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Derivation of Composite Rules for Numerical Calculation of Triple Integrals with Continuous Integrands and Improve Results Using Aitken's Acceleration

Rehab A. Shaaban[1], Safaa M. Aljassas[2], Neeran Tahir Abd Alameer[3]

Iraq\ University of Kufa \ College of Education for girls \ Mathematics Dep.

Email: rihaba.khudair@uokufa.edu.iq[1], safaam.musa@uokufa.edu.iq[2], niran.abdulameer@uokufa.edu.iq[3]

Abstract: *The main goal of this article to calculate the triple integrals with continuous integrand numerically by derivation rules and its correction terms .Improve results using method that obtained by* ***Aitken's acceleration*** *with the composite rule from (Trapezoidal rule on the exterior dimension Z and interior dimension X and the suggested method on the middle dimension Y, denoted by TSuT) When the number of divisions on the three dimensions are equals. We presented theorem with its proof to evaluate this rule and the correction term for it. Moreover, we depended accelerations mentioned with TSuT rule to improve our results and we called the method by* $AI(TSuT)$ *. the results value are better if accuracy and number of sub-intervals are taken in account .*

Keyword: Trapezoidal rule , Suggested method , Aitken's acceleration

1-Introduction

The triple integrals are very useful to find volumes, middle centres and the inertia of volumes for instance the volume inside $x^2+y^2=4x$, up $z=0$ and under $x^2+y^2=4z$. And the volume within the cylinder $\rho=4\cos(\theta)$ that determines by the sphere $p^2+z^2=16$ from the top and by z=0 down. In addition, calculate the middle centre for volume under $z^2=xy$ and up the tringle $y=x, y=0, x=4$. As well as calculating the moment of inertia for volume which is located inside $x^2+y^2=9$, up z=0 and down the plane $x+z=4$. The importance of triple integral stands out in finding the mass with unsteady density, Frank Ayres [6].

There are many researchers was interested in evaluating the triple integrals such as Dheyaa [3], in 2009, he used numerical composite method ($RMRM(RS)$, $RMRM(RM)$, $RMRS(RM)$ and $RMRS(RS)$). These methods have obtained from Romberg acceleration method with Midpoint method (RM) on the exterior dimension (Z). $RS(RS), RS(RM), RM(RM), RM(RS)$ on the middle dimension (Y) and interior dimension (X). He reached that the composite method from Simpson's rule with Romberg acceleration on interior and middle dimension and Midpoint method with Romberg acceleration on exterior dimension RMRS(RS) was better method for evaluating the triple integrals with continuous integrand in terms of accuracy, the number of sub intervals used and time.

In 2010, Eghaar [5], introduced numerical method to calculate the value of triple integrals by Romberg acceleration method on the resulting values from applying Midpoint method on three dimensions X , Y and Z when the number of sub intervals which obtained from divided the interior dimension interval equals to the number of sub intervals that obtained from divided the middle dimension interval and also equals to the number of sub intervals which obtained from divided the exterior dimension interval. She got good results in terms of accuracy and a relatively few sub intervals.

Mohammed et al. [1] presented in 2013 numerical method to evaluate the value of triple integrals with continuous integrands by RSSS method that obtained from Romberg acceleration with Simpson's rule on three dimensions X,Y and Z as the same approach of Eghaar [5].

In 2015, Aljassas [11] introduced a numerical method $RM(RMM)$ to calculating triple integrals with continuous integrands by using Romberg acceleration with Mid-point rule on the three dimensions when the number of divisions on the interior dimension is equal to the number of divisions on the middle dimension, but both of them are deferent from the number of divisions on the exterior dimension and she got a high accuracy in the results in a little sub-intervals relatively and a short time. Also in 2015, Sarada et al. [9] use the generalized Gaussian Quadrature to evaluate triple integral and got a good results.
Additionally, in 2018 Safaa et al [10]. introduced two numerical methods R(MSM) and R(SMM) to evaluate the value of triple integral with continuous integrands, these methods obtained from Romberg acceleration with two rules from Newton–Cotes

formulas (Midpoint and Simpson) and they got good results in terms of accuracy and the access of approximate values to the real values was fast in a relatively few sub intervals.

