

# On Cyclic Decomposition of Group $(Q_{2m} \times C_2)$ when m is an Even Number

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**Abstract—** The main objective of this paper, is determination of the cyclic decomposition of the abelian factor group  $AC(G) = \overline{R}(G)/T(G)$  where  $G = Q_{2m} \times C_2$  when m is an Even Number (the group of all Z-valued characters of G over the group of induced unit characters from all cyclic subgroups of G). We have found that the cyclic, decomposition  $AC(Q_{2m} \times C_2)$  depends on the elementary divisor of m as follows. if  $m = 2^h$ , h any positive integer, then:  $AC(Q_{2m} \times C_2) = \bigoplus_{i=1}^{2(h+1)} C_2$

**Keywords—** ; Group; The Cyclic Decomposition of the Group G; The Cyclic Decomposition of the Group  $Q_{2m}$ .

## 1. INTRODUCTION

The problem of determining the cyclic decomposition of  $AC(G)$  seem to be untouched. We use the concepts of invariant matrix in linear algebra to find the cyclic decomposition of  $AC(G)$ , G is considered to be the group  $Q_2^{h+1} \times C_2$ .

In 1968 T.Y Lam [2] defined  $AC(G)$  and he studied  $AC(G)$ , when G is a cyclic group.

In 2000 H.R .Yassin [4] studied the cyclic decomposition of  $AC(G)$  when G is an elementary abelian group . In 2006 R.H. Abass [9] found  $Ar(Q_{2m} \times C_2)$  When m is an even Number .

In this paper ,we find the cyclic decomposition of the factor group  $AC(Q_2^{h+1} \times C_2)$  where  $Q_{2m}$  is the Quaternion group of order  $4m$  When m is an Even Number and  $C_2$  is the Cyclic group of order 2

## 2. The Factor Group $AC(G)$

This section is devoted to the study of the factor group  $AC(G)$  of a group G and the cyclic decomposition of the group  $AC(Q_{2m})$  and Artin's characters table of the group  $(Q_{2m} \times C_2)$  when m is an Even Number .

**2.1 Definition:[1]** Let  $T(G)$  be the subgroup of  $\overline{R}(G)$  generated by Artin's characters.  $T(G)$  is normal subgroup of  $\overline{R}(G)$  and denotes the factor abelian group  $\overline{R}(G)/T(G)$  by  $AC(G)$  which is called **Artin cokernel of G**.

**2.2 Definition:[5]** Let M be a matrix with entries in a principal domain R. A **k-minor of M** is the determinant of  $k \times k$  sub matrix preserving row and column order.

**2.3 Definition:[5]** A **k-th determinant divisor of M** is the greatest common divisor (g.c.d) of all the k-minors of M. This is denoted by  $D_k(M)$ .

**2.4 Lemma:[5]** Let M, P and W be matrices with entries in a principal ideal domain R, let P and W be invertible matrices ,Then  $D_k(PM^{-1}W) = D_k(M)$  module the group of unites of R.

**2.5 Theorem:[5]** Let M be an  $n \times n$  matrix with entries in principal ideal domain R, then there exist two matrices P and W such that:

- 1- P and W are invertible.
- 2-  $P^{-1}M^{-1}W = D$ .
- 3- D is diagonal matrix.
- 4- if we denote  $D_{ii}$  by  $d_i$  then there exists a natural number m ;  $0 \leq m \leq n$  such that  $j > m$  implies  $d_j = 0$  and  $j \leq m$  implies  $d_j \neq 0$  and  $1 \leq j \leq m$  implies  $d_j | d_{j+1}$ .

