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Hyper *SA*-algebra

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Abstract: *In this paper is to introduce the concept of hyper SA-algebras is a generalization of SA-algebras and study a hyper structure SA-algebra and investigate some of its properties. Also, hyper some types of hyper SA-algebras and hyper SA-ideal of hyper SA-algebras are studied. We study on the fuzzy theory of hyper SA-ideal of hyper SA-algebras hyper SA-algebra. We study homomorphism of hyper SA-algebras which are a common generalization of SA-algebras*

Key words: *SA*-algebra, hyper *SA*-algebra, hyper *SA*-ideal , some types of hyper *SA*-algebras, homomorphism.

1. Introduction

Areej Tawfeeq Hameed and et al ([1]) introduced a new algebraic structure, called SA-algebra, They have studied a few properties of these algebras, the notion of SA-ideals on SA-algebras was formulated and some of its properties are investigated. The concept of a fuzzy set, was introduced by L.A. Zadeh [4]. In [6], S.M. Mostafa and A.T. Hameed made an extension of the concept of fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval-valued membership function). This interval-valued fuzzy KUS-ideals on KUS-algebras is referred to as an i-v fuzzy KUS-ideals on KUS-algebras. they constructed a method of approximate inference using his i-v fuzzy KUS-ideals on KUS-algebras. In this paper, using the notion of interval-valued fuzzy set , we introduce the concept of an interval-valued fuzzy SA-ideals (briefly, i-v fuzzy SA-ideals) of a SA-algebra, and study some of their properties. Using an i-v level set of an i-v fuzzy set, we state a characterization of an i-v fuzzy SA-ideals. We prove that every SA-ideals of a SA-algebra X can be realized as an i-v level SA-ideals of an i-v fuzzy SA ideals of X. In connection with the notion of homomorphism, we study how the images and inverse images of i-v fuzzy SA-ideals become i-v fuzzy SA-ideals.

2. Hyper of The SA-algebra

In this section, some properties of hyper *SA*-algebra are discussed and preliminaries lemmas of *SA*-ideals and fuzzy *SA*-ideals of *SA*-algebra .

Remark 2.1 [10].

Let H be a nonempty set and $p^*(H) = p(H) \setminus \{\emptyset\}$ the family of the nonempty subsets of H. A multi valued operation (said also hyper operation) " $\circ$ " on H is a function, which associates with every pair $(x,y) \in H \times H = H^2$ a nonempty subset of H denoted $x \circ y$. An algebraic hyper structure or simply a hyper structure is a nonempty set H endowed with one or more hyper operations.

Remark 2.2 .

In this work, let H be a nonempty set and $p^*(H) = p(H) \setminus \{\emptyset\}$ the family of the nonempty subsets of H. A hyper operations $\circ$ and $*$ on H is a function which a ssociates every pair $(x,y) \in H \times H = H^2$ a nonempty subset of H denoted $x \circ y$ and $x * y$. An H endowed with two hyper operations.

Definition 2.3.

Let H be a nonempty set and $\circ$ and $*$ be two hyper operations on H such that $\circ, * : H \times H \to p^*(H)$. Then H is called **hyper *SA*-algebra** if it contains a constant 0 and satisfies the following axioms: for all $x, y, z \in H$,

(HSA_1) $x * x = \{0\} = x \circ x$,
(HSA_2) $x * 0 = \{x\} = x \circ 0$,
(HSA_3) $(x * y) * z \ll x * (z \circ y)$,
(HSA_4) $(x \circ y) * (x \circ z) \ll y * z$.

Remark 2.4.

(1) Where $x \ll y$ is defined by $0 \in x \circ y \wedge 0 \in x * y$, for every $I, J \subseteq H, I \ll J$ is defined by $\forall\, a \in I, \exists\, b \in J$ such that $a \ll b$. In such case , we call $\ll$ the hyper order in H.

(2) We shall use the $x \circ y$ in stead of $x \circ \{y\}$ or $\{x\} \circ \{y\}$ and we shall use the $x * y$ in stead of

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$x * \{y\}$ or $\{x\} * \{y\}$.

(3) If $I, J \subseteq H$, then by $I \circ J$, we mean the subset $\bigcup_{\substack{a\in I \\ b\in J}} a \circ b$ of H and $I * J$, we mean the subset $\bigcup_{\substack{a\in I \\ b\in J}} a * b$ of H.

Example 2.5.

