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On The Special Ideals in BZ-algebras

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Abstract: *In this paper, we define the notions of BZ-ideal, a-ideal and p-ideal in BZ-algebras. We give several characterizations and the extensive theorems for the BZ-ideal, a-ideal and p-ideal.*

Keywords: ideal, BZ-ideal, a-ideal, p-ideal, BZ-algebras

1. Introduction

In [5], R. F. Ye introduced a new algebraic structure which is called BZ-algebras. And we described the relation between ideals and congruences. Furthermore, we defined quotient BZ-algebra and studied its properties [5,6]. In addition, they introduced the notion of BZ-ideals and a-ideals in BCI-algebras. They gave several characterizations and the extensive theorems about q-ideals and a-ideals. In this paper, we define the notion of BZ-ideals, p-ideals and a-ideals in BZ-algebras and investigated some related properties. The purpose of this paper is to derive some straightforward consequences of the relations between BZ-ideals, a-ideals and p-ideal. We also investigate some of its properties.

2. Preliminaries

In this section we introduced an algebraic structure called a BZ-algebra which is an algebra (X, $*$, 0) with a binary operation $*$ and a nullary operation 0 such that for all x, y, z ∈ X, satisfies the following properties:

(BZ-1) $((x * z) * (y * z)) * (x * y) = 0$;

(BZ-2) $x * 0 = x$;

(BZ-3) $x * y = 0$ *and* $y * x = 0$ implies that $x = y$.

On BZ-algebra (X, $*$, 0), we defined a binary relation ≤ on X by putting x ≤ y if and only if x $*$ y = 0. Then (X, ≤) is a partially ordered set. It is easy to show that the following properties are true for a BZ-algebra. For any x, y, z in X :

(P-1) $x * ((x * y) * y) = 0$;

(P-2) $x * x = 0$;

(P-3) $x * (y * z) = y * (x * z)$;

(P-4) $((x * y) * y) * y = x * y$;

(P-5) $(x * y) * 0 = (x * 0) * (y * 0)$;

(P-6) $(x * y) * ((z * x) * (z * y)) = 0$;

(P-7) $x \leq y$ *implies* $y * z \leq x * z$;

(P-8) $x \leq y$ *implies* $z * x \leq z * y$.

A subset S of a BZ-algebra X is called **subalgebra of X** if x $*$ y ∈ S whenever x, y ∈ S. A non-empty subset I of a BZ-algebra X is called **ideal of X** if it satisfies the following conditions:

(I-1) 0 ∈ I

(I-2) For any x, y ∈ X, $x * y \in I$ and $x \in I$ imply $y \in I$.

Let I be an ideal of BZ-algebra X. Define **the relation ~ on X** by x ~ y if and only if x $*$ y ∈ I and y $*$ x ∈ I. Then the relation ~ is an equivalence relation on X and

$[0]_I$ = {x ∈ X | x ~ 0} is an ideal of X.

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Let ∼ be an equivalence relation on a BZ-algebra X and I be an ideal of X. Define $[x]_I$ by $[x]_I = \{y \in X \mid x \sim y\} = \{y \in X \mid x * y \in I, y * x \in I\}$. Then the family $\{[x]_I \mid x \in X\}$ gives a partition of X which is denoted by X/I. For any x, y ∈ X, we defined $[x]_I \circ [y]_I = [x * y]_I$, then the binary operation ∘ is a mapping from $X/I \times X/I$ to X/I. It is easily checked that $(X/I, ◦, [0]_I)$ is a BZ-algebra. Moreover, the set X/I is called **the quotient BZ-algebra**. And if I is a closed ideal of BZ-algebra X, then it is clear that $[a]_I = I$ for all a in I.

3. BZ-ideals and Its Properties

In this section, we describe properties of BZ-ideals.

Definition 3.1. A non-empty subset I of BZ-algebras X is said to be **a BZ-ideal of X** if it satisfies the following conditions (I-1) and

(I-3) for any x, y, z ∈ X, $(x * y) * z \in I$ and $y \in I$ imply $x * z \in I$.

First, give example shows that the BZ-ideal of X exists.

Example 3.2. Let X = {0, 1, 2}. Define an operation ∗ on X with the Cayley table given by:

*	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Then the Cayley table clearly (X, ∗, 0) is a BZ-algebras and it is easily checked that I = {0, 1} is a BZ-ideal.