**2.6 Definition:[5]** Let M be matrix with entries in a principal domain R, be equivalent to a matrix  $D = \text{diag}\{d_1, d_2, \dots, d_m, 0, 0, \dots, 0\}$  such that  $d_j | d_{j+1}$  for  $1 \leq j < m$ . We call D **the invariant factor matrix of M** and  $d_1, d_2, \dots, d_m$  the invariant factors of M

**2.7 Theorem:[5]** Let  $K$  be a finitely generated module over a principal domain  $R$ , then  $K$  is the direct sum of cyclic sub module with an annihilating ideal

$$\langle d_1 \rangle, \langle d_2 \rangle, \dots, \langle d_m \rangle, d_j | d_{j+1} \text{ for } j = 1, 2, \dots, K-1.$$

**2.8 Proposition:[1]**  $AC(G)$  is a finitely generated  $\mathbb{Z}$ -module.

**2.9 Theorem (Artin's ):[6]** Every rational valued character of  $G$  can be written as a linear combination of Artin characters with coefficient rational numbers.

### 3. The Matrix $M(G)$ :[3]

Let  $l$  be the number of all distinct  $\Gamma$ - classes then  $Ar(G)$  and  $\equiv(G)$  are of rank 1 according to the Artin's theorem there exists an invertible matrix

$M^{-1}(G)$  with entries in  $\mathbb{Q}$  such that :

$$\overset{*}{\equiv}(G) = M^{-1}(G) \cdot Ar(G)$$

and this implies,

$$M(G) = Ar(G) \cdot (\overset{*}{\equiv}(G))^{-1}$$

$M(G)$  is the matrix expressing the  $T(G)$  basis in terms of the  $\overline{R}(G)$  basis.

By Theorem (2.5) there exist two matrix  $P(G)$  and  $W(G)$  with determinant  $\pm 1$  such that:

$$\begin{aligned} P(G) \cdot M(G) \cdot W(G) &= \text{diag} \{ d_1, d_2, \dots, d_l \} \\ &= D(G) \end{aligned}$$

where  $d_i = \pm D_i / D_{i-1}$ . This process yields a new basis for  $T(G)$  and

$\overline{R}(G), \{v_1, v_2, \dots, v_l\}$  and  $\{u_1, u_2, \dots, u_l\}$  respectively, with the property  $v_j = d_j u_j$ .

Hence, by Theorem (2.7) and Proposition (2.8) the  $\mathbb{Z}$ -module  $AC(G)$  is the direct sum of cyclic sub modules with annihilating ideals

$$\langle d_1 \rangle, \langle d_2 \rangle, \dots, \langle d_l \rangle$$

**3.1 Theorem:[4]**  $AC(G) = \bigoplus_{i=1}^l C_{d_i}$  where  $d_i = \pm D_i / D_{i-1}$  where  $l$  is the number of all distinct  $\Gamma$ -classes.

**3.2 Corollary:[1]**  $|AC(G)| = |\det(M(G))|$ .

**3.3 Lemma:[1]** If  $A$  and  $B$  are two matrices of degree  $m$  and  $t$  respectively, then:

$$\det(A \otimes B) = (\det(A))^t \cdot (\det(B))^m.$$

**3.4 Lemma:[1]** Let  $A$  and  $B$  be two non-singular matrices of rank 1 and  $m$  respectively, over a principal domain  $R$  and let:

$$P_1 A W_1 = D(A) = \text{diag} \{ d_1(A), d_2(A), \dots, d_l(A) \}$$

and

$$P_2 A W_2 = D(B) = \text{diag} \{ d_1(B), d_2(B), \dots, d_m(B) \}$$

The invariant factor matrices of  $A$  and  $B$  then:

$$(P_1 \otimes P_2)(A \otimes B)(W_1 \otimes W_2) = D(A) \otimes D(B)$$

and from this the invariant factor matrices of  $A \otimes B$  can be obtained.

**3.5 Proposition:[4]** Let  $H_1$  and  $H_2$  be  $p_1$  and  $p_2$  - groups respectively where  $p_1$  and  $p_2$  are distinct primes and if  $M_1$  is the matrix from all cyclic subgroups of  $\overline{R}(H_1)$  basis and  $M_2$  is the matrix which expresses the  $T(H_2)$  basis terms of  $\overline{R}(H_2)$  basis then the matrix which expresses the  $T(H_1 \times H_2)$  basis of  $\overline{R}(H_1 \times H_2)$  basis is  $M_1 \otimes M_2$ .

