International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 4, Issue 8, August – 2020, Pages: 110-115

A Structure of BZ-Algebras and its Properties

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Abstract *- In this paper, we define the notions of BZ-algebras, quotient BZ-algebras and investigate its properties. Moreover we show the relation between ideals and congruences on BZ-algebras.*

Keywords— BZ-algebra, ideal, quotient BZ-algebra, congruences**.**

1. Introduction

In [2,3], W.A. Dudek and X. Zhang were introduced an algebraic model of BCC-algebras, i.e., implicational logic. Many authors have tried to construct some generalizations of this and similar algebras. One such an algebraic system have the same partial order as BCC-algebras and BCK-algebras but has no minimal element. Such obtained system is called a BZ-algebra [6,7] or a weak BCC-algebra [8]. From the mathematical point of view the last name is more corrected but more popular is the first ([1,4]). All these algebras have one distinguished element and satisfy some common identities playing a crucial role in these algebras and, in fact, are generalization or a special case of weak BCC-algebras. So, results obtained for weak BCC-algebras are in some sense fundamental for these algebras, especially for BCC/BCH/BCI/BCK-algebras. In this paper is introduction to the general theory of BZ-algebra. We give the notion of BZ-algebra, quotient BZ-algebra and investigate elementary and fundamental properties.

2. BZ-algebras

In this section, we do define some familiar concepts as BZ-algebras, both for illustration and for review of the concept. First, we give a few definitions and some notation.

Definition 2.1. An algebra (X; ∗, 0) with a binary operation ∗ and a nullary operation 0. Then X is called **BZ-algebra** if it satisfies for all x, y, z ∈ X :

(BZ-1) $((x * z) * (y * z)) * (x * y) = 0$;

(BZ-2) $x * 0 = x$;

(BZ-3) $x * y = 0 \; and \; y * x = 0$ implies that $x = y$.

First, give example of BZ-algebra.

Examples 2.2. Let ∗ be defined on an abelian group G by letting $x*y = x^{-1}\cdot y$, where x, y in G, with e is unity element of G. Then (G; ·, e) is a BZ-algebra.

Examples 2.3. Let X = {0, 1} and let ∗ be defined by:

*	0	1
0	0	1
1	1	0

Then (G; ∗, 0) is a BZ-algebra.

Theorem 2.3. Let (X; ∗, 0) be a BZ-algebra if and only if it satisfies the following conditions: for all x, y, z ∈ X,

(1) $((x * y) * (y * z)) * (x * z) = 0$;

(2) $(x * (x * y)) * y = 0$;

(3) $x * x = 0$;

(4) $x * y = 0 \; and \; y * x = 0$ implies that $x = y$.

Proof. Assume that (X; ∗, 0) is a BZ-algebra. From definition of BZ-algebra, (1) and (4) holds. Then we see that

$(x * (x * y)) * y = ((x * 0) * (x * y)) * (y * 0) = 0$, and

$x * x = (x * x) * 0 = ((x * 0) * (x * 0)) * (0 * 0) = 0$, so (2) and (3) holds.

Conversely, we need to show BZ-2. By (1), (2) and (3), we see that

$$((x * 0) * x) * 0 = ((x * 0) * x) * (0 * ((0 * x) * x))$$
$$= ((x * 0) * x) * ((x * x) * ((x * 0) * x)) = 0.$$ And since

$((x * 0) * x) * 0 = 0$. From (4), it follows that $(x * 0) * x = 0$ and
$x * (0 * x) = x * ((x * x) * x) = 0$. Therefore $x * 0 = x$, proving our theorem. ■

Definition 2.4. Define a binary relation ≤ on BZ-algebra X by letting $x \leq y$ if and only if $x * y = 0$.

Proposition 2.5. If (X; ∗, 0) is a BZ-algebra, then (X; ≤) is a partially order set. **Proposition 2.6.** If (X; ∗, 0) be a BZ-algebra and $0 \leq x$, then $x = 0$, for any x ∈ X. Moreover, 0 is called a minimal element in X.

Proof. Let x ≤ 0, then 0 ∗ x = 0. By BZ-2, x ∗ 0 = x, and thus x = 0.

It is easy to show that the following properties are true for a BZ-algebra.

