\overline{\mu}(\tau)]$, then:

(a) If m is an even number then $\mu^{(n)}(\tau) = [\underline{\mu^{(n)}}(\tau),\overline{\mu^{(n)}}(\tau)]$.

(b) If m is an odd number then $\mu^{(n)}(\tau) = \left[\overline{\mu^{(n)}}(\tau),\underline{\mu}^{(n)}(\tau)\right]$.

Theorem 3.

Suppose that $\mu(\tau),\mu^{`}(\tau),\dots,\mu^{(n-1)}(\tau)$ continuous fuzzy valued functions [0,∞) and of exponential order and that $\mu^{(n)}(\tau)$ is piecewise continuous fuzzy-valued function on [0,∞).$\mu^{i_1}(\tau),\mu^{i_2}(\tau),\dots,\mu^{i_m}(\tau)$ are (ii)-differentiable functions for $0\le i_1\le i_2\le\cdots\le i_m\le n-1$, and $\mu^{(p)}(\tau)$ is (i)-differentiable for $p\ne i_j, j = 1,2,\dots,m$, and if θ-cut representation of fuzzy- valued function $\mu(\tau)$ is denoted by $\mu(\tau) = [\underline{\mu}(\tau),\overline{\mu}(\tau)]$, then

1) m is an even number ,we have $\tilde{T}[\mu^{(n)}(\tau)] = \frac{T[\mu(\tau)]}{u^{2n}}\ominus\frac{\mu(0)}{u^{2n-1}}\otimes\sum_{k=1}^{n-1}\frac{\mu^{(k)}(0)}{(u)^{2n-2k-1}}$,

------------------------ (1)

such that

$$\otimes\begin{cases}\ominus, & if\ the\ number\ of\ (ii)-differentiable\ functions\\ & \mu^{(i)} provided\ i<k\ is\ an\ even\ number\\ -, & if\ the\ number\ of\ (ii)-differentiable\ functions\\ & \mu^{(i)}, provided\ i<k\ is\ an\ odd\ number\end{cases}\qquad(2)$$

2) m is an odd number ,we have

$$\tilde{T}[\mu^{(n)}(\tau)] = \frac{-\mu(0)}{u^{2n-1}}\ominus\frac{-T[\mu(\tau)]}{u^{2n}}\otimes\sum_{k=1}^{n-1}\frac{\mu^{(k)}(0)}{(u)^{2n-2k-1}}\qquad(3)$$

Such that

$$\otimes\begin{cases}\ominus, & if\ the\ number\ of\ (ii)-differentiable\ functions\\ & \mu^{(i)} provided\ i<k\ is\ an\ even\ number\\ -, & if\ the\ number\ of\ (ii)-differentiable\ functions\\ & \mu^{(i)}, provided\ i<k\ is\ an\ odd\ number\end{cases}\qquad(4)$$

Proof (1): Let $\mu^{i_1}(\tau),\mu^{i_2}(\tau),\dots,\mu^{i_m}(\tau)$ be (ii)-differentiable functions and m be an even number, then by theorem (2/a), we get

$$\mu^{(n)}(\tau) = [\underline{\mu}^{(n)}(\tau,\theta),\overline{\mu}^{(n)}(\tau,\theta)].$$

Therefore, $\underline{\mu}^{(n)}(\tau,\theta) = \underline{\mu^{(n)}}(\tau,\theta), \overline{\mu}^{(n)}(\tau,\theta) = \overline{\mu^{(n)}}(\tau,\theta)$. Thus

$\tilde{T}\left[\mu^{(n)}(\tau)\right]=\tilde{T}(\underline{\mu}^{(n)}(\tau,\theta),\overline{\mu}^{(n)}(\tau,\theta)) = (T(\underline{\mu}^{(n)}(\tau,\theta)),T(\overline{\mu}^{(n)}(\tau,\theta))),$ (5)

by the relationship between fuzzy Laplace and Tarig transformations :
we know $\mathcal{G}(\mathfrak{U},\theta)=\tilde{T}[\mu(\tau)]$, $\mathcal{F}(p)=L[\mu(\tau)]$.
In general, $\mathcal{G}_n(\mathfrak{U},\theta)=\tilde{T}[\mu^{(n)}(\tau)]$, $\mathcal{F}_n(p)=L[\mu^{(n)}(\tau)]$:

$$\mathcal{G}_n(\mathfrak{U},\theta)=\tilde{T}\left[\mu^{(n)}(\tau)\right]=\frac{\mathcal{F}_n\left(\frac{1}{\mathfrak{U}^2}\right)}{\mathfrak{U}}$$

$$\mathcal{G}_n(\mathfrak{U},\theta)=\frac{1}{\mathfrak{U}}[(\frac{1}{\mathfrak{U}^2})^{2n}\mathcal{F}\left(\frac{1}{\mathfrak{U}^2}\right)\ominus(\frac{1}{\mathfrak{U}^2})^{2n-1}\mu(0)\otimes\sum_{k=1}^{n-1}\frac{\mu^{(k)}(0)}{(u^2)^{n-k-1}}]$$

$$=\frac{\mathcal{G}(\mathfrak{U},\theta)}{\mathfrak{U}^{2n}}\ominus\frac{\mu(0)}{\mathfrak{U}^{2n-1}}\otimes\sum_{k=1}^{n-1}\frac{\mu^{(k)}(0)}{(\mathfrak{U})^{2n-2k-1}}.$$