**3.6 Proposition:[7]** If  $p$  is prime number and  $s$  is positive integer, then:

$$M(C_{P^s}) = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

which is of the order  $(s+1) \times (s+1)$

**3.7 Proposition:[7]**The general form of the matrices  $P(C_{P^s})$  and  $W(C_{P^s})$  is:

$$P(C_{P^s}) = \begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

which is  $(s+1) \times (s+1)$  square matrix.

$W(C_{P^s}) = I_{s+1}$ , where  $I_{s+1}$  is  $(s+1) \times (s+1)$  identity matrix

and  $D(C_{P^s}) = \text{diag } \underbrace{\{1,1,\dots,1\}}_{s+1}$ .

**3.8 Remarks: [6]**If m is an Even Number then we can write  $M(C_m)$  as the following :

$$M(C_m) = \begin{bmatrix} & 1 & 1 \\ R_1(C_m) & \vdots & \vdots \\ & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 & 1 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}$$

which is  $(h+1) \times (h+1)$  square matrix , $R_1(C_m)$  is the matrix obtained by omitting the last two rows  $\{0,0,\dots,1,1\}$  and  $\{0,0,\dots,0,0,1\}$  and the last two columns  $\{1,1,\dots,1,0\}$  and  $\{1,1,\dots,1,1\}$  from the matrix  $M(C_{2^h})$  in the Proposition (3.6).

**3.9 Proposition:[8]**If  $m=2^h$ ,h any positive integers ,then the matrix  $M(Q_{2m})$  of the quaternion group  $Q_{2m}$  is :

$$M(Q_{2m}) = \left[ \begin{array}{c|ccccc} 2R_1(C_{2m}) & 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & \dots & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & \dots & 1 & 1 & 0 & 0 & 1 \end{array} \right]$$

which is  $(h+4) \times (h+4)$  square matrix , $R(C_{2m})$  is similar to the matrix in the remarks (3.8).

**3.10 Example:** By the proposition (3.9) we can find matrix  $M(Q_{128})$  as follow

$$M(Q_{128}) = M(Q_{2^7}) =$$

$$R_1(C_{2^7}) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which 6×6 square matrix .

Then

$$M(Q_{2^7}) = \begin{bmatrix} 2R_1(C_{2^7}) & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

which is 10×10 square matrix .

**3.11 Proposition:[8]** If m is an Even Number then the matrices  $P(Q_{2m})$  and  $W(Q_{2m})$  are taking the forms:

$$P(Q_{2m}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ -1 & 1 \\ 0 & -1 \\ 0 & 0 & \dots & \dots & 0 & 1 & -1 \\ 0 & 0 & \dots & \dots & 0 & 0 & 1 \end{bmatrix}$$

And

$$W(Q_{2m}) = \begin{bmatrix} I_{h+1} & & & & & & & \\ & 0 & 0 & 0 & & & & \\ & 0 & 0 & 0 & & & & \\ & \vdots & \vdots & \vdots & & & & \\ & \vdots & \vdots & \vdots & & & & \\ & 0 & 0 & 0 & & & & \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & \cdots & -1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

where  $I_{h+1}$  is the identity matrix .They are  $(h+4) \times (h+4)$  square matrix .

**3.12 Example:** To find  $P(Q_{128})$  and  $W(Q_{128})$ , by the proposition (3.11) and to find  $AC(Q_{128})$

$$P(Q_{128}) = P(Q_{2^7}) = \left[ \begin{array}{cccccccc|cc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

which is  $10 \times 10$  square matrix

And

$$W(Q_{128}) = W(Q_{2^7}) = \left[ \begin{array}{cccccccc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 1 \end{array} \right]$$

which is  $10 \times 10$  square matrix

$$D(Q_{128}) = P(Q_{128}).M(Q_{128}).W(Q_{128}) = \text{diag}\{2, 2, 2, 2, 2, 2, 2, 1, 1, 1\}$$

**3.13 Theorem:** [8] If m is an Even Number then the cyclic decomposition of  $AC(Q_{2m})$  is:

$$AC(Q_{2m}) = \bigoplus_{i=1}^{h+1} C_2$$

#### 4 The Main Results

In this section we give the general form of the cyclic decomposition of the factor group  $AC(Q_{2m} \times C_2)$  when m is an Even Number .