In this paper, we introduced theorem with its proof to find new numerical rule to evaluate approximate values for triple integrals with continuous integrands and the correction term for it. This rule obtained from applied Trapezoidal rule on the exterior dimension Z and interior dimension X and the suggested method on the middle dimension Y, when $u=u_1=u_2$ (such that u is the number of divisions on the exterior dimension Z, u_1 the number of divisions on the middle dimension Y and u_2 the number of divisions on the interior dimension X), we denoted by TST. Then we could improve the results by using accelerations is Aitken (denoted it by AI(TSuT)

2- Trapezoidal rule

Suppose that the integral I defined as the following

$$I=\int_{o}^{p} h(x)dx = T(k)+\lambda_T(k)+R_T \qquad \ldots(1)$$

such that $T(k)$ is Trapezoidal rule to evaluate the value of integral I and $\lambda_T(k)$ is correction terms for $T(k)$ and R_T is the reminder that related by truncation from $\lambda_T(k)$ after using specified terms from $\lambda_T(k)$. The general formula for the Trapezoidal rule T(k) is:

$$T(k)=\frac{k}{2}\left[h(o)+2\sum_{i=1}^{u-1}h(o+ik)+h(p)\right] \qquad \ldots(2)$$

And the correction terms of Trapezoidal rule for continuous integrations is

$$\lambda_T(k)=-\frac{1}{12}k^2(h_u'-h_0')+\frac{1}{720}k^4(h_u^{(3)}-h_0^{(3)})-\frac{1}{30240}k^6(h_u^{(5)}-h_0^{(5)})+\cdots \qquad \ldots(3)$$

Fox [7].

By using the mean value theorem for the formulas (3) we get:

$$\lambda_T(k)=\frac{-(p-o)}{12}k^2h^{(2)}(\varpi_1)+\frac{(p-o)}{720}k^4h^{(4)}(\varpi_2)-\frac{(p-o)}{30240}k^6h^{(6)}(\varpi_3)-\cdots \qquad \ldots(4)$$

Such that $i=1,2,3,\ldots$, $\varpi_i \in (o,p)$, Eghaar [5]

3- Suggested method (Su)

Suggested method which is introduced by Mohammed et al. [2], it considers one of the singular integral method. The general form is

$$Su(k)=\frac{k}{4}\left[h(0)+h(p)+2h(o+(u-0.5)k)+2\sum_{i=1}^{u-1}(h(o+(i-0.5)k)+h(o+ik))\right] \qquad \ldots(5)$$

And the correction terms of this rule is

$$\delta_{Su}(k)=\frac{1}{24}k^2(h_u'-h_0')-\frac{5}{1440}k^4(h_u^{(3)}-h_0^{(3)})+\frac{61}{60480}k^6(h_u^{(5)}-h_0^{(5)})-\cdots \qquad \ldots(6)$$

By using the mean value theorem for above form, we will get

$$\delta_{Su}(k)=\frac{(p-o)}{24}k^2h^{(2)}(\sigma_1)-\frac{5(p-o)}{1440}k^4h^{(4)}(\sigma_2)+\frac{61(p-o)}{60480}k^6h^{(6)}(\sigma_3)-\cdots \qquad \ldots(7)$$

4-Aitken's delta – Squared Process

In 1926, Alexander Aitken (1985-1926) found a new approach to accelerate the sequence convergence rate. To explain this method, we suppose the sequence $\{x_n\}$ such that $\{x_n\} = \{x_1, x_2, \ldots x_k \ldots\}$ linearly convergence to a certain final value β, so $\beta - x_{i+1} = C_i(\beta - x_i)$, Ralston [4], such that $|C_i| < 1$ and $C_i \to C$.

We can see that C_i will be approximately steady and we can write

$$\beta - x_{i+1} \approx \bar{C}\ (\beta - x_i) \qquad \ldots(8)$$

Such that $|\bar{C}| = C$

We also can see that

$$\frac{\beta - x_{i+2}}{\beta - x_{i+1}} \approx \frac{\beta - x_{i+1}}{\beta - x_i} \qquad \ldots(9)$$

i.e, $$\beta \cong \frac{x_i x_{i+2} - x_{i+1}^2}{x_{i+2} - 2x_{i+1} + x_i} = x_{i+2} - \frac{(\Delta x_{i+1})^2}{\Delta^2 x_i} \qquad \ldots(10)$$

such that $\Delta x_i = (x_{i+1} - x_i)$ and $\Delta^2 x_i = x_i - 2x_{i+1} - x_{i-2}$

when using u from elements of the sequence $\{x_u\}$, we can get u-2 of another sequence {S} Approaching faster than $\{x_u\}$

$$s_{i+2} = x_{i+2} - \frac{(\Delta x_{i+1})^2}{\Delta^2 x_i} \qquad \ldots(11)$$

where $i = 1, 2, \ldots, u-2$

This process is accelerating the convergence to the final value β .