From the ordinary differential equations, we have

$$T\left(\underline{\mu}^{(n)}(\tau,\theta)\right)=\frac{T[\underline{\mu}(\tau,\theta)]}{u^{2n}}-\frac{\underline{\mu}(0,\theta)}{u^{2n-1}}-\sum_{k=1}^{n-1}\frac{\underline{\mu}^{(k)}(0,\theta)}{(u)^{2n-2k-1}}$$

which can be written as:

$$T\left(\underline{\mu}^{(n)}(\tau,\theta)\right)=\frac{T\left[\underline{\mu}(\tau,\theta)\right]}{u^{2n}}-\frac{\underline{\mu}(0,\theta)}{u^{2n-1}}-\sum_{k=1}^{i_1}\frac{\underline{\mu}^{(k)}(0,\theta)}{(u)^{2n-2k-1}}-\sum_{k=i_1+1}^{i_2}\frac{\underline{\mu}^{(k)}(0,\theta)}{(u)^{2n-2k-1}}-\sum_{k=i_2+1}^{i_3}\frac{\underline{\mu}^{(k)}(0,\theta)}{(u)^{2n-2k-1}}-\dots-\sum_{k=i_m+1}^{n-1}\frac{\underline{\mu}^{(k)}(0,\theta)}{(u)^{2n-2k-1}}$$

----------------------------------- (6)

In a similar way, we can get

$$T\left(\overline{\mu}^{(n)}(\tau,\theta)\right)=\frac{T[\overline{\mu}(\tau,\theta)]}{u^{2n}}-\frac{\overline{\mu}(0,\theta)}{u^{2n-1}}-\sum_{k=1}^{i_1}\frac{\overline{\mu}^{(k)}(0,\theta)}{(u)^{2n-2k-1}}-\sum_{k=i_1+1}^{i_2}\frac{\overline{\mu}^{(k)}(0,\theta)}{(u)^{2n-2k-1}}-\sum_{k=i_2+1}^{i_3}\frac{\overline{\mu}^{(k)}(0,\theta)}{(u)^{2n-2k-1}}-\dots-\sum_{k=i_m+1}^{n-1}\frac{\overline{\mu}^{(k)}(0,\theta)}{(u)^{2n-2k-1}}$$

----------------------------------- (7)

Since $0\le i_1\le i_2\le\cdots\le i_m\le n-1$ we can apply theorem (2/a) for each $[\mu^{(k+1)}(\tau)]$ where $1\le k\le n-1$ as following:

$\underline{\mu}^{(k)}(0,\theta)=\underline{\mu^{(k)}}(0,\theta)$, $\overline{\mu}^{(k)}(0,\theta)=\overline{\mu^{(k)}}(0,\theta), 1\le k\le i_1,$
$\overline{\mu}^{(k)}(0,\theta)=\underline{\mu^{(k)}}(0,\theta)=,\underline{\mu}^{(k)}(0,\theta)=\overline{\mu^{(k)}}(0,\theta), i_1+1\le k\le i_2,$
$\underline{\mu}^{(k)}(0,\theta)=\underline{\mu^{(k)}}(0,\theta),\overline{\mu}^{(k)}(0,\theta)=\overline{\mu^{(k)}}(0,\theta), i_2+1\le k\le i_3$,
⋮
$\underline{\mu}^{(k)}(0,\theta)=\underline{\mu^{(k)}}(0,\theta)$, $\overline{\mu}^{(k)}(0,\theta)=\overline{\mu^{(k)}}(0,\theta), i_3+1\le k\le n-1$.

The last equations yields from theorem (2/a) due to that m is an even number. Substituting (6) and (7) in (5) to get:

$$\tilde{T}\left[\mu^{(n)}(\tau)\right]=$$
$$\left(\frac{T\left[\underline{\mu}(\tau,\theta)\right]}{u^{2n}}-\frac{\underline{\mu}(0,\theta)}{u^{2n-1}}-\sum_{k=1}^{i_1}\frac{\underline{\mu^{(k)}}(0,\theta)}{(u)^{2n-2k-1}}-\sum_{k=i_1+1}^{i_2}\frac{\overline{\mu^{(k)}}(0,\theta)}{(u)^{2n-2k-1}}-\sum_{k=i_2+1}^{i_3}\frac{\underline{\mu^{(k)}}(0,\theta)}{(u)^{2n-2k-1}}-\dots-\sum_{k=i_m+1}^{n-1}\frac{\underline{\mu^{(k)}}(0,\theta)}{(u)^{2n-2k-1}},\frac{T[\overline{\mu}(\tau,\theta)]}{u^{2n}}-\frac{\overline{\mu}(0,\theta)}{u^{2n-1}}-\sum_{k=1}^{i_1}\frac{\overline{\mu^{(k)}}(0,\theta)}{(u)^{2n-2k-1}}-\sum_{k=i_1+1}^{i_2}\frac{\underline{\mu^{(k)}}(0,\theta)}{(u)^{2n-2k-1}}-\sum_{k=i_2+1}^{i_3}\frac{\overline{\mu^{(k)}}(0,\theta)}{(u)^{2n-2k-1}}\dots-\sum_{k=i_m+1}^{n-1}\frac{\overline{\mu^{(k)}}(0,\theta)}{(u)^{2n-2k-1}}.$$