**4.1 Proposition:** If m is an Even Number then the matrix  $M(Q_{2m} \times C_2)$  of the group  $(Q_{2m} \times C_2)$  is :

$$M(Q_{2m} \times C_2) = \begin{bmatrix} M(Q_{2m}) & M(Q_{2m}) \\ 0 & M(Q_{2m}) \end{bmatrix}$$

Which is  $2(h+4) \times 2(h+4)$  square matrix , $M(Q_{2m})$  is similar to the matrix in Proposition (3.9).

**Proof :** first find Artin's characters table of  $(Q_{2m} \times C_2)$  when m is an Even Number and from the theorem (1.6) we get the rational valued characters  $(\cong(Q_{2m} \times C_2))$  table of the group  $(Q_{2m} \times C_2)$  when  $m=2^h, h \in \mathbb{Z}^+$ .

Thus , by definition of  $M(G)$  we can find the matrix  $M(Q_{2m} \times C_2)$  when m is an Even Number.

$$M(Q_{2m} \times C_2) = Ar(Q_{2m} \times C_2) \cdot (\cong(Q_{2m} \times C_2))^{-1}$$

$$= \begin{bmatrix} 2 & 2 & 2 & \cdots & \cdots & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & \cdots & \cdots & 2 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & \cdots & \cdots & 2 & 1 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & \cdots & \cdots & 2 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & \cdots & \cdots & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 2 & \cdots & \cdots & 2 & 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots & \cdots & 2 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \cdots & \cdots & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & \cdots & \cdots & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 2 & 2 & 2 & \cdots & \cdots & 2 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & 2 & 2 & \cdots & \cdots & 2 & 1 & 1 & 1 & 1 & 1 \\ \ddots & & & & & & & & 0 & 0 & 2 & \cdots & \cdots & 2 & 1 & 1 & 1 & 1 & 1 \\ \vdots & \vdots & & & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & & & \vdots & \ddots & \vdots \\ \vdots & \vdots & & & & \vdots & \vdots & 0 & 0 & 0 & \cdots & \cdots & 2 & 1 & 1 & 1 & 1 & 1 \\ & & & & & & & & 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 1 & 1 & 1 & 1 \\ & & & & & & & & \ddots & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & 1 & 1 & \cdots & \cdots & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned}
 & \left[ \begin{array}{cccc|cccc|cccc|cccc}
 & 1 & 1 & 1 & 1 & & 1 & 1 & 1 & 1 \\
 & 1 & 1 & 1 & 1 & & 1 & 1 & 1 & 1 \\
 2.R_1(C_{2m}) & 1 & 1 & 1 & 1 & 2.R_1(C_{2m}) & 1 & 1 & 1 & 1 \\
 & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
 & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
 & 1 & 1 & 1 & 1 & & 1 & 1 & 1 & 1 \\
 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 1 & 1 & 1 \\
 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & 0 & 1 \\
 0 & 1 & 1 & \cdots & \cdots & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & \cdots & \cdots & 1 & 0 & 0 & 1 & 1 \\
 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 0 & 0 & 1 \\
 = & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & & & & & & & 1 & 1 & 1 & 1 \\
 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & & & & & & & 1 & 1 & 1 & 1 \\
 & \ddots & & & & & & & & & & & & & 1 & 1 & 1 & 1 \\
 & \vdots & \vdots & & & & \vdots & \vdots & & & & & & & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \vdots & \vdots & & & & \ddots & \vdots & \vdots & & & & & & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \vdots & \vdots & & & & \vdots & \vdots & & & & & & & 1 & 1 & 1 & 1 \\
 & & & & & & 0 & 0 & 0 & & & & & & 0 & 1 & 1 & 1 & 1 \\
 & & & & & & \ddots & 0 & 0 & 0 & & & & & 0 & 0 & 1 & 0 & 1 \\
 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & 1 & 1 & \cdots & \cdots & 1 & 0 & 0 & 1 & 1 \\
 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 & 0 & 0 & 1
 \end{array} \right] \\
 & = \left[ \begin{array}{c|c}
 M(Q_{2m}) & M(Q_{2m}) \\
 \hline
 0 & M(Q_{2m})
 \end{array} \right] = M(Q_{2m} \times C_2)
 \end{aligned}$$