Next, we give the relations between BZ-ideal and ideal and subalgebra are considered as the following theorem.

Theorem 3.3. An ideal of BZ-algebras (X, ∗, 0) is a BZ-ideal.

Proof. Suppose that I is an ideal of BZ-algebra and let $(x * y)*z \in I$ and $y \in I$. It follows that $(x * 0) * z \in I$ imply $x * z \in I$. Thus (I-2) holds. Combining (I-1), we conclude that I is a BZ-ideal of X.

Next, I will show example the converse of Theorem 3.3 is not true.

Example 3.4. Let X = {0, 1, 2, 3}. Define an operation ∗ on X with the Cayley table given by:

*	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	3	1	0

Then it is easily checked that (X, ∗, 0) is a BZ-algebra and I = {0,2} is an ideal of X, but not a BZ-ideal of X. Since $(3 * 2) * 1 = 1 * 1 = 0 \in I$ and $2 \in I$, but $3 * 1 = 3 \notin I$.

Now, we investigate the characterization of BZ-ideal.

Theorem 3.5. If I is an ideal of BZ-algebras X, then the following are equivalent:

(1) I is an BZ-ideal of X;

(2) For any x, y ∈ X, ($(x * 0) * y$) ∈ I implies $x * y \in I$;

(3) For any x, y, z ∈ X, $(x * y) * z \in I$ implies $x * (y * z) \in I$.

Proof. Assume that I is an ideal of BZ-algebra X and x, y, z ∈ X.

(1) ⇒ (2) Let I be an BZ-ideal of X and $(x * 0) * y \in I$. Since $0 \in I$, by (I-3), $x * y \in I$.

(2) ⇒ (3) Suppose that (2) holds and $(x * y) * z \in I$. We see that $(x * y) * z) * ((x * 0) * (y * z)) \leq (x * 0) * (y * (x * y))$ $= (x * 0) * (x * (y * y)) = (x * 0) * (x * 0) = 0 \in I$. Since I is an ideal of X, so $((x * 0) * (y * z)) \in I$. By (2), So $x * (y * z) \in I$.

(3) ⇒ (1) Let $(x * y) * z \in I$ and $y \in I$. From (3), we obtain that $x * (y * z) \in I$. Thus $y * (x * z) \in I$ by (P-3). Since $y \in I$ and I is an ideal, hence $x * z \in I$, proving that I is a BZ-ideal of X.

Theorem 3.6. Let J and I be ideals of a BZ-algebra X with $I \subseteq J$. If I is a BZ-deal of X, then so is J.

Proof. Let I is a BZ-deal of a BZ-algebra X and set $s = (x * 0) * y \in J$. Since $(x * 0) * (s * y) = s * ((x * 0) * y) = 0 \in I$. By Theorem (3.5(2)), we get that $x * (s * y) \in I$. And since I is a BZ−ideal, then $s * (x * y) \in I$. Thus $s * (x * y) \in J$ and J is an ideal, so $x * y \in J$. Therefore J is a BZ-ideal.

Corollary 3.7. If zero ideal $\{0\}$ of BZ-algebra X is a BZ-ideal, then every ideal of X is a BZ-ideal.

Theorem 3.8. Let I be an ideal of BZ-algebra X. If for any $x \in I$ and y ∈ X imply $x * y \in I$, then I is BZ-ideal of X.

Proof. Assume that $(x * y) * z \in I$ and $y \in I$. By hypothesis, we obtain $x * ((x * z) * z) \in I$ and $x * y \in I$. Then $(x * y) * (x * z) \in I$, since I is an ideal of X, so $x * z \in I$, proving that I is BZ-ideal of X.

Lemma 3.9. If I is a BZ-ideal of BZ-algebra X, then $x * (x * 0) \in I$ for all x ∈ X.

Proof. Assume that I is a BZ-ideal of BZ-algebra X.

Since $(x * 0) * (x * 0) = 0 \in I$, then it follows that $x * (x * 0) \in I$ because Theorem (3.5(2).

4. a-Ideals and Its Properties

Definition 4.1. A non-empty subset I of BZ-algebra X is called **an a-ideal of X** if it satisfies the following conditions (I-1) and (I-4) for all x, y, z ∈ X, $(x * 0) * (z * y) \in I$ and $z \in I$ imply $y * x \in I$.

