

Isomorphism of BZ-algebras

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Abstract: In this paper, we introduce homomorphisms of BZ-algebra and investigate its properties. Moreover, the relations between quotient BZ-algebra and isomorphism also provided.

Keywords: homomorphism, isomorphism, ideal, BZ-algebras

1. Introduction

In [7], R. F. Ye introduced a new algebraic structure which is called BZ-algebras. And we described the relation between ideals and congruences[6]. Furthermore, They defined quotient BZ-algebra and study its properties [8]. Several authors [1,2,5] have studied homomorphism of BCI-algebras, BCK-algebras and binary algebra. In [3,4] W.A. Dudek, X. Zhang have defined BCC-algebras and studied investigate some of its properties and homomorphism of BCC-algebras. In this paper, we apply the notion of homomorphism of BCI-algebras, BCK-algebras and binary algebra theory to BZ-algebras, and as a result. We introduce a new concept, called BZ-isomorphism. Using this concept as defined we investigated some of its properties. The purpose of this paper is to derive some straightforward consequences of the relations between quotient BZ-algebras and isomorphisms and also investigate some of its properties.

2. Preliminaries

In this section we introduced an algebraic structure called a BZ-algebra which is an algebra $(X, *, 0)$ with a binary operation $*$ and a nullary operation 0 such that for all $x, y, z \in X$, satisfies the following properties:

$$(BZ-1) ((x * z) * (y * z)) * (x * y) = 0;$$

$$(BZ-2) x * 0 = x;$$

$$(BZ-3) x * y = 0 \text{ and } y * x = 0 \text{ implies that } x = y.$$

On BZ-algebra $(X, *, 0)$, we defined a binary relation \leq on X by putting $x \leq y$ if and only if $x * y = 0$. Then (X, \leq) is a partially ordered set. It is easy to show that the following properties are true for a BZ-algebra. For any x, y, z in X :

$$(P-1) x * ((x * y) * y) = 0;$$

$$(P-2) x * x = 0;$$

$$(P-3) x * (y * z) = y * (x * z);$$

$$(P-4) ((x * y) * y) * y = x * y;$$

$$(P-5) (x * y) * 0 = (x * 0) * (y * 0);$$

$$(P-6) (x * y) * ((z * x) * (z * y)) = 0;$$

$$(P-7) x \leq y \text{ implies } y * z \leq x * z;$$

$$(P-8) x \leq y \text{ implies } z * x \leq z * y.$$

A subset S of a BZ-algebra X is called **subalgebra of X** if $x * y \in S$ whenever $x, y \in S$. A non-empty subset I of a BZ-algebra X is called **ideal of X** if it satisfies the following conditions:

$$(I-1) 0 \in I$$

(I-2) For any $x, y \in X$, $x * y \in I$ and $x \in I$ imply $y \in I$.

Let I be an ideal of BZ-algebra X . Define the relation \sim on X by $x \sim y$ if and only if $x * y \in I$ and $y * x \in I$. Then the relation \sim is an equivalence relation on X and $[0]_I = \{x \in X \mid x \sim 0\}$ is an ideal of X .

Let \sim be an equivalence relation on a BZ-algebra X and I be an ideal of X . Define $[x]_I$ by $[x]_I = \{y \in X \mid x \sim y\} = \{y \in X \mid x * y \in I, y * x \in I\}$. Then the family $\{[x]_I \mid x \in X\}$ gives a partition of X which is denoted by X/I . For any $x, y \in X$, we defined $[x]_I \circ [y]_I = [x * y]_I$, then the binary operation \circ is a mapping from $X/I \times X/I$ to X/I . It is easily checked that $(X/I, \circ, [0]_I)$ is a BZ-algebra. Moreover, the set X/I is called the **quotient BZ-algebra**. And if I is a closed ideal of BZ-algebra X , then it is clear that $[a]_I = I$, for all a in I .