(2) Let $\mu^{i_1}(\tau),\mu^{i_2}(\tau),\dots,\mu^{i_m}(\tau)$ be (ii)-differentiable functions and m be an odd number, then by theorem (2/b), we get

$$\mu^{(n)}(\tau)=[\overline{\mu}^{(n)}(\tau,\theta),\underline{\mu}^{(n)}(\tau,\theta)].$$

Therefore, $\underline{\mu}^{(n)}(\tau,\theta)=\overline{\mu^{(n)}}(\tau,\theta),\underline{\mu^{(n)}}(\tau,\theta)=\overline{\mu}^{(n)}(\tau,\theta)$.

$$\tilde{T}\left[\mu^{(n)}(\tau)\right]=\tilde{T}(\underline{\mu}^{(n)}(\tau,\theta),\overline{\mu}^{(n)}(\tau,\theta))=(T(\overline{\mu}^{(n)}(\tau,\theta)),T(\underline{\mu}^{(n)}(\tau,\theta)))$$

---------------------------------------(3.8)

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by the same procedure for (1) we have

$$\mathcal{G}_n(\mathfrak{U},\theta) = \tilde{T}\left[\mu^{(n)}(\tau)\right] = \frac{\mathcal{F}_n\left(\frac{1}{\mathfrak{U}^2}\right)}{\mathfrak{U}}$$

$$\mathcal{G}_n(\mathfrak{U},\theta) = \tfrac{1}{\mathfrak{U}}[-(\tfrac{1}{\mathfrak{U}^2})^{2n-1}\mu(0) \ominus -(\tfrac{1}{\mathfrak{U}^2})^{2n}\mathcal{F}\left(\tfrac{1}{\mathfrak{U}^2}\right) \otimes \textstyle\sum_{k=1}^{n-1} \frac{\mu^{(k)}(0)}{(u^2)^{n-k-1}}]$$

$$= \frac{-\mathcal{G}(\mathfrak{U},\theta)}{\mathfrak{U}^{2n}} \ominus \frac{-\mu(0)}{\mathfrak{U}^{2n-1}} \otimes \sum_{k=1}^{n-1} \frac{\mu^{(k)}(0)}{(\mathfrak{U})^{2n-2k-1}}.$$

which can be written as:

$$T\left(\underline{\mu}^{(n)}(\tau,\theta)\right) = \frac{T\left[\underline{\mu}(\tau,\theta)\right]}{u^{2n}} - \frac{\underline{\mu}(0,\theta)}{u^{2n-1}} - \sum_{k=1}^{i_1} \frac{\underline{\mu}^{(k)}(0,\theta)}{(u)^{2n-2k-1}} - \sum_{k=i_1+1}^{i_2} \frac{\underline{\mu}^{(k)}(0,\theta)}{(u)^{2n-2k-1}} - \sum_{k=i_2+1}^{i_3} \frac{\underline{\mu}^{(k)}(0,\theta)}{(u)^{2n-2k-1}} - \dots - \sum_{k=i_m+1}^{n-1} \frac{\underline{\mu}^{(k)}(0,\theta)}{(u)^{2n-2k-1}}. \text{------------------------------------} (9)$$

In a similar way, we can get

$$T\left(\overline{\mu}^{(n)}(\tau,\theta)\right) = \frac{T[\overline{\mu}(\tau,\theta)]}{u^{2n}} - \frac{\overline{\mu}(0,\theta)}{u^{2n-1}} - \sum_{k=1}^{i_1} \frac{\overline{\mu}^{(k)}(0,\theta)}{(u)^{2n-2k-1}} - \sum_{k=i_1+1}^{i_2} \frac{\overline{\mu}^{(k)}(0,\theta)}{(u)^{2n-2k-1}} - \sum_{k=i_2+1}^{i_3} \frac{\overline{\mu}^{(k)}(0,\theta)}{(u)^{2n-2k-1}} - \dots - \sum_{k=i_m+1}^{n-1} \frac{\overline{\mu}^{(k)}(0,\theta)}{(u)^{2n-2k-1}} \text{------------------------------------} (10)$$