(1) Let $H = \{0, a\}$ and hyper operation $\circ$ and $*$ as the following table:

$\circ$	0	a
0	{0}	{0}
a	{a}	{0}

$*$	0	a
0	{0}	{0,a}
a	{a}	{0}

Then $(H;\circ,*,0)$ is a hyper *SA*-algebra.

(2) Let $H = \{0,1,2\}$ and hyper operation $\circ$ and $*$ as the following table:

$\circ$	0	1	2
0	{0}	{0,2}	{0,1,2}
1	{1}	{0}	{0}
2	{2}	{0,2}	{0}

$*$	0	1	2
0	{0}	{0,1}	{0,2}
1	{1}	{0}	{0}
2	{2}	{0}	{0}

Then $(H;\circ,*,0)$ is a hyper *SA*-algebra.

(3) Let $(H;+,-,0)$ be a *SA*-algebra and define a hyper operation $\circ$ and $*$ on H
$x \circ y = \{x + y\} \wedge x * y = \{x - y\}$, for all $x, y \in H$, then $(H;\circ,*,0)$ is a hyper *SA*-algebra.

(4) If $H = \{0,1,2,\ldots\}$ and hyper operations $\circ$ and $*$ on H is defind as follows:

$$x * y = x \circ y = \begin{cases} \{0, x\} & if\ x \le y \\ \{x\} & if\ x > y \end{cases}, \text{ for all } x, y \in H.$$

Then $(H;\circ,*,0)$ is a hyper *SA*-algebra.

Proposition 2.6.

Let $(H;\circ,*,0)$ be a hyper *SA*-algebra. Then for all $x, y, z \in H$ and for all nonempty subset I, J of H the following statements hold:

$(\boldsymbol{p_1})$ $I \subseteq J$ implies $I \ll J$;
$(\boldsymbol{p_2})$ $0 \circ 0 = \{0\} = 0 * 0$;
$(\boldsymbol{p_3})$ $x \ll x$;
$(\boldsymbol{p_4})$ $I \circ 0 = I = I * 0$;
$(\boldsymbol{p_5})$ $x \circ y \ll y * x$;
$(\boldsymbol{p_6})$ $x \circ (0 \circ 0) = \{x\} = x * (0 * 0)$;
$(\boldsymbol{p_7})$ $x * (x \circ z) \ll z$;
$(\boldsymbol{p_8})$ $0 \circ 0 = \{0\} = 0 * 0$;

Proof:

$(\boldsymbol{p_1})$ By Definition of $\ll$.
$(\boldsymbol{p_2})$ By (HSA_2), put x=0, then $0 \circ 0 = 0 * 0 = 0$.
$(\boldsymbol{p_3})$ By (HSA_3), let $y = z = 0$. Then by (p_2) and (HSA_2) then $x \ll x$.
$(\boldsymbol{p_4})$ The proof follows by (HSA_2).
$(\boldsymbol{p_5})$ By (HSA_4) *and* (HSA_1), put $z = x$, then, $x \circ y \ll y * x$.
$(\boldsymbol{p_6})$ The proof follows by $(HSA_1), (HSA_2)$.
$(\boldsymbol{p_7})$ By (HSA_4), put $y = 0$, then, $x * (x \circ z) \ll z$.
$(\boldsymbol{p_8})$ The proof follows by (HSA_1). ■

Lemma 2.7.

In hyper *SA*-algebra $(H;\circ,*,0)$ the following hold: for all $x, y, z \in H$, $y \ll z \Leftrightarrow x \circ y \ll x \circ z$.

Proof:

Since $y \ll z$, it follows that $0 \in y \circ z \wedge 0 \in y * z$. By (HSA_4), we obtain
$(x \circ y) * (x \circ z) \ll (y * z)$. Then $0 \in (x \circ y) * (x \circ z) \circ (y * z)$, and $0 \in (x \circ y) * (x \circ z) * (y * z)$, but $0 \in y \circ z \wedge 0 \in y * z$. Hence, $0 \in (x \circ y) * (x \circ z)$, $(i.e.)$, $x \circ y \ll x \circ z$. ■

Lemma 2.8.

In hyper SA-algebra $(H;\circ,*,0)$, the following statements hold: for all $x, y, z \in H$,

(1) $x * y \ll z \Leftrightarrow x \ll z \circ y$,

(2) $I \ll 0 \Leftrightarrow 0 \in I$,

(3) $y \in (x \circ 0) \wedge y \in (x * 0) \Longrightarrow x \ll y$.