3. homomorphism of BZ-algebras

In this section, first we will define BZ-homomorphism and next we can describe properties of BZ-homomorphism.

Definition 3.1. Let $(X, *, 0_X)$ and $(Y, \square, 0_Y)$ be BZ-algebras. A homomorphism is a map $f : X \rightarrow Y$ satisfying $f(x * y) = f(x) \square f(y)$ for all $x, y \in X$.

For example, the zero mapping $g : X \rightarrow Y$ where $g(x) = 0_Y$, for any $x \in X$, then g is a BZ-algebra.

In general a homomorphism $f : X \rightarrow Y$ may not be surjective or injective. An injective homomorphism is called **monomorphism**, a surjective homomorphism is called **epimorphism**, and a bijective homomorphism is called **isomorphism**.

Moreover, we say X is isomorphic to Y , symbolically, $X \cong Y$. The kernel of the homomorphism f , denoted by $\ker f$, is the set of elements of X that map to 0_Y in Y . **Definition 3.2.** Let f be a mapping of a BZ-algebra X into a BZ-algebra Y , and let $I \subseteq X$ and $J \subseteq Y$. The image of I in X under f is $f(I) = \{f(x) \mid x \in I\}$ and the inverse image of J in Y is $f^{-1}(J) = \{x \in X \mid f(x) \in J\}$.

Next, the basic properties of homomorphism are considered as the following theorem.

Theorem 3.3. Let f be a homomorphism of a BZ-algebra X into a BZ-algebra Y . Then:

- (1) $f(0_X) = 0_Y$.
- (2) If 0_X is the identity in X , then $f(0_X)$ is the identity in Y .
- (3) f is injective if and only if $\ker f = \{0_X\}$.
- (4) $x \leq_X y$ implies $f(x) \leq_Y f(y)$.

Proof. Assume that $f : X \rightarrow Y$ is a homomorphism.

(1) Since $0_X * 0_X = 0_X$, then $f(0_X) = f(0_X * 0_X) = f(0_X) \square f(0_X) = 0_Y$.

(2) Assume that 0_X is the identity in X and 0_Y is the identity in Y . From BZ-2, $f(0_X) \circ 0_Y = 0_Y$ and $f(0_X) \square 0_Y = f(0_X) \square [f(0_X) \square f(0_X)] = f(0_X) \square f(0_X * 0_X) = f(0_X) \square f(0_X) = 0_Y$. By BZ-3, we get that $f(0_X) = 0_Y$. This show that $f(0_X)$ is the identity in Y .

(3) Suppose that f is injective and $x \in \ker f$. It follows that $f(x) = 0_Y$. Since $f(0_X) = 0_Y$, so $f(x) = f(0_X)$. By assumption, $x = 0_X$. Thus $\ker f = \{0_X\}$.

Conversely, suppose that $\ker f = \{0_X\}$. Let $x, y \in X$ be such that $f(x) = f(y)$. Then we get that $f(x * y) = f(x) \square f(y) = 0_Y$ and $f(y * x) = f(y) \square f(x) = 0_Y$, thus

$x * y, y * x \in \ker f$, this means that $x * y = 0_X = y * x$. From BZ-3, $x = y$, and shows that f is injective.

(4) Let $x \leq_X y$. It follows that $x * y = 0_X$. So, (1) implies $f(x) \square f(y) = f(x * y) = f(0_X) = 0_Y$. Hence $f(x) \leq_Y f(y)$.

Theorem 3.4. Let $f : X \rightarrow Y$ be a homomorphism. Then:

- (2) If I is a subalgebra of X , then $f(I)$ is a subalgebra of Y .
- (1) If I is an ideal of X , then $f(I)$ is an ideal of Y .
- (4) If J is a subalgebra of Y , then $f^{-1}(J)$ is a subalgebra of X .
- (3) If J is an ideal in Y , then $f^{-1}(J)$ is an ideal in X .
- (5) $\ker f$ is a subalgebra of X .
- (6) $\ker f$ is an ideal of X .
- (7) $\text{Im} f$ is a subalgebra of Y .