**4.2 Example :** Consider the group  $(Q_{128} \times C_2)$ , we can find the matrix  $M(Q_{128} \times C_2)$  by using two ways :

**First :** By the definition of  $M(G)$

$$M(Q_{128} \times C_2) = M(Q_{2^7} \times C_2) = Ar(Q_{2^7} \times C_2) \cdot (\overset{*}{\equiv}(Q_{2^7} \times C_2))^{-1}$$

512	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
256	256	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
128	128	128	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
64	64	64	64	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
32	32	32	32	32	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	16	16	16	16	16	0	0	0	0	0	0	0	0	0	0	0	0	0
8	8	8	8	8	8	8	0	0	0	0	0	0	0	0	0	0	0	0
4	4	4	4	4	4	4	4	4	0	0	0	0	0	0	0	0	0	0
128	128	0	0	0	0	0	0	0	4	0	0	0	0	0	0	0	0	0
128	128	0	0	0	0	0	0	0	0	4	0	0	0	0	0	0	0	0
=	256	0	0	0	0	0	0	0	0	256	0	0	0	0	0	0	0	0
128	128	0	0	0	0	0	0	0	0	128	128	0	0	0	0	0	0	0
64	64	64	0	0	0	0	0	0	0	64	64	64	0	0	0	0	0	0
32	32	32	32	32	0	0	0	0	0	32	32	32	32	0	0	0	0	0
16	16	16	16	16	0	0	0	0	0	16	16	16	16	16	0	0	0	0
8	8	8	8	8	8	0	0	0	0	8	8	8	8	8	8	0	0	0
4	4	4	4	4	4	4	4	0	0	0	4	4	4	4	4	4	0	0
2	2	2	2	2	2	2	2	0	0	2	2	2	2	2	2	2	2	0
64	64	0	0	0	0	0	0	2	0	64	64	0	0	0	0	0	2	0
64	64	0	0	0	0	0	0	0	2	64	64	0	0	0	0	0	0	2