Example 4.2. Let X = {0, 1, 2, 3}. Define an operation ∗ on X with the Cayley table given by:

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then it is easily checked that (X, ∗, 0) is a BZ-algebras and I = {0, 1} an a-ideal of X. The following theorem show the relations between a-ideals and ideals and between a-ideals and subalgebra.

Theorem 4.3. If I is an a-ideal of BZ-algebra X, then I is an ideal.

Proof. Let I be an a-deal of X. First, we will show that I is an ideal of X. Assume that $x * y \in I$ and $x \in I$. It follows that

$(0 * 0) * (x * y) \in I$ and $x \in I$, by (I-4), we obtain $y * 0 \in I$. Substituting x = 0 = z in (I-4), we get that $y \in I$ and $0 \in I$, then $y * 0 \in I$. So, it follows that $(y * 0) * 0 \in I$. Putting y = z = 0 in (I-4), it follows that if $(x * 0) * (0 * 0) \in I$ and $0 \in I$, then $0 * x \in I$.

Now, $(x * 0) * 0 \in I$, implies that $x \in I$. Since $(y * 0) * 0 \in I$, so $y \in I$. Proving that I is an ideal of X. Finally, to show that I is a subalgebra. Now, assume that $x \in I$ and $y \in I$. We see that $x * 0 \in I$ and $y * 0 \in I$.

Since $x * (y * x) = y * 0 \in I$ and $x \in I$, then $y * x \in I$. Similarly, $x * y \in I$. Therefore I is a subalgebra, proving our theorem.

The following theorem gives us some equivalences of a-ideals.

Theorem 4.4. Let I be an ideal of BZ-algebra X. The following conditions are equivalent:

(1) I is an a-ideal of X;

(2) $(x * 0) * (z * y) \in I$ implies $(z * y) * x \in I$, for any x, y, z ∈ X;

(3) $(x * 0) * y \in I$ implies $y * x \in I$, for any x, y ∈ X.

Proof. Let I be an ideal of BZ-algebra X.

(1) ⇒ (2), Assume that I is an a-ideal of X and set $s = (x * 0) * (z * y) \in I$. We can write

$(x * 0) * (s * (z * y)) = s * ((x * 0) * (z * y)) = 0 \in I$. By (I-4) and $s \in I$, thus $(z * y) * x \in I$. Proving that (2) holds.

(2) ⇒ (3), Putting z = 0 in (2), we obtain (3).

(3) ⇒ (1), Let $(x * 0) * (z * y) \in I$ and $z \in I$. We see that $\big((x * 0) * (z * y)\big) * \big((x * 0) * y\big) \leq (z * y) * y \leq z \in I$. Since I is an ideal of X, thus $((x * 0) * (z * y)) * ((x * 0) * y) \in I$, and imply that $(x * 0) * y \in I$. By (3), $y * x \in I$, and so I is an a-ideal of X.

Definition 4.5. An ideal I of BZ-algebra X is called **a p-ideal of X** if it satisfies $(x * 0) * 0 \in I$ implies $x \in I$.

Next, we give the relations between a-ideal and p-ideal are considered as the following theorem.

Theorem 4.6. Any a-ideal of BZ-algebra is a p-ideal.

Proof. Suppose that I is an a-ideal of BZ-algebras. By Theorem (4.3), it follows that I is an ideal. Putting y = z = 0 in Theorem (4.2(2)), so $(x * 0) * (0 * 0) \in I$ implies $(x * 0) * 0 \in I$. By Theorem (4.4(2)), we get that $(x * 0) * 0 \in I$ implies $x \in I$. Therefore I is an BZ-ideal.

Next, I will show example the converse of Theorem (4.6) is not true.

Example 4.7. Let X = {0, 1, 2}. Define an operation ∗ on X with the Cayley table given by:

*	0	1	2
0	0	2	2
1	1	0	1
2	2	1	0

Then it is easily checked that (X, ∗, 0) is a BZ-algebras and I = {0} is a p-ideal of X, but it is not a-ideal of X. Since (2∗0)∗(0∗1) = 1∗1 = 0 ∈ {0} and 0 ∈ {0}, but 1 ∗ 2 = 1 ∉ {0}. The proof is complete.