Proof. Assume that $f : X \rightarrow Y$ is a homomorphism.

(1) Let I be a subalgebra of X and $x, y \in f(I)$. Then there exist $a, b \in I$ such that $x = f(a)$ and $y = f(b)$.

Since $x \boxtimes y = f(a) \boxtimes f(b) = f(a * b) \in f(I)$. Thus $f(I)$ is a subalgebra of Y .

(2) Let I be an ideal of X . We see that $0_X \in I$, and by Theorem (3.3(1)), $0_Y = f(0_X) \in f(I)$, so $0_Y \in f(I)$. Now, assume that $f(x) \boxtimes f(y) \in f(I)$ and $f(x) \in f(I)$, it follows that $f(x * y) \in f(I)$, so $x * y, x \in I$. Since I is an ideal of X , $y \in I$, it follows that $f(y) \in f(I)$. Hence $f(I)$ is an ideal of Y .

(3) Let J be a subalgebra of Y and $x, y \in f^{-1}(J)$. Then $f(x) = a$ and $f(y) = b$ for some $a, b \in J$. Thus

$f(x * y) = f(x) \boxtimes f(y) = a * b \in J$, as J is a subalgebra. Hence $x * y \in f^{-1}(J)$.

(4) Let J be an ideal in Y . Then $0_Y \in J$, we get that $0_X = f^{-1}(0_Y) \in f^{-1}(J)$.

For any $x, y \in X$, let $x * y \in f^{-1}(J)$ and $x \in f^{-1}(J)$. It follows that $f(x) \boxtimes f(y) = f(x * y) \in J$ and $f(x) \in J$. Since J is an ideal of Y , we obtain that $f(y) \in J$. Consequently $y \in f^{-1}(J)$, proving that $f^{-1}(J)$ is an ideal of X .

(5) Now, we will show $\ker f$ is subalgebra of X . Let $x, y \in \ker f$. Then $f(x * y) = f(x) \boxtimes f(y) = 0_Y \boxtimes 0_Y = 0_Y$, these imply that $x * y \in \ker f$. Therefore $\ker f$ is a subalgebra of X .

(6) It is clear that $\ker f \subseteq X$. Since $f(0_X) = 0_Y$, so $0_X \in \ker f$. It follows that $\ker f \neq \emptyset$. Let $x * y \in \ker f$ and $x \in \ker f$ implies $f(x * y) = 0_Y$ and $f(x) = 0_Y$. We get that $f(y) = f(y) \boxtimes 0_Y = f(x) \boxtimes f(y) = f(x * y) = 0_Y$. Thus $y \in \ker f$.

(7) Let $a, b \in \text{Im}(f)$, then there exist $x, y \in X$ such that $a = f(x)$ and $b = f(y)$, so $a \boxtimes b = f(x) \boxtimes f(y) = f(x * y) \in \text{Im}(f)$. This proves that $\text{Im}(f)$ is a subalgebra of Y .

In general, $\text{Im}(f)$ may not be an ideal.

Example 3.5. Let $X = \{0, 1, 2\}$. Define an operation $*$ on X by:

*	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

Then, it can be easily show that $(X, *, 0)$ is a BZ-algebras. Now, let f be the mapping from X to itself such that $f(0) = 0, f(1) = 0$ and $f(2) = 2$, then we see that $\text{Im}(f) = \{0, 2\}$. So $\text{Im}(f)$ is not an ideal of X , since $2 \in \text{Im}(f)$ and $2 * 1 = 0 \in \text{Im}(f)$, but $1 \notin \text{Im}(f)$.

The next proposition holds, whose verification is routine and omitted.