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5-The numerical rule TSuT and its corrections terms

Theorem:

Let $h(x,y,z)$ is continuous function and derivable at each point of the region $[o,p]\times[q,r]\times[s,t]$, then the approximate value of $I=\int_s^t\int_q^r\int_o^p h(x,y,z)dxdydz$ can be evaluated by the following rule:

$$I=\int_s^t\int_q^r\int_o^p h(x,y,z)dxdydz=TSuT(k)=\frac{h^3}{16}[h(o,q,s)+h(o,q,t)+h(o,r,s)+h(o,r,t)+h(p,q,s)+h(p,q,t)$$
$$+h(p,r,s)+h(p,r,t)+2h(o,q+(u-0.5)k,s)+2h(o,q+(u-0.5)k,t)+2h(r,q+(u-0.5)k,s)$$
$$+2h(r,q+(u-0.5)k,t)+2\sum_{\ell=1}^{u-1}(h(a,c,e+\ell k)+h(o,r,s+\ell k)+h(p,q,s+\ell k)+h(p,r,s+\ell k)$$
$$+2h(o,q+(u-0.5)k,s+\ell k)+2h(r,q+(u-0.5)k,s+\ell k))+2\sum_{i=1}^{u-1}(h(o+ik,q,s)+h(o+ik,q,t)$$
$$+h(o+ik,r,s)+h(o+ik,r,t)+2h(o+ik,q+(u-0.5)k,s)+2h(o+ik,q+(u-0.5)k,t)$$
$$+2\sum_{\ell=1}^{u-1}(h(o+ik,q,s+\ell k)+h(o+ik,r,e+\ell k)+2h(o+ik,q+(u-0.5)k,s+\ell k))$$
$$+2\sum_{j=1}^{u-1}[h(o,q+(j-0.5)k,s)+h(o,q+(j-0.5)k,t)+2\sum_{\ell=1}^{u-1}h(o,q+(j-0.5)k,s+\ell k)$$
$$+h(p,q+(j-0.5)k,s)+h(p,q+(j-0.5)k,t)+2\sum_{\ell=1}^{u-1}h(p,q+(j-0.5)k,s+\ell k)$$
$$+2\sum_{i=1}^{u-1}(h(o+ik,q+(j-0.5)k,s)+h(o+ik,q+(j-0.5)k,t)+2\sum_{\ell=1}^{u-1}h(o+ik,q+(j-0.5)k,s+\ell k))$$
$$+h(o,q+jk,s)+h(o,q+jk,t)+2\sum_{\ell=1}^{u-1}h(o,q+jk,s+\ell k)+h(p,q+jk,s)+h(p,q+jk,t)$$
$$+2\sum_{\ell=1}^{u-1}h(p,q+jk,s+\ell k)+2\sum_{i=1}^{u-1}(h(o+ik,q+jk,s)+h(o+ik,q+jk,t)+2\sum_{\ell=1}^{u-1}h(o+ik,q+jk,\ell+\ell k))]$$

And the correction form (error formula)is : $I-TSuT(k)=\xi_1 k^2+\xi_2 k^4+\zeta_3 k^6+\cdots$

Where $\xi_1,\xi_2,\zeta_3,\ldots$ are constants

Proof :

The integral I can be written as

$$I=\int_{s}^{t}\int_{q}^{r}\int_{o}^{p}h(x,y,z)dxdydz=TSuT\,(k)+\lambda_{TSuT}\,(k) \qquad ...(12)$$

Such that $TSuT\,(k)$ is the approximate integral value that is calculated numerically by Trapezoidal rule on exterior dimension Z and interior dimension X. And the suggested method on middle dimension Y, $\lambda_{TSuT}\,(k)$ is the correction terms series that can be add to the $TSuT\,(k)$ values and if $(u=u_1=u_2)$ then $k=\frac{t-s}{u}=\frac{r-q}{u_1}=\frac{p-o}{u_2}$, where u,u_1,u_2 are the number of divided on Z,Y,X respectively.

The singular integral $\int_{o}^{p}h(x,y,z)dx$ can be evaluated numerically by Trapezoidal rule on X and suppose (Y and Z are constants) as following:

$$\int_{o}^{p}h(x,y,z)dx=\frac{h}{2}\left(h(o,y,z)+h(p,y,z)+2\sum_{i=1}^{u-1}h(o+ik,y,z)\right)-\frac{(p-o)}{12}k^{2}h^{(2)}(\theta_1,y,z)$$
$$+\frac{(p-o)}{720}k^{4}h^{(4)}(\theta_2,y,z)-\frac{(p-o)}{30240}k^{6}h^{(6)}(\theta_3,y,z)-\cdots \qquad \cdots(13)$$