Since $0 \le i_1 \le i_2 \le \dots \le i_m \le n-1$ we can apply theorem (2/b) for each $[\mu^{(k)}(\tau)]$ where $1 \le k \le n-1$ as following:

$$\underline{\mu}^{(k)}(0,\theta) = \underline{\mu^{(k)}}(0,\theta)\,, \overline{\mu}^{(k)}(0,\theta) = \overline{\mu^{(k)}}(0,\theta), 1 \le k \le i_1$$

$$\overline{\mu}^{(k)}(0,\theta) = \underline{\mu^{(k)}}(0,\theta) =, \underline{\mu}^{(k)}(0,\theta) = \overline{\mu^{(k)}}(0,\theta), i_1 + 1 \le k \le i_2$$

$$\underline{\mu}^{(k)}(0,\theta) = \underline{\mu^{(k)}}(0,\theta), \overline{\mu}^{(k)}(0,\theta) = \overline{\mu^{(k)}}(0,\theta), i_2 + 1 \le k \le i_3$$

$$\vdots$$

$$\underline{\mu}^{(k)}(0,\theta) = \overline{\mu^{(k)}}(0,\theta)\,, \overline{\mu}^{(k)}(0,\theta) = \underline{\mu^{(k)}}(0,\theta), i_3 + 1 \le k \le n-1\,.$$

The last equations yields from theorem (2/b) due to that m is an odd number. Substituting (9), (10) in (8) to get:

$$\tilde{T}\left[\mu^{(n)}(\tau)\right] = \Big(\frac{T[\overline{\mu}(\tau,\theta)]}{u^{2n}} - \frac{\overline{\mu}(0,\theta)}{u^{2n-1}} - \sum_{k=1}^{i_1} \frac{\overline{\mu^{(k)}}(0,\theta)}{(u)^{2n-2k-1}} - \sum_{k=i_1+1}^{i_2} \frac{\underline{\mu^{(k)}}(0,\theta)}{(u)^{2n-2k-1}} - \sum_{k=i_2+1}^{i_3} \frac{\overline{\mu^{(k)}}(0,\theta)}{(u)^{2n-2k-1}} \dots - \sum_{k=i_m+1}^{n-1} \frac{\underline{\mu^{(k)}}(0,\theta)}{(u)^{2n-2k-1}}\,, \frac{T\left[\underline{\mu}(\tau,\theta)\right]}{u^{2n}} - \frac{\underline{\mu}(0,\theta)}{u^{2n-1}} - \sum_{k=1}^{i_1} \frac{\underline{\mu^{(k)}}(0,\theta)}{(u)^{2n-2k-1}} - \sum_{k=i_1+1}^{i_2} \frac{\overline{\mu^{(k)}}(0,\theta)}{(u)^{2n-2k-1}} - \sum_{k=i_2+1}^{i_3} \frac{\underline{\mu^{(k)}}(0,\theta)}{(u)^{2n-2k-1}} - \dots - \sum_{k=i_m+1}^{n-1} \frac{\overline{\mu^{(k)}}(0,\theta)}{(u)^{2n-2k-1}}\,.$$

Remark .1

By theorem (3), there are 2^n cases for $\tilde{T}$-transformation for $\mu^{(n)}(\tau), n \in Z^+$. So, we have $\sum_{k=0}^{n}\binom{n}{k} = 2^n$, where $\binom{n}{k}$ is the number of cases that contains k functions of the type (ii)-differentiable among the functions $\mu(\tau), \mu`(\tau), \dots, \mu^{(n-1)}(\tau)$.

4. Ilustrative examples

Following are examples of order 5 derivative, that solved to show the validity of theorem (3).

1. Example

Consider the following second-order FIVP:

$\mu``(\tau) = \mu(\tau)$

$\mu(0) = \mu`(0) = (2\theta - 2, 2 - 2\theta)$

note that $\underline{\mu}(0) = \underline{\mu}`(0) = 2\theta - 2$

$\overline{\mu}(0) = \overline{\mu}`(0) = 2 - 2\theta$

$f(\tau, \mu(\tau), \mu^{`}(\tau), \theta) = \mu(\tau) = (\underline{\mu}(\tau), \overline{\mu}(\tau))$,
$\underline{f}(\tau, \mu(\tau), \mu^{`}(\tau), \theta) = \underline{\mu}(\tau)$
and $\overline{f}(\tau, \mu(\tau), \mu^{`}(\tau), \theta) = \overline{\mu}(\tau)$.There are $2^2 = 4$ cases:

Case (1): Let $\mu(\tau)$, $\mu^{`}(\tau)$ **i- differentiable**. Then :
$\underline{\mu}^{`}(\tau,\theta)\ \underline{\mu}^{`}(\tau,\theta), \overline{\mu}^{`}(\tau,\theta) = \overline{\mu}^{`}(\tau,\theta)$,
Thus:
$T[\underline{f}(\tau, \mu(\tau), \mu^{`}(\tau), \theta)] = T[\underline{\mu}(\tau)]$,
$T[\overline{f}(\tau, \mu(\tau), \mu^{`}(\tau), \theta)] = T[\overline{\mu}(\tau)]$.
Using theorem (3) when m is an even number, we get:
$\frac{G(u)}{u^{10}} - \frac{\underline{\mu}(0)}{u^9} - \frac{\underline{\mu}^{`}(0)}{u^7} = T(\underline{\mu}(\tau))$
$T(\underline{\mu}(\tau))$- $(2\theta - 2)(u+u^3 +) = u^{10}T(\underline{\mu}(\tau))$
$T(\underline{\mu}(\tau)) = \frac{(2\theta-2)(u+u^3)}{1-u^{10}} = \frac{u}{1-u^2}$ which implies that $\underline{\boldsymbol{\mu}}(\boldsymbol{\tau}) = \boldsymbol{e^{\tau}}$,
$T(\overline{\mu}(\tau)) = \frac{(2-2\theta)(u+u^3)}{1-u^{10}} = \frac{u}{1-u^2}$ which implies that $\overline{\boldsymbol{\mu}}(\boldsymbol{\tau}) = \boldsymbol{e^{\tau}}$.

Case (2): Let $\mu(\tau)$, $\mu^{`}(\tau)$ **ii- differentiable**. Then:
$\underline{\mu}^{`}(\tau,\theta) = \overline{\mu}^{`}(\tau,\theta)$, $\overline{\mu}^{`}(\tau,\theta) = \underline{\mu}^{`}(\tau,\theta)$,
Thus:
$T[\underline{f}(\tau, \mu(\tau), \mu^{`}(\tau), \theta)] = T[\overline{\mu}(\tau)]$,
$T[\overline{f}(\tau, \mu(\tau), \mu^{`}(\tau), \theta)] = T[\underline{\mu}(\tau)]$.
Using theorem (3) when m is an even number, we get:
$\frac{G(u)}{u^{10}} - \frac{\underline{\mu}(0)}{u^9} - \frac{\overline{\mu}^{`}(0)}{u^7} = T(\overline{\mu}(\tau))$
$T\left[\underline{\mu}(\tau)\right] = u^{10}T[\overline{\mu}(\tau)] + (2\theta - 2)(u - u^3)$, (1)
and
$T[\overline{\mu}(\tau)] = u^{10}T\left[\underline{\mu}(\tau)\right] + (2 - 2\theta)(u - u^3)$ (2)
solving (1),(2)
$T\left[\underline{\mu}(\tau)\right] = \frac{(2-2\theta)(u-u^3)(u^{10}-1)}{1-u^{20}} = \frac{u}{1-u^4}$ and
$T[\overline{\mu}(\tau)] = \frac{(2\theta-2)(u-u^3)(u^{10}-1)}{1-u^{20}} = \frac{u}{1-u^4}$ which implies that $\underline{\boldsymbol{\mu}}(\boldsymbol{\tau}) = \boldsymbol{cosh\tau} = \overline{\boldsymbol{\mu}}(\boldsymbol{\tau})$ **.**
Other cases are solved by the same way.

2. Example

Consider the following fifth-order FIVP:

$\mu^{(5)}(\tau) = \mu(\tau)$
$\mu(0) = \mu^{`}(0) = \mu^{``}(0) \ldots = \mu^{(4)}(0) = (2\theta - 2, 2 - 2\theta)$
note that $\underline{\mu}(0) = \underline{\mu}^{`}(0) = \underline{\mu}^{``}(0) = \underline{\mu}^{```}(0) = \underline{\mu}^{(4)}(0) = 2\theta - 2$
$\overline{\mu}(0) = \overline{\mu}^{`}(0) = \overline{\mu}^{``}(0) = \overline{\mu}^{```}(0) = \overline{\mu^{(4)}}(0) = 2 - 2\theta$
$f(\tau, \mu(\tau), \mu^{`}(\tau), \mu^{``}(\tau), \ldots, \mu^{(4)}(\tau), \theta) = \mu(\tau) = (\underline{\mu}(\tau), \overline{\mu}(\tau))$,
$\underline{f}(\tau, \mu(\tau), \mu^{`}(\tau), \mu^{``}(\tau), \ldots, \mu^{(4)}(\tau), \theta) = \underline{\mu}(\tau)$
and $\overline{f}(\tau, \mu(\tau), \mu^{`}(\tau), \mu^{``}(\tau), \ldots, \mu^{(4)}(\tau), \theta) = \overline{\mu}(\tau)$.There are $2^5 = 32$ cases:

Case (1): Let $\mu(\tau)$, $\mu^{`}(\tau)$ and $\mu^{``}(\tau)$, $\mu^{```}(\tau)$, $\mu^{(4)}(\tau)$ be **i- differentiable**. Then :

$\underline{\mu}`(\tau,\theta)\ \underline{\mu}`(\tau,\theta), \overline{\mu`}(\tau,\theta) = \overline{\mu`}(\tau,\theta), \underline{\mu}``(\tau,\theta) = \underline{\mu}``(\tau,\theta), \overline{\mu``}(\tau,\theta) = \overline{\mu``}(\tau,\theta)$.