Theorem 2.7. Let (X; ∗, 0) be a BZ-algebra if and only if it satisfies the following conditions: for all x, y, z ∈ X,
(1) $((x * y) * (y * z)) \leq (x * z)$;
(2) $((x * y) * y) \leq x$;
(3) $x \leq y$ if and only if $x * y = 0$.

Proposition 2.8. Let x, y, z be any element in a BZ-algebra X. Then
(1) $x \leq y$ implies $z * x \leq z * y$.
(2) $x \leq y$ implies $y * z \leq x * z$.

Proposition 2.9. Let x, y, z be any element in a BZ-algebra X. Then
$x * (y * z) = y * (x * z)$.

Proof. Since Theorem (2.7(2)), $(x * z) * z \leq x$, and by Proposition (2.8(2)), we get that $x * (y * z) \leq ((x * z) * z) * (y * z)$. Putting $x = y$ and $y = x * z$ in Theorem (2.7(1)), it follows that $((x * z) * z) * (y * z) \leq y * (x * z)$. By the transitivity of ≤ gives $x * (y * z) \leq y * (x * z)$. And we replacing x by y and y by x, we obtain $y * (x * z) \leq x * (y * z)$. By the anti-symmetry of ≤, thus
$x * (y * z) = y * (x * z)$ and finishing the proof.

Corollary 2.10. Let x, y, z be any element in a BZ-algebra X. Then
(1) $y * z \leq x$ if and only if $x * z \leq y$.
(2) $(z * x) * (z * y) \leq x * y$.

Proposition 2.11. Let x, y, z be any element in a BZ-algebra X. Then
(1) $((x * y) * y) * y = x * y$.
(2) $(x * y) * 0 = (x * 0) * (y * 0)$.

Proof.
(1) From Theorem (2.3(2)) and Theorem (2.7(1)),
$(((x * y) * y) * y) * (x * y) \leq x * ((x * y) * y) = 0$. Thus
$(((x * y) * y) * y) * (x * y) = 0$. Since $(x * y) * (((x * y) * y) * y) = ((x * y) * y) * ((x * y) * y) = 0$. So, by BZ-3, $(x * y) * y = x * y$.
(2) Since

$$(x * 0) * (y * 0) = (x * 0) * (y * ((x * y) * (x * y)))$$
$$= (x * 0) * ((x * y) * (y * (x * y)))$$
$$= (x * 0) * ((x * y) * (x * (y * y)))$$
$$= (x * y) * ((x * 0) * (x * 0))$$
$$= (x * y) * 0.$$ The proof is complete.

In this paper we will denote N for the set of all nonnegative integers, i.e.,
0, 1, 2, ..., and N* for the set of all natural numbers, i.e., 1, 2, 3, ..., and we will also use the following notation in brevity:
$y^0 * x = x$, $y^n * x = \underbrace{y * (\ldots * (y * (y * x)))}_{n-\text{times}}$, where x, y are any elements in a BZ-algebra and n ∈ N*.

Proposition 2.12. Let x, y be any element in a BZ-algebra X. Then
(1) $((y * x) * x)^n * x = y^n * x$, for any n ∈ N.
(2) $(x^n * 0) * 0 = (x * 0)^n * 0$, for any n ∈ N.

Proof. Let X be a BZ-algebra and x, y ∈ X and n, m ∈ N.
(1) Proceed by induction on n and defined the statement P(n),
$((y * x) * x)^n * x = y^n * x$.

We see that P(0) is true, $((y * x) * x)^0 * x = x = y^0 * x$. Assume that P(k) is true for some arbitrary k ≥ 0, that is $((y * x) * x)^k * x = y^k * x$. Since

$$\begin{aligned}((y * x) * x)^{k+1} * x &= ((y * x) * x) * \left(((y * x) * x)^k * x\right)\\ &= ((y * x) * x) * (y^k * x)\\ &= y^k * \left(((y * x) * x) * x\right)\\ &= y^k * (y * x)\\ &= y^{k+1} * x.\end{aligned}$$

This show that P(k + 1) is true and by the principle of mathematical induction, P(n) is true for each n ∈ N*.