By integrate the form (17) numerical on the interval $[q,r]$ by using the suggested method on Y, then we get:

$$i)\int_{q}^{r}h(x_0,y,z)dy=\frac{k}{4}\Bigg[h(o,q,z)+h(o,r,z)+2h(o,q+(u-0.5)k,z)+2\sum_{j=1}^{u-1}(h(o,q+(j-0.5)k,z)$$
$$+h(o,q+jk,z))\Bigg]+\frac{(p-o)}{24}k^{2}h^{(2)}(o,\mu_1,z)-\frac{5(p-o)}{1440}k^{4}h^{(4)}(o,\mu_2,z)+\frac{61(p-o)}{60480}k^{6}h^{(6)}(o,\mu_3,z)-\cdots$$
$$...(14)$$

$$ii)\int_{q}^{r}h(p,y,z)dy=\frac{k}{4}\Bigg[h(p,q,z)+h(p,r,z)+2h(p,q+(u-0.5)k,z)+2\sum_{j=1}^{u-1}(h(p,q+(j-0.5)k,z)$$
$$+h(p,q+jk,z))\Bigg]+\frac{(p-o)}{24}k^{2}h^{(2)}(p,\mu_1,z)-\frac{5(p-o)}{1440}k^{4}h^{(4)}(p,\mu_2,z)+\frac{61(p-o)}{60480}k^{6}h^{(6)}(p,\mu_3,z)-\cdots$$
$$...(15)$$

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$$iii)\int_q^r h(o+ik,y,z)dy=\frac{k}{4}\Big[h(o+ik,q,z)+h(o+ik,r,z)+2h(o+ik,q+(u-0.5)k,z)+$$
$$2\sum_{j=1}^{u-1}(h(o+ik,q+(j-0.5)k,z)+h(o+ik,q+jk,z))\Big]+\frac{(p-o)}{24}k^2h^{(2)}(o+ik,\mu_1,z)$$
$$-\frac{5(p-o)}{1440}k^4h^{(4)}(o+ik,\mu_2,z)+\frac{61(p-o)}{60480}k^6h^{(6)}(o+ik,\mu_3,z)-\cdots \quad ...(16)$$
$$v)\int_q^r\left[-\frac{(p-o)}{12}k^2h^{(2)}(\theta_1,y,z)+\frac{(p-o)}{720}k^4h^{(4)}(\theta_2,y,z)-\frac{(p-o)}{30240}k^6h^{(6)}(\theta_3,y,z)-\cdots\right]dy$$

…(17)

And by integrate the above formulas numerically on the interval $[s,t]$ by using Trapezoidal rule on Z, we will get

$$i)\int_s^t\int_q^r h(x_0,y,z)dydz=\frac{k^2}{8}[h(o,q,s)+h(o,q,t)+h(o,r,s)+h(o,r,s)+2(h(o,q+(u-0.5)k,s)$$
$$+h(o,q+(u-0.5)k,t))+2\sum_{\ell=1}^{u-1}(h(o,q,s+\ell k)+h(o,r,s+\ell k)+2h(o,q+(u-0.5)k,s+\ell k))$$
$$+2\sum_{j=1}^{u-1}(h(o,q+(j-0.5)k,s)+h(o,q+(j-0.5)k,t)+(h(o,q+jk,s)+(h(o,q+jk,t)$$
$$+2\sum_{\ell=1}^{u-1}(h(o,q+(j-0.5)k,s+\ell k)+h(o,q+jk,s+\ell k)))\ ]+\int_s^t\left[\frac{(p-o)}{24}k^2h^{(2)}(o,\mu_1,z)\right.$$
$$\left.-\frac{5(p-o)}{1440}k^4h^{(4)}(o,\mu_2,z)+\frac{61(p-o)}{60480}k^6h^{(6)}(o,\mu_3,z)-\cdots\right]dz-\frac{(t-s)}{12}k^2h^{(2)}(o,q,\eta_1)$$
$$+\frac{(t-s)}{720}k^4h^{(4)}(o,q,\eta_2)-\frac{(t-s)}{30240}k^6h^{(6)}(o,q,\eta_3)-\cdots-\frac{(t-s)}{12}k^2h^{(2)}(o,r,\eta_1)+\frac{(t-s)}{720}k^4h^{(4)}(o,r,\eta_2)$$
$$-\frac{(t-s)}{30240}k^6h^{(6)}(o,r,\eta_3)-\cdots-\frac{(t-s)}{12}k^2h^{(2)}(o,q+(u-0.5)k,\eta_1)+\frac{(t-s)}{720}k^4h^{(4)}(o,q+(u-0.5)k,\eta_2)$$
$$-\frac{(t-s)}{30240}k^6h^{(6)}(o,q+(u-0.5)k,\eta_3)-\cdots+\sum_{j=1}^{u-1}\left[-\frac{(t-s)}{12}k^2h^{(2)}(o,q+(j-0.5)k,\eta_{1j})\right.$$
$$+\frac{(t-s)}{720}k^4h^{(4)}(o,q+(j-0.5)k,\eta_{2j})-\frac{(t-s)}{30240}k^6h^{(6)}(o,q+(j-0.5)k,\eta_{3j})-....$$
$$\left.-\frac{(t-s)}{12}k^2h^{(2)}(o,q+jk,\eta_{1j})+\frac{(t-s)}{720}k^4h^{(4)}(o,q+jk,\eta_{2j})-\frac{(t-s)}{30240}k^6h^{(6)}(o,q+jk,\eta_{3j})-...\right]$$