$\underline{\mu}```(\tau,\theta) = \underline{\mu}```(\tau,\theta), \overline{\mu```}(\tau,\theta) = \overline{\mu```}(\tau,\theta), \underline{\mu}^{(4)}(\tau,\theta) = \underline{\mu}^{(4)}(\tau,\theta), \overline{\mu^{(4)}}(\tau,\theta) = \overline{\mu^{(4)}}(\tau,\theta)$.

Thus:

$T[\underline{f}(\tau, \mu(\tau), \mu`(\tau), \mu``(\tau), \dots, \mu^{(4)}(\tau), \theta)] = T[\underline{\mu}(\tau)]$,

$T[\overline{f}(\tau, \mu(\tau), \mu`(\tau), \mu``(\tau), \dots, \mu^{(4)}(\tau), \theta)] = T[\overline{\mu}(\tau)]$.

Using theorem (3) when m is an even number, we get:

$\frac{G(u)}{u^{10}} - \frac{\underline{\mu}(0)}{u^9} - \frac{\underline{\mu}`(0)}{u^7} - \frac{\underline{\mu}``(0)}{u^5} \frac{\underline{\mu}```(0)}{u^3} - \frac{\underline{\mu}^{(4)}(0)}{u} = T(\underline{\mu}(\tau))$

$T(\underline{\mu}(\tau)) - (2\theta - 2)(u + u^3 + u^5 + u^7 + u^9) = u^{10} T(\underline{\mu}(\tau))$

$T(\underline{\mu}(\tau)) = \frac{(2\theta-2)(u+u^3+u^5+u^7+u^9)}{1-u^{10}} = \frac{u}{1-u^2}$ which implies that $\underline{\boldsymbol{\mu}}(\boldsymbol{\tau}) = \boldsymbol{e^{\tau}}$,

$T(\overline{\mu}(\tau)) = \frac{(2-2\theta)(u+u^3+u^5+u^7+u^9)}{1-u^{10}} = \frac{u}{1-u^2}$ which implies that $\overline{\boldsymbol{\mu}}(\boldsymbol{\tau}) = \boldsymbol{e^{\tau}}$.

Case (2): Let $\mu`(\tau), \mu``(\tau), \mu```(\tau), \mu^{(4)}(\tau)$ be **(i)-differentiable** and $\mu(\tau)$ be **(ii)-differentiable**. Then:

$\underline{\mu}`(\tau,\theta) = \overline{\mu`}(\tau,\theta), \overline{\mu`}(\tau,\theta) = \underline{\mu}`(\tau,\theta), \underline{\mu}``(\tau,\theta) = \overline{\mu``}(\tau,\theta), \overline{\mu``}(\tau,\theta) = \underline{\mu}``(\tau,\theta)$.

$\underline{\mu}```(\tau,\theta) = \overline{\mu```}(\tau,\theta), \overline{\mu```}(\tau,\theta) = \underline{\mu}```(\tau,\theta), \underline{\mu}^{(4)}(\tau,\theta) = \overline{\mu^{(4)}}(\tau,\theta), \overline{\mu^{(4)}}(\tau,\theta) = \underline{\mu}^{(4)}(\tau,\theta)$.

Thus:

$T[\underline{f}(\tau, \mu(\tau), \mu`(\tau), \mu``(\tau), \dots, \mu^{(4)}(\tau), \theta)] = T[\overline{\mu}(\tau)]$,

$T[\overline{f}(\tau, \mu(\tau), \mu`(\tau), \mu``(\tau), \dots, \mu^{(4)}(\tau), \theta)] = T[\underline{\mu}(\tau)]$.

Using theorem (3) when m is an odd number, we get:

$\frac{G(u)}{u^{10}} - \frac{\underline{\mu}(0)}{u^9} - \frac{\overline{\mu`}(0)}{u^7} - \frac{\overline{\mu``}(0)}{u^5} \frac{\overline{\mu```}(0)}{u^3} - \frac{\overline{\mu^{(4)}}(0)}{u} = T(\overline{\mu}(\tau))$

$T\left[\underline{\mu}(\tau)\right] = u^{10} T[\overline{\mu}(\tau)] + (2\theta - 2)(u - u^3 - u^5 - u^7 - u^9)$, (3)

and

$T[\overline{\mu}(\tau)] = u^{10} T\left[\underline{\mu}(\tau)\right] + (2 - 2\theta)(u - u^3 - u^5 - u^7 - u^9)$ (4)

solving (3),(4)

$T\left[\underline{\mu}(\tau)\right] = \frac{(2-2\theta)(u-u^3-u^5-u^7-u^9)(u^{10}-1)}{1-u^{20}} = \frac{u^3}{(1-u^2)^2}$ and

$T[\overline{\mu}(\tau)] = \frac{(2\theta-2)(u-u^3-u^5-u^7-u^9)(u^{10}-1)}{1-u^{20}} = \frac{u^3}{(1+400u^2)^2}$ which implies that $\underline{\boldsymbol{\mu}}(\boldsymbol{\tau}) = \boldsymbol{\tau e^{\tau}}, \overline{\boldsymbol{\mu}}(\boldsymbol{\tau}) = \boldsymbol{\tau e^{400\tau}}$.