Proof:

(1) Let $x, y, z \in H$ such that $x * y \ll z$, then there exists $t \in x * y$ such that $t \ll z \Leftrightarrow 0 \in t \circ z \wedge 0 \in t * z \Longrightarrow 0 \in t * z \subseteq (x * y) * z \ll x * (z \circ y)$ by $(HSA_3) \Leftrightarrow 0 \in x * (z \circ y) \Leftrightarrow x \ll (z \circ y)$.

(2) Let $I \ll 0$. It means that there $a \in I$ such that $a \ll 0$. By (HSA_2) a =0, and so $0 \in I$.

(3) Let $y \in (x \circ 0) \wedge y \in (x * 0)$, then by (2), $(x \circ 0) \ll y \wedge (x * 0) \ll y$, then $0 \in (x \circ 0) \circ y = x \circ y \wedge 0 \in (x * 0) * y = x * y$ $(i.e.)$, $0 \in x \circ y \wedge 0 \in x * y$. Hence $x \ll y$. ■

Definition 2.9.

Let S be a nonempty subset of a hyper SA-algebra H. Then S is said to be a **hyper *SA*-subalgebra of *H*** if $x \circ y \subseteq S \wedge x * y \subseteq S, \forall x, y \in S$.

Proposition 2.10.

Let S be a nonempty subset of a hyper SA-algebra $(H,*,\circ,0)$, if S is hyper SA-subalgebra of H, then $0 \in S$.

Proof:

Since if S is hyper SA-subalgebra of H , then $x \circ y \in S \wedge x * y \in S$.
Let $a \in S$, since $a \ll a$, we have $0 \in a \circ a \subseteq S$ and $0 \in a * a \subseteq S$ **,** then $0 \in S$. ■

3. *Some Types of Hyper SA-algebras.*

In this section, the notion of some types of hyper SA-algebras is introduced. Several theorems and properties are stated and proved.

Definition 3.1.

A hyper SA-algebra $(H;\circ,*,0)$ is said

(1) **Column hyper *SA*-algebra** (briefly,C-hyper SA-algebra),
if $x * 0 = \{x\} = x \circ 0$, for all $x \in H$;

(2) **Diagonal hyper *SA*-algebra** (briefly,D-hyper SA-algebra),
if $x \circ x = \{0\} = x * x$, for all $x \in H$;

(3) **Very thin hyper *SA*-algebra** (briefly,V-hyper SA-algebra), if it is a CD-hyper SA-algebra.

Example 3.2.

(1) Let $H = \{0, a, b\}$ be a set. Define a hyper operations $\circ, *$ on H as following table :

(2)

$\circ$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{0,a}	{0}

$*$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{0,b}	{0}

Then, $(H;\circ,*,0)$ is an V-hyper SA-algebra.

(3) Let $H = \{0, a\}$ be a set. Define a hyper operations $\circ, *$ on H

(4) as follows table:

$\circ$	0	a
0	{0}	{0}
a	{a}	{0,a}

$*$	0	a
0	{0}	{0}
a	{a}	{0}

Then, $(H;\circ,*,0)$ is an C-hyper SA-algebra.

(5) Let $H = \{0, a\}$ be a set. Define a hyper operations $\circ,*$ on H as follows table:

$\circ$	0	a
0	{0}	{a}
a	{a}	{0}

$*$	0	a
0	{0}	{0}
a	{0,a}	{0}

Then, $(H;\circ,*,0)$ is an D-hyper SA-algebra.

Theorem 3.3.

Let $(H;\circ,*,0)$ be a D-hyper SA-algebra.Then, for all $x, y, z \in H$

(1) $a \in 0 * x$ implies $a * x \ll \{0\}$,

(2) $(x * y) * y \ll x * \{0\}$.

Proof:

(1) By Definition (3.1(1,3)) and (HSA_3), $(0 * x) * x \ll 0 * (x \circ x) = 0 * 0 = \{0\}$. It follows that,for all $a \in 0 * x$, $a * x \ll \{0\}$.

(2) By (HSA_3) and Definition (3.1(3)), $(x * y) * y \ll x * (y \circ y) = x * 0$, then $(x * y) * y \ll x * \{0\}$. ■

Theorem 3.4.

Let $(H;\circ,*,0)$ be a V-hyper SA-algebra. Then, for all $x, y, z \in H$

(1) $\{y\} \ll x * (y \circ x)$.

(2) If $z \in x \circ y$, then $\{x\} \ll y * z$.

Proof:

(1) By (HSA_3) and (p_2) and Definition (3.1(1,3)), $\{y\} = y * 0 = y * (x * x) \ll x * (y \circ x)$, then $\{y\} \ll x * (y \circ x)$.