…(18)

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$$ii)\int_s^t\int_q^r h(p,y,z)dy\,dz = \frac{k^2}{8}[h(p,q,s)+h(p,q,t)+h(p,r,s)+h(p,r,s)+2(h(p,q+(u-0.5)k,s)$$

$$+h(p,q+(u-0.5)k,t))+2\sum_{\ell=1}^{u-1}(h(p,q,s+\ell k)+h(p,r,s+\ell k)+2h(p,q+(u-0.5)k,s+\ell k))$$

$$+2\sum_{j=1}^{u-1}(h(p,q+(j-0.5)k,s)+h(p,q+(j-0.5)k,t)+(h(p,q+jk,s)+(h(p,q+jk,t)$$

$$+2\sum_{\ell=1}^{u-1}(h(p,q+(j-0.5)k,s+\ell k)+h(p,q+jk,s+\ell k)))\ \ ]+\int_s^t\left[\frac{(p-o)}{24}k^2h^{(2)}(p,\mu_1,z)\right.$$

$$\left.-\frac{5(p-o)}{1440}k^4h^{(4)}(p,\mu_2,z)+\frac{61(p-o)}{60480}k^6h^{(6)}(p,\mu_3,z)-\cdots\right]dz-\frac{(t-s)}{12}k^2h^{(2)}(p,q,\eta_1)$$

$$+\frac{(t-s)}{720}k^4h^{(4)}(p,q,\eta_2)-\frac{(t-s)}{30240}k^6h^{(6)}(p,q,\eta_3)-\cdots-\frac{(t-s)}{12}k^2h^{(2)}(p,r,\eta_1)+\frac{(t-s)}{720}k^4h^{(4)}(p,r,\eta_2)$$

$$-\frac{(t-s)}{30240}k^6h^{(6)}(p,r,\eta_3)-\cdots-\frac{(t-s)}{12}k^2h^{(2)}(p,q+(u-0.5)k,\eta_1)+\frac{(t-s)}{720}k^4h^{(4)}(p,q+(u-0.5)k,\eta_2)$$

$$-\frac{(t-s)}{30240}k^6h^{(6)}(p,q+(u-0.5)k,\eta_3)-\cdots+\sum_{j=1}^{u-1}\left[-\frac{(t-s)}{12}k^2h^{(2)}(p,q+(j-0.5)k,\eta_{1j})\right.$$

$$+\frac{(t-s)}{720}k^4h^{(4)}(p,q+(j-0.5)k,\eta_{2j})-\frac{(t-s)}{30240}k^6h^{(6)}(p,q+(j-0.5)k,\eta_{3j})-....$$

$$\left.-\frac{(t-s)}{12}k^2h^{(2)}(p,q+jk,\eta_{1j})+\frac{(t-s)}{720}k^4h^{(4)}(p,q+jk,\eta_{2j})-\frac{(t-s)}{30240}k^6h^{(6)}(p,q+jk,\eta_{3j})-...\right]$$

...(19)

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$$iii)\int_s^t\int_q^r h(p,y,z)dydz=\frac{k^2}{8}[h(o+ik,q,s)+h(o+ik,q,t)+h(o+ik,r,s)+h(o+ik,r,s)+2(h(o+ik,q+(u-0.5)k,s)$$

$$+h(o+ik,q+(u-0.5)k,t))+2\sum_{\ell=1}^{u-1}(h(o+ik,q,s+\ell k)+h(o+ik,r,s+\ell k)+2h(o+ik,q+(u-0.5)k,s+\ell k))$$

$$+2\sum_{j=1}^{u-1}(h(o+ik,q+(j-0.5)k,s)+h(o+ik,q+(j-0.5)k,t)+(h(v,q+jk,s)+(h(o+ik,q+jk,t)$$