1/256	1/256	1/256	1/256	1/256	1/256	1/512	1/512	1/512	1/512	1/256	1/256	1/256	1/256	1/256	1/512	1/512	1/512			
-1/256	1/256	1/256	1/256	1/256	1/256	1/512	1/512	1/512	1/512	-1/256	1/256	1/256	1/256	1/256	1/512	1/512	1/512	1/512		
0	-1/128	1/128	1/128	1/128	1/128	1/256	1/256	1/256	1/256	0	-1/128	1/128	1/128	1/128	1/128	1/256	1/256	1/256	1/256	
0	0	-1/64	1/64	1/64	1/64	1/128	1/128	1/128	1/128	0	0	-1/64	1/64	1/64	1/64	1/128	1/128	1/128	1/128	
0	0	0	-1/32	1/32	1/32	1/64	1/64	1/64	1/64	0	0	0	-1/32	1/32	1/32	1/64	1/64	1/64	1/64	
0	0	0	0	-1/16	1/16	1/32	1/32	1/32	1/32	0	0	0	-1/16	1/16	1/32	1/32	1/32	1/32	1/32	
0	0	0	0	0	-1/8	1/16	1/16	1/16	1/16	0	0	0	0	-1/8	1/16	1/16	1/16	1/16	1/16	
0	0	0	0	0	0	-1/8	1/8	1/8	1/8	0	0	0	0	0	-1/8	1/8	-1/8	1/8		
0	0	0	0	0	0	-1/8	-1/8	1/8	1/8	0	0	0	0	0	-1/8	-1/8	1/8	1/8		
0	0	0	0	0	0	1/8	-1/8	-1/8	1/8	0	0	0	0	0	0	1/8	-1/8	-1/8	1/8	
-1/256	-1/256	-1/256	-1/256	-1/256	-1/256	-1/512	-1/512	-1/512	-1/512	1/256	1/256	1/256	1/256	1/256	1/512	1/512	1/512	1/512		
1/256	-1/256	-1/256	-1/256	-1/256	-1/256	-1/512	-1/512	-1/512	-1/512	-1/256	1/256	1/256	1/256	1/256	1/512	1/512	1/512	1/512		
0	1/128	-1/128	-1/128	-1/128	-1/128	-1/256	-1/256	-1/256	-1/256	0	-1/128	1/128	1/128	1/128	1/128	1/256	1/256	1/256	1/256	
0	0	1/64	-1/64	-1/64	-1/64	-1/128	-1/128	-1/128	-1/128	0	0	-1/64	1/64	1/64	1/64	1/128	1/128	1/128	1/128	
0	0	0	1/32	-1/32	-1/32	-1/64	-1/64	-1/64	-1/64	0	0	0	-1/32	1/32	1/32	1/64	1/64	1/64	1/64	
0	0	0	0	1/16	-1/16	-1/32	-1/32	-1/32	-1/32	0	0	0	-1/16	1/16	1/32	1/32	1/32	1/32	1/32	
0	0	0	0	0	1/8	-1/16	-1/16	-1/16	-1/16	0	0	0	0	0	-1/8	1/16	1/16	1/16	1/16	
0	0	0	0	0	0	1/8	-1/8	1/8	-1/8	0	0	0	0	0	0	-1/8	1/8	-1/8	1/8	
0	0	0	0	0	0	1/8	1/8	-1/8	-1/8	0	0	0	0	0	0	-1/8	-1/8	1/8	1/8	
0	0	0	0	0	0	-1/8	1/8	1/8	-1/8	0	0	0	0	0	0	0	1/8	-1/8	-1/8	1/8

**Second :** By the proposition (3.9) ,from example (3.10) then

And by the proposition (4.1) then  $M(Q_{128} \times C_2)$  equal to:

$$M(Q_{128} \times C_2) = \left[ \begin{array}{c|c} M(Q_{128}) & M(Q_{128}) \\ \hline 0 & M(Q_{128}) \end{array} \right] =$$

2	2	2	2	2	2	1	1	1	1	2	2	2	2	2	2	1	1	1	1
0	2	2	2	2	2	1	1	1	1	0	2	2	2	2	2	1	1	1	1
0	0	2	2	2	2	1	1	1	1	0	0	2	2	2	2	1	1	1	1
0	0	0	2	2	2	1	1	1	1	0	0	0	2	2	2	1	1	1	1
0	0	0	0	2	2	1	1	1	1	0	0	0	0	2	2	1	1	1	1
0	0	0	0	0	2	1	1	1	1	0	0	0	0	0	2	1	1	1	1
0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	1	0	1
0	1	1	1	1	1	0	0	1	1	0	1	1	1	1	1	0	0	1	1
0	1	1	1	1	1	0	0	0	1	0	1	1	1	1	1	1	0	0	1
0	0	0	0	0	0	0	0	0	0	2	2	2	2	2	2	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	2	2	2	2	2	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	2	2	2	2	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	2	2	2	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	2	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	1
0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0

**4.3 Proposition:** If m is an Even Number then the matrices  $P(Q_{2m} \times C_2)$  and  $W(Q_{2m} \times C_2)$  are taking the forms :

$$P(Q_{2m} \times C_2) = \left[ \begin{array}{c|c} P(Q_{2m}) & -P(Q_{2m}) \\ \hline 0 & P(Q_{2m}) \end{array} \right]$$

which is  $2(h+4) \times 2(h+4)$  square matrix .