(2) Since

$$\begin{aligned}(x^n * 0) * 0 &= \left(x * (x^n - 1 * 0)\right) * 0\\ &= (x * 0) * \left((x^n - 1 * 0) * 0\right)\\ &= (x * 0) * \left(\left(x * (x^{n-2} * 0)\right) * 0\right)\\ &= (x * 0) * \left((x * 0) * \left((x^{n-2} * 0) * 0\right)\right)\\ &= (x * 0)^2 * \left((x^{n-2} * 0) * 0\right)\\ &= \ldots = (x * 0)^n * 0 \ \text{✷}\end{aligned}$$

Given x ∈ X if it satisfies $0 * x = 0$, that is x ≤ 0, the element x is called a positive element of X. By definition, the zero element 0 of X is positive.

Proposition 2.12. Let x be any element in a BZ-algebra X. Then $x * (0 * (0 * x))$ is a positive element of X, for every x ∈ X.

Proof. Since $0 * (x * (0 * (0 * x)) = (0 * x) * (0 * (0 * (0 * x)))$ $= (0 * x) * (0 * 0x) = 0$. Therefore $x * (0 * (0 * x)$ is a positive element of X.

3. Ideals of BZ-algebra

Definition 3.1. A non-empty subset S of a BZ-algebra X is called **a subalgebra of X** on condition that x ∗ y ∈ S, whenever x, y ∈ S.

Definition 3.2. A non-empty subset *I* of a BZ-algebra X is called **an ideal of X** if it satisfies the following conditions:

(I-1) $0 \in I$,

(I-2) for any x, y ∈ X, $x * y \in I$ and $x \in I$ imply $y \in I$.

Examples 3.3. Let X = {0, 1, 2, 3} and let ∗ be defined by the table:

*	0	1	2	3
0	0	0	3	3
1	1	0	3	2
2	2	3	0	1
3	3	3	0	0

Thus, it can be easily shown that X is a BZ- algebra. And we see that I = {0, 1} and J = {0, 3} are subalgebras and ideals of X.

Lemma 3.4. If *I* is an ideal of BZ-algebra X, then *I* is an subalgebra. The convers is not true in general.

Lemma 3.5. Let *I* be a subalgebra of BZ-algebra X. Then A is an ideal of X if and only if x ∈ *I* and z ∗ y ∈ *I* imply z ∗ (x ∗ y) ∈ *I* for all x, y, z ∈ X.

Proof. Let *I* be an ideal of X and let x ∈ *I* whereas z ∗ y ∈ *I*. Suppose that $z * (x * y) \in I$. By Proposition (2.9), we see that $x * (z * y) \in I$. Since *I* is an ideal of X and $x \in I, z * y \in I$, a contradiction. So $z * (x * y) \in I$.

Conversely, assume that if $x \in I$ and $z * y \in I$ imply $z * (x * y) \in I$ for all x, y, z ∈ X. Since *I* is a subalgebra of X, then there is $x \in I$ which $0 = x * x \in I$ That is, $0 \in I$.

Now, let $x * y \in I$ and $x \in I$. Assume that $y \in I$. We have that $0 * y = y \in I$. It follows that $(x * y) * 0 \in I$. Hence $x * y \in I$, contradiction. Therefore *I* is an ideal of X. This completes the proof.

Corollary 3.6. Let *I* be a subalgebra of BZ-algebra X. Then *I* is an ideal of X if and only if $x \in I$ and $y \in I$ imply $x * y \in I$ for all x, y ∈ X.

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 4, Issue 8, August – 2020, Pages: 110-115

Lemma 3.7. Let I be a subalgebra of BZ-algebra X. Then I is an ideal of X if and only if $x * (y * z) \in I$ and $x * z \in I$ imply $y \in I$ for all x, y, z ∈ X.

Proof. Let I be an ideal of X and let $x * (y * z) \in I, x * z \in I$. Suppose that $y \in I$. By Proposition (2.9), we have $y * (x * z) \in I$. Since I is an ideal of X, thus $x * z \in I$, contradiction, this shows that $y \in I$.