Proposition 3.6. Let f be a homomorphism from a BZ-algebra X to a BZ-algebra Y . Then:

(1) f is an epimorphism if and only if $\text{Im}(f) = Y$.

(2) f is an isomorphism if and only if the inverse mapping f^{-1} is an isomorphism.

Theorem 3.7. Let I be an ideal of BZ-algebra X . Defined the map $f : X \rightarrow X/I$ by $f(x) = [x]_I$, for all $x \in X$. Then f is epimorphism, we call f is the natural homomorphism of X onto X/I . Furthermore, $\ker f = I$.

Proof. Let I be an ideal of BZ-algebra X and $x, y \in X$. Since $f(x * y) = [x * y]_I = [x]_I \circ [y]_I = f(x) \boxtimes f(y)$, proving that f is a homomorphism.

Next we will show f is surjective, let $[x]_I \in X/I$ and $x \in X$. Then $f(x) = [x]_I$, so f is surjective.

Finally, to show that $\ker f = I$, let $x \in \ker f$. We get that $[x]_I = f(x) = [0]_I$, then $x \sim 0$. It follows that $x * 0 \in I$ and $0 * x \in I$. By hypothesis, $0 \in I$. Hence, $x \in I$, this mean $\ker f \subseteq I$. To show that $I \subseteq \ker f$, let $x \in I$. Since I is an ideal of X , we have $0 \in I$. Thus $x * 0 \in I$ and $0 * x \in I$. It follows that $x \sim 0$, so $[x]_I = [0]_I$. Since $f(x) = [x]_I = [0]_I$, then $x \in \ker f$. Accordingly, $\ker f = I$.

Corollary 3.8. Let I be a subalgebra of BZ-algebra X . Defined the map $f : X \rightarrow X/I$ by $f(x) = [x]_I$, for all $x \in X$. Then f is epimorphism, we call f is the natural homomorphism of X onto X/I . Furthermore, $\ker f = I$.

Theorem 3.9. Let f be a homomorphism of a BZ-algebra $(X, *, 0_X)$ onto a BZ-algebra $(Y, \cdot, 0_Y)$ and I be an ideal of X contain in $\ker f$. Let g be the natural homomorphism of X onto X/I then there exists a unique homomorphism h of X/I onto Y such that $f = h \circ g$. Furthermore, h is an injective if and only if $I = \ker f$.

Proof. Define the map $h : X/I \rightarrow Y$ by $h([a]_I) = f(a)$ for all $[a]_I \in X/I$. We first show that, h is well-defined, let $[a]_I, [b]_I \in X/I$ be such that $[a]_I = [b]_I$. We get that $a \sim b$, so $a * b \in I$ and $b * a \in I$. Since $I \subseteq \ker f$, $a * b \in \ker f$ and $b * a \in \ker f$. Thus $f(a) \cdot f(b) = f(a * b) = 0_Y$ and $f(b) \cdot f(a) = f(b * a) = 0_Y$. From BZ-3, $f(a) = f(b)$. Hence h is well-defined. We will show that h is homomorphism. Let $[a]_I, [b]_I \in X/I$. Then $h([a]_I \circ [b]_I) = h([a * b]_I) = f(a * b) = f(a) \cdot f(b) = h([a]_I) \cdot h([b]_I)$, proving that h is a homomorphism.

Next, to show that $f = hog$. For any $a \in X$, then $(hog)(a) = h(g(a)) = h([a]_I) = f(a)$. Hence $hog = f$.

Finally, if $h' : X/I \rightarrow Y$ is another function such that $f = h'og$. Let $[a]_I \in X/I$. The equation $h([a]_I) = f(a) = (h'og)(a) = h'(g(a)) = h'([a]_I)$. Thus $h([a]_I) = h'([a]_I)$, for all $[a]_I \in X/I$.