$$+2\sum_{\ell=1}^{u-1}(h(o+ik,q+(j-0.5)k,s+\ell k)+h(o+ik,q+jk,s+\ell k)))\ \ ]+\int_s^t\left[\frac{(p-o)}{24}k^2h^{(2)}(o+ik,\mu_1,z)\right.$$

$$\left.-\frac{5(p-o)}{1440}k^4h^{(4)}(o+ik,\mu_2,z)+\frac{61(p-o)}{60480}k^6h^{(6)}(o+ik,\mu_3,z)-\cdots\right]dz-\frac{(t-s)}{12}k^2h^{(2)}(o+ik,q,\eta_1)$$

$$+\frac{(t-s)}{720}k^4h^{(4)}(o+ik,q,\eta_2)-\frac{(t-s)}{30240}k^6h^{(6)}(o+ik,q,\eta_3)-\cdots-\frac{(t-s)}{12}k^2h^{(2)}(o+ik,r,\eta_1)+\frac{(t-s)}{720}k^4h^{(4)}(o+ik,r,\eta_2)$$

$$-\frac{(t-s)}{30240}k^6h^{(6)}(o+ik,r,\eta_3)-\cdots-\frac{(t-s)}{12}k^2h^{(2)}(o+ik,q+(u-0.5)k,\eta_1)+\frac{(t-s)}{720}k^4h^{(4)}(o+ik,q+(u-0.5)k,\eta_2)$$

$$-\frac{(t-s)}{30240}k^6h^{(6)}(o+ik,q+(u-0.5)k,\eta_3)-\cdots+\sum_{j=1}^{u-1}\left[-\frac{(t-s)}{12}k^2h^{(2)}(o+ik,q+(j-0.5)k,\eta_{1j})\right.$$

$$+\frac{(t-s)}{720}k^4h^{(4)}(o+ik,q+(j-0.5)k,\eta_{2j})-\frac{(t-s)}{30240}k^6h^{(6)}(o+ik,q+(j-0.5)k,\eta_{3j})-....$$

$$\left.-\frac{(t-s)}{12}k^2h^{(2)}(o+ik,q+jk,\eta_{1j})+\frac{(t-s)}{720}k^4h^{(4)}(o+ik,q+jk,\eta_{2j})-\frac{(t-s)}{30240}k^6h^{(6)}(o+ik,q+jk,\eta_{3j})-...\right]$$

$$v)\int_s^t\int_q^r\left[-\frac{(p-o)}{12}k^2h^{(2)}(\theta_1,y,z)+\frac{(p-o)}{720}k^4h^{(4)}(\theta_2,y,z)-\frac{(p-o)}{30240}k^6h^{(6)}(\theta_3,y,z)-\cdots\right]dydz$$

….(20)

And since $h_{x^2},h_{x^4},h_{x^6},...$, $h_{y^2},h_{y^4},h_{y^6},...$ and $h_{z^2},h_{z^4},h_{z^6},...$ are continuous functions at each point of the region $[o,p]\times[q,r]\times[s,t]$. By add (17),(18),(19)and (20) together we get

$$I=\int_s^t\int_q^r\int_o^p h(x,y,z)dxdydz=TSuT(k)=\frac{h^3}{16}[h(o,q,s)+h(o,q,t)+h(o,r,s)+h(o,r,t)+h(p,q,s)+h(p,q,t)$$

$$+h(p,r,s)+h(p,r,t)+2h(o,q+(u-0.5)k,s)+2h(o,q+(u-0.5)k,t)+2h(r,q+(u-0.5)k,s)$$

$$+2h(r,q+(u-0.5)k,t)+2\sum_{\ell=1}^{u-1}(h(a,c,e+\ell k)+h(o,r,s+\ell k)+h(p,q,s+\ell k)+h(p,r,s+\ell k)$$

$$+2h(o,q+(u-0.5)k,s+\ell k)+2h(r,q+(u-0.5)k,s+\ell k))+2\sum_{i=1}^{u-1}(h(o+ik,q,s)+h(o+ik,q,t)$$

$$+h(o+ik,r,s)+h(o+ik,r,t)+2h(o+ik,q+(u-0.5)k,s)+2h(o+ik,q+(u-0.5)k,t)$$

$$+2\sum_{\ell=1}^{u-1}(h(o+ik,q,s+\ell k)+h(o+ik,r,e+\ell k)+2h(o+ik,q+(u-0.5)k,s+\ell k))$$

$$+2\sum_{j=1}^{u-1}[h(o,q+(j-0.5)k,s)+h(o,q+(j-0.5)k,t)+2\sum_{\ell=1}^{u-1}h(o,q+(j-0.5)k,s+\ell k)$$