Conversely, assume that $x * (y * z) \in I$ and $x * z \in I$ imply $y \in I$ for all x, y, z ∈ X. Since I is a subalgebra of X, then there is $y \in I$ which $0 = y * y \in I$. Then $0 \in I$. Let $y * z \in I, y \in I$ and suppose that $z \in I$. By BZ-2, $(y * z) * 0 \in I$ and $z * 0 \in I$. By assumption, so $y \in I$, a contradiction. This proves that I is an ideal of X.

Corollary 3.8. Let I be a subalgebra of BZ-algebra X. Then I is an ideal of X if and only if $x * y \in I$ and $y \in I$ imply $x \in I$ for all x, y, z ∈ X. This Lemma gives some properties of ideal of BZ-algebra.

Lemma 3.9. If I is an ideal of BZ-algebra X and J is an ideal of I, then J is an ideal of X.

Proof. Since J is an ideal of I, then $0 \in J$. Let x, y ∈ X such that $x * y \in J$ and $x \in J$. It follows that that $x * y \in I$ and $x \in I$. By assumption, I is an ideal of X, so $y \in I$ and $x \in J$ From J is an ideal of I, so $y \in J$. Therefore, J is an ideal of X.

Theorem 3.10. Let $\{I_j : j \in J\}$ be a family of subalgebras of a BZ-algebra X. Then $\cap_{j\in J} I_j$ is a subalgebra of X.

Proof. Let $\{I_j : j \in J\}$ be a family of subalgebras of X. It is obvious that $\cap_{j\in J} I_j \subseteq X$. Since $0 \in I_j$ for all j ∈ J, it follows that $0 \in \cap_{j\in J} I_j$. Let $x*y \in \cap_{j\in J} I_j$ and $x \in \cap_{j\in J} I_j$.

We will show that $\cap_{j\in J} I_j$ is a subalgebra of X. Let $x, y \in \cap_{j\in J} I_j$. It follows that x, y ∈ I_j for all j ∈ J. Since I_j is a subalgebra of X and $x * y \in I_j$, for all j ∈ J, then $x * y \in \cap_{j\in J} I_j$. This show that $\cap_{j\in J} I_j$ is a subalgebra

Theorem 3.11. Let $\{I_j : j \in J\}$ be a family of ideals of a BZ-algebra X. Then $\cap_{j\in J} I_j$ is an ideal of X.

Proof. Let $\{I_j : j \in J\}$ be a family of ideals of X. It is obvious that $\cap_{j\in J} I_j \subseteq X$. Since $0 \in I_j$ for all j ∈ J, it follows that $0 \in \cap_{j\in J} I_j$. Let $x*y \in \cap_{j\in J} I_j$ and $x \in \cap_{j\in J} I_j$. We get that $x * y \in I_j$ and $x \in I_j$ for all j ∈ J, then $y \in I_j$ for all j ∈ J. Because I_j is an ideal of X. So $y \in \cap_{j\in J} I_j$, proving our theorem.

Theorem 3.12. Let $\{J_i : i \in N\}$ be a family of subalgebras of a BZ-algebra X where $J_n \subseteq J_{n+1}$ for all n ∈ N. Then $\cup_{n=1}^{\infty} J_n$ is a subalgebra of X.

Proof. Let $\{J_i : i \in N\}$ be a family of subalgebras of X. We will show that $\cup_{n=1}^{\infty} J_n$ is a subalgebra of X. Let $x, y \in \cup_{n=1}^{\infty} J_n$. It follows that $x \in J_j$ for some j ∈ N and $y \in J_k$ for some k ∈ N. Furthermore, we assume that $j \le k$, we obtain $J_j \subseteq J_k$. That is, $x \in J_k$ and $x \in J_k$. Since J_k is a subalgebra of X, we get $x * y \in J_k \subseteq \cup_{n=1}^{\infty} J_n$. This proves that $\cup_{n=1}^{\infty} J_n$ is a subalgebra of X, proving our theorem.

Theorem 3.13. Let $\{J_i : i \in N\}$ be a family of ideals of a BZ-algebra X where $J_n \subseteq J_{n+1}$, for all n ∈ N. Then $\cup_{n=1}^{\infty} J_n$ is an ideal of X.

Proof. Let $\{J_i : i \in N\}$ be a family of ideals of X. It can be proved easily that $\cup_{n=1}^{\infty} J_n \subseteq X$. Since J_i is an ideal of X for all i, so $0 \in \cup_{n=1}^{\infty} J_n$.