Case (3): Let $\mu(\tau), \mu``(\tau), \mu```(\tau), \mu^{(4)}(\tau)$ be **(i)-differentiable** and $\mu`(\tau)$ be **(ii)-differentiable**. Then we have:

$\underline{\mu}`(\tau,\theta) = \underline{\mu}`(\tau,\theta), \overline{\mu`}(\tau,\theta) = \overline{\mu`}(\tau,\theta), \underline{\mu}``(\tau,\theta) = \overline{\mu``}(\tau,\theta), \overline{\mu``}(\tau,\theta) = \underline{\mu}``(\tau,\theta)$.

$\underline{\mu}```(\tau,\theta) = \overline{\mu```}(\tau,\theta), \overline{\mu```}(\tau,\theta) = \underline{\mu}```(\tau,\theta), \underline{\mu}^{(4)}(\tau,\theta) = \overline{\mu^{(4)}}(\tau,\theta), \overline{\mu^{(4)}}(\tau,\theta) = \underline{\mu}^{(4)}(\tau,\theta)$.

Thus:

$T[\underline{f}(\tau, \mu(\tau), \mu`(\tau), \mu``(\tau), \dots, \mu^{(4)}(\tau), \theta)] = T[\overline{\mu}(\tau)]$,

$T[\overline{f}(\tau, \mu(\tau), \mu`(\tau), \mu``(\tau), \dots, \mu^{(4)}(\tau), \theta)] = T[\underline{\mu}(\tau)]$.

Using theorem (3) when m is an odd number, we get:

$\frac{G(u)}{u^{10}} - \frac{\underline{\mu}(0)}{u^9} - \frac{\underline{\mu}`(0)}{u^7} - \frac{\overline{\mu``}(0)}{u^5} \frac{\overline{\mu```}(0)}{u^3} - \frac{\overline{\mu^{(4)}}(0)}{u} = T(\overline{\mu}(\tau))$

$T\left[\underline{\mu}(\tau)\right] = u^{10} T[\overline{\mu}(\tau)] + (2\theta - 2)(u + u^3 - u^5 - u^7 - u^9)$, (5)

and

$T[\overline{\mu}(\tau)] = u^{10} T\left[\underline{\mu}(\tau)\right] + (2 - 2\theta)(u + u^3 - u^5 - u^7 - u^9)$ (6)

solving (5),(6)

$T\left[\underline{\mu}(\tau)\right] = \frac{(2-2\theta)(u+u^3-u^5-u^7-u^9)(u^{10}-1)}{1-u^{20}} = \frac{u^3}{(1-2u^2)^2}$ and

$T[\overline{\mu}(\tau)] = \frac{(2\theta-2)(u+u^3-u^5-u^7-u^9)(u^{10}-1)}{1-u^{20}} = \frac{u^3}{(1-400u^2)^2}$ which implies that $\underline{\boldsymbol{\mu}}(\boldsymbol{\tau}) = \boldsymbol{\tau e^{2\tau}}, \overline{\boldsymbol{\mu}}(\boldsymbol{\tau}) = \boldsymbol{\tau e^{400\tau}}$.