$$+h(p,q+(j-0.5)k,s)+h(p,q+(j-0.5)k,t)+2\sum_{\ell=1}^{u-1}h(p,q+(j-0.5)k,s+\ell k)$$

$$+2\sum_{i=1}^{u-1}(h(o+ik,q+(j-0.5)k,s)+h(o+ik,q+(j-0.5)k,t)+2\sum_{\ell=1}^{u-1}h(o+ik,q+(j-0.5)k,s+\ell k))$$

$$+h(o,q+jk,s)+h(o,q+jk,t)+2\sum_{\ell=1}^{u-1}h(o,q+jk,s+\ell k)+h(p,q+jk,s)+h(p,q+jk,t)$$

$$+2\sum_{\ell=1}^{u-1}h(p,q+jk,s+\ell k)+2\sum_{i=1}^{u-1}(h(o+ik,q+jk,s)+h(o+ik,q+jk,t)+2\sum_{\ell=1}^{u-1}h(o+ik,q+jk,\ell+\ell k))]$$

$$+(p-o)(r-q)(t-s)\left(\frac{-k^2}{12}\frac{\partial^2 h\left(\overline{\beta_1},\overline{\sigma_1},\overline{\psi_1}\right)}{\partial x^2}+\frac{k^4}{720}\frac{\partial^4 h\left(\overline{\beta_2},\overline{\sigma_2},\overline{\psi_2}\right)}{\partial x^4}-\frac{k^6}{30240}\frac{\partial^6 h\left(\overline{\beta_3},\overline{\sigma_3},\overline{\psi_3}\right)}{\partial x^6}-\cdots\right)$$

$$+(p-o)(r-q)(t-s)\left(\frac{k^2}{24}\frac{\partial^2 h\left(\overline{\overline{\beta_1}},\overline{\overline{\sigma_1}},\overline{\overline{\psi_1}}\right)}{\partial y^2}-\frac{5k^4}{1440}\frac{\partial^4 h\left(\overline{\overline{\beta_2}},\overline{\overline{\sigma_2}},\overline{\overline{\psi_2}}\right)}{\partial y^4}+\frac{61k^6}{60480}\frac{\partial^6 h\left(\overline{\overline{\beta_3}},\overline{\overline{\sigma_3}},\overline{\overline{\psi_3}}\right)}{\partial y^6}-\cdots\right)\ldots(21)$$

$$+(p-o)(r-q)(t-s)\left(\frac{-k^2}{12}\frac{\partial^2 h\left(\overline{\overline{\overline{\beta_1}}},\overline{\overline{\overline{\sigma_1}}},\overline{\overline{\overline{\psi_1}}}\right)}{\partial z^2}+\frac{k^4}{720}\frac{\partial^4 h\left(\overline{\overline{\overline{\beta_2}}},\overline{\overline{\overline{\sigma_2}}},\overline{\overline{\overline{\psi_2}}}\right)}{\partial z^4}-\frac{k^6}{30240}\frac{\partial^6 h\left(\overline{\overline{\overline{\beta_3}}},\overline{\overline{\overline{\sigma_3}}},\overline{\overline{\overline{\psi_3}}}\right)}{\partial z^6}-\cdots\right)$$

$$\lambda_{TSuT}=(p-o)(r-q)(t-s)\frac{k^2}{12}\left(-\frac{\partial^2 h\left(\overline{\beta_1},\overline{\sigma_1},\overline{\psi_1}\right)}{\partial x^2}+\frac{1}{2}\frac{\partial^2 h\left(\overline{\overline{\beta_1}},\overline{\overline{\sigma_1}},\overline{\overline{\psi_1}}\right)}{\partial y^2}-\frac{\partial^2 h\left(\overline{\overline{\overline{\beta_1}}},\overline{\overline{\overline{\sigma_1}}},\overline{\overline{\overline{\psi_1}}}\right)}{\partial z^2}\right)$$

$$+(p-o)(r-q)(t-s)\frac{k^4}{720}\left(\frac{\partial^4 h\left(\overline{\beta_2},\overline{\sigma_2},\overline{\psi_2}\right)}{\partial x^4}-\frac{5}{2}\frac{\partial^4 h\left(\overline{\overline{\beta_2}},\overline{\overline{\sigma_2}},\overline{\overline{\psi_2}}\right)}{\partial y^4}+\frac{\partial^4 h\left(\overline{\overline{\overline{\beta_2}}},\overline{\overline{\overline{\sigma_2}}},\overline{\overline{\overline{\psi_2}}}\right)}{\partial z^4}\right)$$