Let $x * y \in \cup_{n=1}^{\infty} J_n$ and $x \in \cup_{n=1}^{\infty} J_n$. It follows that $x * y \in J_j$ for some j ∈ N and $x \in J_k$ for some k ∈ N. Furthermore, let $J_j \subseteq J_k$. Hence $x * y \in J_k$ and $x \in J_k$. By assumption, J_k is an ideal of X, it follows that $y \in J_k$. Therefore, $y \in \cup_{n=1}^{\infty} J_n$, proving that $\cup_{n=1}^{\infty} J_n$ is an ideal of X, proving our theorem.

4. Quotient BZ-Algebras

In this section, we describe congruence on BZ-algebras.

Definition 4.1. Let I be an ideal of a BZ-algebra X. Define a relation ∼ on X by:
x ∼ y if and only if $x * y \in I$ and $y * x \in I$.

Theorem 4.2. If I is an ideal of BZ-algebra X, then the relation ∼ is an equivalence relation on X.

Proof. Let I be an ideal of X and x, y, z ∈ X. By Theorem (2.3), $x * x = 0$ and assumption, $x * x \in I$. That is, x ∼ x. Hence ∼ is reflexive.

Next, suppose that x ∼ y. It follows that $x * y \in I$ and $y * x \in I$. Then y ∼ x, so ∼ is symmetric.

Finally, let x ∼ y and y ∼ z. Then $x * y, y * x, y * z, z * y \in I$ and $\big((x * z) * (y * z)\big) * (x * y) = 0 \in I$. It follows that $(x * z) * (y * z) \in I$, and since $z * y \in I$, so $z * x \in I$. Similarly, $x * z \in I$. Thus ∼ is transitive.

Therefore, ∼ is an equivalence relation.

Lemma 4.3. Let I be an ideal of BZ-algebra X. For any x, y, u, v ∈ X, if u ∼ v and x ∼ y, then u ∗ x ∼ v ∗ y.

Proof. Assume that u ∼ v and x ∼ y, for any x, y, u, v ∈ X, then $u * v, v * u,$

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 4, Issue 8, August – 2020, Pages: 110-115

$x * y, y * x \in I$ and by BZ-1, we see that $((u * x) * (v * x)) * (u * v) = 0$ and $((v * x) * (u * x)) * (v * u) = 0$. From assumption and I is an ideal of X, these imply that $(v * x) * (u * x) \in I$ and $(u * x) * (v * x) \in I$. This shows that $v * x \sim u * x$.

On the other hand, by Corollary (2.10), we have that $((y * v) * (x * v)) * (y * x) = 0$ and $((x * v) * (y * v)) * (x * y) = 0$. From assumption and I is an ideal of X, these imply that $(y * v) * (x * v) \in I$ and $(x * v) * (y * v) \in I$. Thus $x * v \sim y * v$. Since $\sim$ is symmetric and transitive, so $u * x \sim v * y$.

Corollary 4.4. If I is an ideal of BZ-algebra X, then the relation $\sim$ is a congruence relation on X.

Proof. By Theorem (4.2) and Lemma (4.3).

Definition 4.5. Let I be an ideal of a BZ-algebra X. Given $x \in X$, the equivalence class $[x]_I$ of x is defined as the set of all element of X that are equivalent to x, that is, $[x]_I = \{y \in X : x \sim y\}$. We define the set $X/I = \{[x]_I : x \in X\}$ and a binary operation $\circ$ on X/I by $[x]_I \circ [y]_I = [x * y]_I$.

Theorem 4.6. If I is an ideal of BZ-algebra X with $X/I = \{[x]_I : x \in X\}$ where a binary operation $\circ$ on a set X/I is defined by $[x]_I \circ [y]_I = [x * y]_I$, then the binary operation $\circ$ is a mapping from $X/I \times X/I$ to X/I.

Proof. Let $[x_1]_I, [x_2]_I, [y_1]_I, [y_2]_I \in X/I$ such that $[x_1]_I = [x_2]_I$ and $[y_1]_I = [y_2]_I$. It follows that $x_1 \sim x_2$ and $y_1 \sim y_2$. By Lemma (4.3), $x_1 * y_1 \sim x_2 * y_2$, proving that $[x_1 * y_1]_I = [x_2 * y_2]_I$.