Now, we will show that h is injective if and only if $I = \ker f$. Suppose firstly that h is injective and $a \in \ker f$. Then $h([0_X]_I) = 0_Y = f(a) = h([a]_I)$ and since h is an injective, thus $[0_X]_I = [a]_I$. It follows that $0_X \sim a$, then $0_X * a \in I$ and $a * 0_X \in I$. By hypothesis, $0_X \in I$. Hence, $a \in I$, this mean $\ker f \subseteq I$. This show that $\ker f = I$.

On the other hand, suppose that $\ker f = I$ and $[a]_I, [b]_I \in X/I$ such that $h([a]_I) = h([b]_I)$. Then $f(a) = f(b)$, it follows that $f(a * b) = f(a) \cdot f(b) = 0_Y$. Thus $a * b \in \ker f$. Since $\ker f = I$, so $a * b \in I$. Similarly, $b * a \in I$. Hence $a \sim b$, proving that $[a]_I = [b]_I$. This show that h is injective. This completes the proof.

Next, we state the first isomorphism of BZ-algebras as the following theorem.

Theorem 3.10. (First Isomorphism Theorem)

If f be a homomorphism of a BZ-algebra $(X, *, 0_X)$ into a BZ-algebra $(Y, \cdot, 0_Y)$, then the quotient BZ-algebra $X/\ker(\varphi)$ is isomorphic to $\varphi(X)$.

Proof. Let $\varphi : X \rightarrow Y$ be a homomorphism and let $K = \ker(\varphi) = \{a \in X : \varphi(a) = 0_Y\}$. We get that $X/K = \{[a]_K : a \in X\}$, where $[a]_K = \{b \in X : a \sim b\}$. From Theorem (3.4(5)), we have $\ker(\varphi)$ is an ideal of X . Thus $(X/K, \circ, [0]_K)$ is a BZ-algebra and $\varphi(X) = \{\varphi(a) : a \in X\}$. Assume that $f : X/K \rightarrow \varphi(X)$ defined by $f([a]_K) = \varphi(a)$, where $[a]_K \in X/K$.

Let $[a]_K, [b]_K \in X/K$ be such that $[a]_K = [b]_K$. Then $a \sim b$, it follows that $a * b \in K$ and $b * a \in K$. Thus $\varphi(a) \cdot \varphi(b) = 0_Y = \varphi(b) \cdot \varphi(a)$. By BZ-3, we get that $\varphi(a) = \varphi(b)$. Hence f is well-defined.

Let $[a]_K, [b]_K \in X/K$. We get that $f([a]_K \circ [b]_K) = f([a * b]_K) = \varphi(a * b) = \varphi(a) \cdot \varphi(b) = f([a]_K) \cdot f([b]_K)$. This show that f is a homomorphism.

Let $[a]_K, [b]_K \in X/K$ be such that $f([a]_K) = f([b]_K)$. Then $\varphi(a) = \varphi(b)$, it follows that $\varphi(a * b) = \varphi(a) \cdot \varphi(b) = 0_Y$. Thus $a * b \in \ker(\varphi) = K$.

Similarly, $b * a \in K$. We see that $a \sim b$, this mean $[a]_K = [b]_K$. Hence f is an injective. Let $a \in \varphi(X)$. Then there exists $b \in X$ such that $a = \varphi(b)$ and $[b]_K \in X/K$. Thus $f([b]_K) = \varphi(b) = a$. Therefore f is a surjective, proving our theorem.

It is easy to check that if I is an ideal of BZ-algebra X and J is an ideal of I , then J is an ideal of X . So, it follows that J is an ideal of $I \cup J$ and $I \cap J$ is an ideal of I .

Theorem 3.11. (Second Isomorphism Theorem)

Let X be a BZ-algebra and A, B be ideals of X . If $A \cup B$ is a BZ-algebra, then the quotient BZ-algebras $A/(A \cap B)$ and $(A \cup B)/B$ are isomorphic.