Theorem 4.8. Any a-ideal of BZ-algebra is a BZ-ideal.

Proof. Suppose that I is an a-ideal of BZ-algebras. It follows that I is an ideal. Now, let $(x * 0) * y \in I$. We obtain that

$((x * 0) * y) * ((((y * 0) * x) * 0) * 0)$

$= ((x * 0) * y) * ((((y * 0) * 0) * (x * 0)) * 0)$

$= ((x * 0) * y) * ((((y * 0) * 0) * 0) * ((x * 0) * 0))$

$= ((x * 0) * y) * ((y * 0) * ((x * 0) * 0))$

$\leq ((x * 0) * y) * ((x * 0) * y) = 0 \in I.$

Since I is an ideal, so $(((y * 0) * x) * 0) * 0 \in I$. It follows that I is a p-ideal as Theorem (4.6). Then $(y * 0) * x \in I$ and by Theorem (4.4(3)), we get $x * y \in I$. The proof is complete.

The converse of Theorem (4.8) is not true. From Example (3.2) we can show that $I = \{0\}$ is not a-ideal of X. And it easy to show $I = \{0\}$ is a BZ-ideal.

The following from Theorem (4.8), that if I is an a-ideals, then it is a BZ-ideals.

For the converse part we need the condition that I is an p-ideals as follows.

Theorem 4.9. A non-empty subset I of BZ-algebra X is an a-ideal if and only if it is both a BZ-ideal and a p-ideal.

Proof. Let I be an a-deal of X. It is clear that I is both a BZ-ideal and a p-ideal because Theorem (4.6) and Theorem (4.8).

On the other hand, suppose that I is both a BZ-ideal and a p-ideal. It follows that I is a subalgebra.

Now, assume that $(x * 0) * y \in I$. By Theorem (3.5(2)), we have that $x * (0 * y) \in I$, and hence $x * y \in I$.

Consider the equation

$(x * y) * [(y * 0) * (x * 0)] = (y * 0) * [(x * y) * (x * 0)]$

$= (y * 0) * [x * ((x * y) * 0)]$

$= (y * 0) * [x * ((x * 0) * (y * 0)$

$= x * [(y * 0) * ((x * 0) * (y * 0)]$

$= x * [(x * 0) * ((y * 0) * (y * 0)]$

$= x * [(x * 0) * 0] = 0 \in I.$

From I is an ideal and $x * y \in I$, implies $(y * x) * 0 \in I$. It follows that $((y * x) * 0) * 0 \in I$ because I is a closed. And since I is a p-ideal, then $y * x \in I$, and so I is an a-ideal.

The extensive theorem of a-ideal was given by the following theorem.

Theorem 4.10. Let J and I be two ideals of BZ-algebra X with $I \subseteq J$. If I is an a-ideal of X, then so is J.

Proof. Let I be an a-ideal of X, then I is both a BZ-ideal and a p-ideal. Now, we need to show J is a BZ-ideal and a p-ideal of X. Assume that $s = (x * 0) * y \in J$, then we have $(x * 0) * (s * y) = s * \big((x * 0) * y\big) = 0 \in I$. By Theorem (3.5(2)), $x * (s * y) \in I$, and it follows $s * (x * y) \in I$. Thus $s * (x * y) \in J$ and since J is an ideal, so $x * y \in J$. Therefore J is a BZ-ideal of X. Finally, to show that J is a p-ideal of X and let $t = (x * 0) * 0 \in J$.

Then $(x * 0) * (t * 0) = t * \big((x * 0) * 0\big) = 0 \in I$, and we can write

$\big((x * 0) * (t * 0)\big) * \Big(\big((t * x) * 0\big) * 0\Big)$

$= \big((x * 0) * (t * 0)\big) * \Big(\big((t * 0) * 0\big) * \big((x * 0) * 0\big)\Big)$

$\leq ((x * 0) * (t * 0)) * ((x * 0) * (t * 0)) = 0 \in I.$

Since I is an ideal, we get that $((t * x) * 0) * 0 \in I$. And since I is a p-ideal, thus $t * x \in I \subseteq J$, and so $x \in J$ as J is an ideal. Hence J is a p-ideal of X. The proof is complete.

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