Theorem 4.7. If I is an ideal of BZ-algebra X, then $(X/I; \circ, [0]_I)$ is a BZ-algebra. Moreover, the set X/I is called the quotient BZ-algebra.

Proof. Let $[x]_I, [y]_I, [z]_I \in X/I$. Then

$(([x]_I \circ [z]_I) \circ ([y]_I \circ [z]_I)) \circ ([x]_I \circ [y]_I) = ([x * z]_I \circ [y * z]_I) \circ [x * y]_I$
$= [(x * z) * (y * z)]_I \circ [x * y]_I = [((x * z) * (y * z)) * (x * y)]_I = [0]_I$. It is clear that $[x]_I \circ [0]_I = [x * 0]_I = [x]_I$. Now, let $[x]_I \circ [y]_I = [0]_I$ and $[y]_I \circ [x]_I = [0]_I$. It follows that $x * y \sim 0$ and $y * x \sim 0$, that is $(x * y) * 0, (y * x) * 0 \in I$. Since I is an ideal of X and $0 \in I$, we get that $x * y, y * x \in I$. Consequently, $x \sim y$, proving that $[x]_I = [y]_I$. Therefore, $(X/I; \circ, [0]_I)$ is a BZ-algebra.

Example 4.8. According to Example (3.3), we can get that $X/I = \{[0]_I, [2]_I\}$, where $[0]_I = [1]_I = \{0, 1\}$ and $[2]_I = [3]_I = \{2, 3\}$. Let $\circ$ be defined on X/I by:

$\circ$	$[0]_I$	$[2]_I$
$[0]_I$	$[0]_I$	$[2]_I$
$[2]_I$	$[2]_I$	$[0]_I$

Then $(X/I; \circ, [0]_I)$ is a BZ-algebra.

Proposition 4.9. Let X be a BZ-algebra and I, J be any sets such that $I \subseteq J \subseteq X$. Suppose that I is a subalgebra of X. Then J is a subalgebra of X if and only if J/I is a subalgebra of X/I.

Proof. Let I be a subalgebra of X with $I \subseteq J \subseteq X$. Suppose firstly that J is a subalgebra of X, then $J/I = \{[x]_I : x \in J\}$, where $[x]_I = \{y \in J : x \sim y\}$, and $X/I = \{[x]_I : x \in X\}$, where $[x]_I = \{y \in X : x \sim y\}$.

Obviously, $J/I \subseteq X/I$ and $[0]_I \in J/I$.

Now, let $[x]_I \in J/I$ and $[x]_I \in J/I$, it follows that $x \in J$ and $y \in J$. By assumption, $x * y \in J$. Accordingly, $[x * y]_I = [x]_I \circ [y]_I \in J/I$, this shows that J/I is a subalgebra of X/I.

Proposition 4.10. Let X be a BZ-algebra and I, J be any sets such that $I \subseteq J \subseteq X$. Suppose that I is an ideal of X, then J is an ideal of X if and only if J/I is an ideal of X/I.

Proof. Let I be an ideal of X with $I \subseteq J \subseteq X$. Suppose firstly that J is an ideal of X, then $J/I = \{[x]_I : x \in J\}$, where $[x]_I = \{y \in J : x \sim y\}$, and $X/I = \{[x]_I : x \in X\}$, where $[x]_I = \{y \in X : x \sim y\}$. Obviously, $J/I \subseteq X/I$ and $[0]_I \in J/I$.

Now, let $[x]_I \circ [y]_I \in J/I$ and $[y]_I \in J/I$. Then $[x * y]_I = [x]_I \circ [y]_I \in J/I$, it follows that $x * y \in J$ and $x \in J$. By assumption, $y \in J$. Accordingly, $[y]_I \in J/I$, this shows that J/I is an ideal of X/I.

On the other hand, suppose that J/I is an ideal of X/I and I is an ideal of X with $I \subseteq J \subseteq X$. Thus, $0 \in J$. Let $x * y \in J$ and $x \in J$.