Proof. Let $\varphi : A \rightarrow (A \cup B)/B$ be a map by $\varphi(x) = [x]_B$ for all $x \in A$. It is obvious that φ is well defined. Let $[x]_B \in (A \cup B)/B$.

If $x \in A$, then $[x]_B = \varphi(x)$. If $x \in B$, then $[x]_B = [0]_B = \varphi(0)$. Thus φ is onto $(A \cup B)/B$. Consider the equation $\varphi(x * y) = [x * y]_B = [x]_B \circ [y]_B = \varphi(x) \circ \varphi(y)$.

Shows that φ is a homomorphism. Now let $x \in \ker(\varphi)$. Then we get $\varphi(x) = [0]_B$, so $[x]_B = [0]_B$. It follows that $x \in B$. Since $\ker(\varphi) \subseteq A$, so $x \in A \cap B$. Hence $\ker(\varphi) \subseteq A \cap B$.

On the other hand, let $x \in A \cap B$. Then $x \in B$. Thus $\varphi(x) = [x]_B = [0]_B$, so $x \in \ker(\varphi)$. Hence $A \cap B \subseteq \ker(\varphi)$. Therefore, $\ker(\varphi) = A \cap B$. From Theorem (3.10), immediately gives us that $A/(A \cap B) \cong (A \cup B)/B$.

Next, we state the third isomorphism theorem of BZ-algebras.

Theorem 3.12. (Third Isomorphism Theorem)

Let X be a BZ-algebra and A, B be ideals of X , with $A \subseteq B \subseteq X$. Then:

- (1) the quotient B/A is an ideal of the quotient X/A , and
- (2) the quotient KU-algebra $(X/A)/(B/A)$ is isomorphic to X/B .

Proof.

(1) To show that B/A is an ideal of X/A . It is clear that $B/A \subseteq X/A$ and $[0]_A \in B/A$. Let $[x]_A \circ [y]_A \in B/A$ and $[x]_A \in B/A$. Then $x * y \in B$ and $x \in B$. Since B is an ideal of X , $y \in B$, so $[y]_A \in B/A$. Therefore, B/A is an ideal of X/A .

(2) Let $\varphi : X/A \rightarrow X/B$ defined by $\varphi([x]_A) = [x]_B$. Assume that $[x]_A = [y]_A$. Then $x \sim y$ determined by A , that is $x*y, y*x \in A$. Since $A \subseteq B$, $x*y, y*x \in B$. Thus $x \sim y$ determined by B , and hence $[x]_B = [y]_B$. Then $\varphi([x]_A) = \varphi([y]_A)$. Therefore, φ is well defined.

Next, to show that φ is onto X/B , let $[x]_B \in X/B$. If $x \in X$ and $x \notin B$, then $[x]_B = \varphi([x]_A)$. If $x \in B$, then $[x]_B = [0]_B = \varphi([0]_A)$. Hence φ is onto.

Consider the equation $\varphi([x]_A \circ [y]_A) = \varphi([x * y]_A) = [x * y]_B = [x]_B \circ [y]_B = \varphi([x]_A) \circ \varphi([y]_A)$. Shows that φ is a homomorphism.

Finally, to show that $\ker(\varphi) = B/A$, let $[x]_A \in \ker(\varphi)$. Then $\varphi([x]_A) = [0]_B$, so $[x]_B = [0]_B$. It follows that $x \in B$. Now we have $[x]_A \in B/A$. Hence $\ker(\varphi) \subseteq B/A$.

Going the other hand, let $[x]_A \in B/A$. We get that $\varphi([x]_A) = [x]_B = [0]_B$, since $x \in B$. Thus $[x]_A \in \ker(\varphi)$, and hence $B/A \subseteq \ker(\varphi)$. Consequently, $\ker(\varphi) = B/A$. By Theorem (3.10), $(X/A)/(B/A)$ is isomorphic to X/B .

It turns out that an analogous result of the third isomorphism theorem for groups is also true for BZ-algebras.

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