And

$$W(Q_{2m} \times C_2) = \left[ \begin{array}{c|c} W(Q_{2m}) & 0 \\ \hline 0 & W(Q_{2m}) \end{array} \right]$$

which is  $2(h+4) \times 2(h+4)$  square matrix .

**Proof :** By using the proposition (4.1) taking the matrix  $M(Q_{2m} \times C_2)$  and the above forms  $P(Q_{2m} \times C_2)$  and  $W(Q_{2m} \times C_2)$  then we have :

$$\begin{aligned} P(Q_{2m} \times C_2) \cdot M(Q_{2m} \times C_2) \cdot W(Q_{2m} \times C_2) &= \text{diag } \{ \underbrace{2, 2, 2, 2, \dots, 2}_{2(h+1)}, 1, 1, 1, 1, 1, 1, 1 \} \\ &= D(Q_{2m} \times C_2) \end{aligned}$$

which is  $2(h+4) \times 2(h+4)$  square matrix .

**4.4 Example:** To find the matrices  $P(Q_{128} \times C_2)$  and  $W(Q_{128} \times C_2)$  by the proposition (4.3) from Example (3.12) to find  $P(Q_{128})$  and  $W(Q_{128})$  :

$$P(Q_{128} \times C_2) = \left[ \begin{array}{c|c} P(Q_{128}) & -P(Q_{128}) \\ \hline 0 & P(Q_{128}) \end{array} \right] =$$

And

$$W(Q_{128} \times C_2) = \left[ \begin{array}{c|c} W(Q_{128}) & 0 \\ \hline 0 & W(Q_{128}) \end{array} \right] =$$

**4.5 Example:** To find  $D(Q_{128} \times C_2)$  and the cyclic decomposition of the factor group

We find the matrices  $P(Q_{128} \times C_2)$  and  $W(Q_{128} \times C_2)$  as in example (4.4) and  $M(Q_{128} \times C_2)$  as in example (4.2), then :

$$P(Q_{128} \times C_2) \cdot M(Q_{128} \times C_2) \cdot W(Q_{128} \times C_2) =$$

$$\text{diag}\{2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,1,1,1,1,1,1\} = D(Q_{128} \times C_2)$$

Then by Theorem (3.1) we have

$$AC(D(Q_{128} \times C_2)) = \bigoplus_{i=1}^{14} C_2$$

The following theorem gives the cyclic decomposition of the factor group  $AC(D(Q_{2m} \times C_2))$  when m is an Even Number .

**4.6 Theorem:** If m is an Even Number then the cyclic decomposition of  $AC(Q_{2m} \times C_2)$  is :

$$AC(D(Q_{2m} \times C_2)) = \bigoplus_{i=1}^{2(h+1)} C_2$$

**Proof :** By using the proposition (4.1),we can find matrix  $M(Q_{2m} \times C_2)$  and by the proposition (4.3),we find  $P(Q_{2m} \times C_2)$  and  $W(Q_{2m} \times C_2)$  :

$P(Q_{2m} \times C_2) \cdot M(Q_{2m} \times C_2) \cdot W(Q_{2m} \times C_2) =$

$$\text{diag}\{2,2,2,2,2,\dots,2,2,2,1,1,1,1,1,1\}$$

Then ,by the theorem (3.1) we have :

$$AC(D(Q_{2m} \times C_2)) = \bigoplus_{i=1}^{2(h+1)} C_2$$

**4.7 Example :** Consider the groups  $(Q_{32768} \times C_2)$  , $(Q_{268435456} \times C_2)$ ,then :

1.  $AC(Q_{32768} \times C_2) = AC(Q_{2^{15}} \times C_2) = \bigoplus_{i=1}^{30} C_2$
2.  $AC(Q_{268435456} \times C_2) = AC(Q_{2^{28}} \times C_2) = \bigoplus_{i=1}^{56} C_2$

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