It follows that $[x*y]_I, [x]_I \in J/I$. Since $[x*y]_I = [x]_I \circ [y]_I$, so $[x]_I \circ [y]_I \in J/I$. By hypothesis, $[y]_I \in J/I$ implies $y \in J$, proving our Lemma.

Corollary 4.11. Let I, J be ideals of a BZ-algebra X with $I \subseteq J$, then I is an ideal of X.

Proof. Obvious.

Next, the basic properties of equivalence classes are considered are as the following Theorem.

Theorem 4.12. Let I be a subalgebra (ideal) of a BZ-algebra X and $a, b \in X$. Then

(1) $[a]_I = I$ if and only if $a \in I$.

(2) $[a]_I = [b]_I$ or $[a]_I \cap [b]_I = \emptyset$.

Proof. Let I be a subalgebra (ideal) of X and $a, b \in X$.

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
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(1) It is clear due to the fact that $a \sim a$ for all $a \in X$ and $a * a = 0 \in I$, so we get that $a \in [a]_I = I$.

Conversely, let $x \in [a]_I$. Then $x \sim a$, it follows that $x * a, a * x \in I$. By hypothesis, $x \in I$. Hence, $[a]_I \subseteq I$. To show that $I \subseteq [a]_I$, choose $x \in I$. Since I is a subalgebra (ideal) of X, we have $x * a, a * x \in I$. Thus, $x \sim a$, this means that $x \in [a]_I$ and shows that $I \subseteq [a]_I$. Consequently, $[a]_I = I$.

(2) Assume that $[a]_I \cap [b]_I = \emptyset$. Then there is $x \in [a]_I \cap [b]_I$ such that $x \in [a]_I$ and $x \in [b]_I$. It follows that $x \sim a$ and $x \sim b$, so $a \sim b$ by the symmetric and transitive properties. Thus $[a]_I = [b]_I$.

Theorem 4.13. If I is a subalgebra of a BZ-algebra X and $y \in I$, then $[y]_I$ is subalgebra of X.

Proof. Let I be a subalgebra of X and $y \in I$. It is clear that $0 \in [y]_I$. Now, suppose that $a \in [y]_I$ and $b \in [y]_I$. We will show that $a * b \in [y]_I$. Then $a \sim y$ and $b \sim y$, it follows that $y * a \in I$ and $y * b \in I$. By assumption, $y \in I$ and I is a subalgebra of X and $a * b \in I$, therefore, $y * (a * b) \in I$. Then $(a * b) \sim y$, this means $a * b \in [y]_I$.

Finally, let $a, b \in [y]_I$. Then $a \sim y$ and $b \sim y$, by Lemma (4.3), $a * b \sim y * y$. By Theorem (2.3), it follows that $a * b \sim 0$. Thus $a * b \in [0]_I$. Now, we have $0, y \in I$ and I is a subalgebra, so $0 * y \in I$ and $y * 0 \in I$. That is, $0 \sim y$. Hence, $[0]_I = [y]_I$. By transitive, $a * b \in [y]_I$, proving our theorem.

Theorem 4.14. If I is an ideal of a BZ-algebra X and $y \in I$, then $[y]_I$ is an ideal of X.

Proof. Let I be an ideal of X and $y \in I$. It is clear that $0 \in [y]_I$. Now, suppose that $a * b \in [y]_I$ and $a \in [y]_I$. We will show that $b \in [y]_I$. Then $a * b \sim y$ and $a \sim y$, it follows that $y * (a * b) \in I$ and $y * a \in I$. By assumption, $a \in I$. From Proposition (2.9), $a * (y * b) = y * (a * b) \in I$, and I is an ideal of X and $a \in I$, therefore, $y * b \in I$.

By properties of X, we get that

$$(a * 0) * \Big(\big((a * b) * y\big) * (b * y)\Big) = \big(a * (b * b)\big) * \Big(\big((a * b) * y\big) * (b * y)\Big)$$
$$= (b * (a * b)) * (((a * b) * y) * (b * y)) = 0.$$

By hypothesis, $(a * 0) * (((a * b) * y) * (b * y)) \in I$, and I is an ideal and $a \in I$, then $b * y \in I$. Hence, $b \sim y$, this means $a \in [y]_I$. Accordingly, $[y]_I$ is an ideal of X, proving our theorem.

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