Where

$$+(p-o)(r-q)(t-s)\frac{k^6}{30240}\left(-\frac{\partial^6 h\left(\overline{\beta_3},\overline{\sigma_3},\overline{\psi_3}\right)}{\partial x^6}+\frac{61k^6}{2}\frac{\partial^6 h\left(\overline{\overline{\beta_3}},\overline{\overline{\sigma_3}},\overline{\overline{\psi_3}}\right)}{\partial y^6}-\frac{\partial^6 h\left(\overline{\overline{\overline{\beta_3}}},\overline{\overline{\overline{\sigma_3}}},\overline{\overline{\overline{\psi_3}}}\right)}{\partial z^6}\right)+\ldots$$

...(22)

$$\left(\overline{\overline{\overline{\beta_v}}},\overline{\overline{\overline{\sigma_v}}},\overline{\overline{\overline{\psi_v}}}\right),\left(\overline{\overline{\beta_v}},\overline{\overline{\sigma_v}},\overline{\overline{\psi_v}}\right),\left(\overline{\beta_v},\overline{\sigma_v},\overline{\psi_v}\right)\in[o,p]\times[q,r]\times[s,t]\ ,v=1,2,3,\ldots$$

If the integrand is continuous function and the partial derivatives is existing at each point of the integral region $[o,p]\times[q,r]\times[s,t]$, then the error form can be written as following

$$I-TSuT(k)=\xi_1 k^2+\xi_2 k^4+\xi_3 k^6+\cdots \qquad \ldots(23)$$

Where $\xi_1,\xi_2,\xi_3,\cdots$ are constants and they depend on partial derivatives for function at the integral region.

6-Examples and results:

Example (1): the integral $I=\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} x\sin(3.2y+1.5z)\,dxdydz$ which its analytical value is 0.13070564809215 (Rounded to 14 decimal places) with integrand is defined for all $(x,y,z)\in\left[0,\frac{\pi}{4}\right]\times\left[0,\frac{\pi}{4}\right]\times\left[0,\frac{\pi}{4}\right]$, results that listed in tables (1) , where $u=32$ the value is correct for three decimal places using TSuT. Then applying $AI(TSuT)$ method, we got a correct value for 9 decimal places with (2^{18} sub intervals).

Example (2): The integral $\int_2^3\int_1^2\int_0^1 xe^{-x-y-z}\,dxdydz$ which its analytical value is 0.00525674345502 (Rounded to 14 decimal places) with integrand is defined for all $(x,y,z)\in[0,1]\times[1,2]\times[2,3]$ We can conclude from table (2), where $u=32$ the value is correct for five decimal places and using TSuT. When applying $AI(TSuT)$ method, we got a correct value for 9 decimal places with (2^{18} sub intervals). Otherwise, we got a correct value for six decimal places by using a mentioned method where $u=64$ by using $AI(TSuT)$ rule, we got correct 10 decimal places with (2^{21} sub intervals).

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U	TSuT values	AI(TSuT)	AI(TSuT)
1	0.09964872364554		
2	0.12270156070941		
4	0.12869045719049	0.13079236799399	
8	0.13020097698957	0.13071046295410	
16	0.13057942591845	0.13070594060543	0.13070567631330
32	0.13067408915178	0.13070566624633	0.13070564852652

Table (1) evaluating the triple integral $I=\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}}\int_0^{\frac{\pi}{4}} x\sin(3.2y+1.5z)\,dx\,dy\,dz=0.13070564809215$ **by using** $\boldsymbol{AI(TSuT)}$

U	TSuT values	AI(TSuT)	AI(TSuT)	AI(TSuT)
1	0.00404136381356			
2	0.00497244560013			
4	0.00518692990479	0.00525112705606		
8	0.00523936951039	0.00525633959489		
16	0.00525240494407	0.00525671724206	0.00525674673960	
32	0.00525565913839	0.00525674180085	0.00525674350901	
64	0.00525647239531	0.00525674335138	0.00525674345597	0.00525674345485

Table (2) evaluating the triple integral $I=\int_2^{3}\int_1^{2}\int_0^{1} x\,e^{-x-y-z}\,dx\,dy\,dz=0.00525674345502$ **by using** $AI(TSuT)$

7-Conclusion

It can be seen from the a above tables:

When we evaluated the approximate value of triple integral with continuous integrand by using composite rule (from Trapezoidal rule on the dimensions X and Z and suggested rule on the dimension Y when the number of divisions are equals). The TSuT rule gives us correct value (for several decimal places) comparing with the real value for integrals by using several sub intervals without using any acceleration method, while we got a correct value for 9 decimal places with (2^{18} sub intervals) for first integral, and in second integral we got a correct value with 11 decimal places with (2^{21} sub intervals) if Aitken acceleration was used

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