Case (4): Let $\mu(\tau), \mu^{`}(\tau), \mu^{```}(\tau), \mu^{(4)}(\tau)$ be **(i)-differentiable** and $\mu^{``}(\tau)$ be **(ii)-differentiable**. Then:
$\underline{\mu}^{`}(\tau,\theta) = \underline{\mu}^{`}(\tau,\theta), \overline{\mu}^{`}(\tau,\theta) = \overline{\mu}^{`}(\tau,\theta), \underline{\mu}^{``}(\tau,\theta) = \underline{\mu}^{``}(\tau,\theta), \overline{\mu}^{``}(\tau,\theta) = \overline{\mu}^{``}(\tau,\theta).$
$\underline{\mu}^{```}(\tau,\theta) = \overline{\mu}^{```}(\tau,\theta), \overline{\mu}^{```}(\tau,\theta) = \underline{\mu}^{```}(\tau,\theta), \underline{\mu}^{(4)}(\tau,\theta) = \overline{\mu}^{(4)}(\tau,\theta), \overline{\mu}^{(4)}(\tau,\theta) = \underline{\mu}^{(4)}(\tau,\theta).$
Thus:
$T[\underline{f}(\tau, \mu(\tau), \mu^{`}(\tau), \mu^{``}(\tau), \ldots, \mu^{(4)}(\tau), \theta)] = T[\overline{\mu}(\tau)],$
$T[\overline{f}(\tau, \mu(\tau), \mu^{`}(\tau), \mu^{``}(\tau), \ldots, \mu^{(4)}(\tau), \theta)] = T[\underline{\mu}(\tau)].$
Using theorem (3) when m is an odd number, we get:
$\frac{G(u)}{u^{10}} - \frac{\underline{\mu}(0)}{u^9} - \frac{\underline{\mu}^{`}(0)}{u^7} - \frac{\underline{\mu}^{``}(0)}{u^5} \frac{\overline{\mu}^{```}(0)}{u^3} - \frac{\overline{\mu^{(4)}}(0)}{u} = T(\overline{\mu}(\tau))$
$$T\left[\underline{\mu}(\tau)\right] = u^{10}T[\overline{\mu}(\tau)] + (2\theta - 2)(u + u^3 + u^5 - u^7 - u^9), \qquad (7)$$
and
$$T[\overline{\mu}(\tau)] = u^{10}T\left[\underline{\mu}(\tau)\right] + (2 - 2\theta)(u + u^3 + u^5 - u^7 - u^9) \qquad (8)$$
solving (7),(8)
$$T\left[\underline{\mu}(\tau)\right] = \frac{(2-2\theta)(u+u^3+u^5-u^7-u^9)(u^{10}-1)}{1-u^{20}} = \frac{u^3}{(1-u^2)^2} \text{ and}$$
$$T[\overline{\mu}(\tau)] = \frac{(2\theta-2)(u+u^3+u^5-u^7-u^9)(u^{10}-1)}{1-u^{20}} = \frac{u^3}{(1-400u^2)^2} \text{ which implies that}$$
$\boldsymbol{\underline{\mu}(\tau) = \tau e^{\tau}, \overline{\mu}(\tau) = \tau e^{400\tau}}$.

Case (5): Let $\mu(\tau), \mu^{`}(\tau), \mu^{``}(\tau), \mu^{(4)}(\tau)$ be **(i)-differentiable** and $\mu^{```}(\tau)$ be **(ii)-differentiable**. Then:
$\underline{\mu}^{`}(\tau,\theta) = \underline{\mu}^{`}(\tau,\theta), \overline{\mu}^{`}(\tau,\theta) = \overline{\mu}^{`}(\tau,\theta), \underline{\mu}^{``}(\tau,\theta) = \underline{\mu}^{``}(\tau,\theta), \overline{\mu}^{``}(\tau,\theta) = \overline{\mu}^{``}(\tau,\theta).$
$\underline{\mu}^{```}(\tau,\theta) = \underline{\mu}^{```}(\tau,\theta), \overline{\mu}^{```}(\tau,\theta) = \overline{\mu}^{```}(\tau,\theta), \underline{\mu}^{(4)}(\tau,\theta) = \overline{\mu^{(4)}}(\tau,\theta), \overline{\mu^{(4)}}(\tau,\theta) = \underline{\mu}^{(4)}(\tau,\theta).$
Thus:
$T[\underline{f}(\tau, \mu(\tau), \mu^{`}(\tau), \mu^{``}(\tau), \ldots, \mu^{(4)}(\tau), \theta)] = T[\overline{\mu}(\tau)],$
$T[\overline{f}(\tau, \mu(\tau), \mu^{`}(\tau), \mu^{``}(\tau), \ldots, \mu^{(4)}(\tau), \theta)] = T[\underline{\mu}(\tau)].$
Using theorem (3) when m is an odd number, we get:
$\frac{G(u)}{u^{10}} - \frac{\underline{\mu}(0)}{u^9} - \frac{\underline{\mu}^{`}(0)}{u^7} - \frac{\underline{\mu}^{``}(0)}{u^5} - \frac{\underline{\mu}^{```}(0)}{u^3} - \frac{\overline{\mu^{(4)}}(0)}{u} = T(\overline{\mu}(\tau))$
$$T\left[\underline{\mu}(\tau)\right] = u^{10}T[\overline{\mu}(\tau)] + (2\theta - 2)(u + u^3 + u^5 + u^7 - u^9), \qquad (7) \text{ and}$$
$$T[\overline{\mu}(\tau)] = u^{10}T\left[\underline{\mu}(\tau)\right] + (2 - 2\theta)(u + u^3 + u^5 + u^7 - u^9) \qquad (8)$$
solving (7),(8)
$$T\left[\underline{\mu}(\tau)\right] = \frac{(2-2\theta)(u+u^3+u^5+u^7-u^9)(u^{10}-1)}{1-u^{20}} = \frac{u^3}{(1-2u^2)^2}, \text{ and}$$
$$T[\overline{\mu}(\tau)] = \frac{(2\theta-2)(u+u^3+u^5+u^7-u^9)(u^{10}-1)}{1-u^{20}} = \frac{u^3}{(1-400u^2)^2} \text{ which implies that}$$
$\boldsymbol{\underline{\mu}(\tau) = \tau e^{2\tau}, \overline{\mu}(\tau) = \tau e^{400\tau}}$. Other cases are solved by the same way.

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