(2) Let $z \in x \circ y$, then by (1), $\{x\} \ll y * (x \circ y)$. Hence $\{x\} \ll y * z$. ■

4. *Hyper SA-ideal of SA-algebra.*

In this section, the notion of hyper SA-ideal of SA-algebra is introduced. Several definitions and theorems and properties are stated and proved.

Definition 4.1.

Let I be a nonempty subset of hyper SA-algebra $(H;\circ,*,0)$. I is called **a hyper SA-ideal of H** if $\forall\, x, y, z \in H$,

$(HSAI_1)$ $0 \in I$,

$(HASI_2)$ $x \circ z \ll I$ and $y * z \ll I$ implies $x \circ y \ll I$.

Example 4.2.

Let $H = \{0,1,2\}$ is a hyper SA-algebra. Define a hyper operations $\circ,*$ on H as follows table:

$\circ$	0	1	2
0	{0}	{1}	{2}
1	{0}	{0}	{0}
2	{0}	{0,2}	{0}

$*$	0	1	2
0	{0}	{0,1}	{0,2}
1	{0}	{0}	{0}
2	{0}	{0,2}	{0}

Then, $I_1 = \{0,1\}$ and $I_2 = \{0,2\}$ are the only hyper SA-ideals of H.

Example 4.3.

Let $H = \{0,1,2,\dots\}$ and hyper operations $\circ$ *and* $*$ on H is defind as follows:

$x * y = x \circ y = \begin{cases} \{0, x\} & if\ x \le y \\ \{0\} & if\ x > y \end{cases}$ for all $x, y \in H$, is a hyper SA-algebra. Then, $I_1 = \{0,1\}, I_2 = \{0,1,2\}, I_3 = \{0,1,2,3\}, \dots, I_n = \{0,1,2,3,\dots,n\}$ are hyper SA-ideals of H.

Definition 4.4.

Let I be a nonempty subset of hyper SA-algebra $(H;\circ,*,0)$.Then I is said to be **a hyper ideal of *H*** if: $\forall\, x, y \in H$,

(HI_1) $0 \in I$,

(HI_2) $x \circ y \ll I$ *and* $x \in I$ imply $y \in I$,

(HI_3) $x * y \ll I$ *and* $x \in I$ imply $y \in I$.

Example 4.5.

Concider a hyper SA-algebra $H = \{0,1,2\}$ with the following table,

$\circ$	0	1	2
0	{0}	{1}	{2}
1	{0}	{0}	{0}
2	{0}	{0,2}	{0}

$*$	0	1	2
0	{0}	{0,1}	{0,2}
1	{0}	{0}	{0}
2	{0}	{0,1}	{0,2}

Then $I = \{0,2\}$ is hyper ideal of H.

Definition 4.6.

Let I be a nonempty subset of a hyper SA-algebra $(H;\circ,*,0)$ and $0 \in I$. I is called **a weak hyper ideal of *H*** , if $\forall\, x, y \in H$,

1) $x \circ y \subseteq I$ and $x \in I$ imply that $y \in I$,
2) $x * y \subseteq I$ and if $x \in I$ imply that $y \in I$.

Definition 4.7.

Let I be a nonempty subset of a hyper SA-algebra $(H;\circ,*,0)$ and $0 \in I$. I is called **a strong hyper ideal of *H*** , if $\forall\, x, y \in H$,

1) $(x \circ y) \cap I \neq \emptyset$ and $x \in I$ imply that $y \in I$,
2) $(x * y) \cap I \neq \emptyset$ and if $x \in I$ imply that $y \in I$.

Theorem 4.8.

Every weak hyper ideal is strong hyper ideal.

Proof:

Let I be a weak hyper ideal, then $x \circ y \subseteq I$ *and* $x \in I$ implies $y \in I$. Thus $(x \circ y) \cap I \neq \emptyset$ and $x \in I$ implies $y \in I$. And $x * y \subseteq I$ *and* $x \in I$ implies $y \in I$. Thus $(x * y) \cap I \neq \emptyset$ *and* $x \in I$ implies $y \in I$. Hence I is a strong hyper ideal. ■

Remark 4.9:

Every strong hyper ideal is not a weak hyper ideal, as the following example:

Example 4.10:

Let $H = \{0, a\}$ be a set. Define a hyper operations $\circ,*$ on H as following table:

$\circ$	0	a	b
0	{0}	{a}	{0,a}
a	{0}	{0,a}	{b}
b	{0}	{b}	{0,b}

$*$	0	a	b
0	{0}	{a}	{0,a}
a	{0}	{0,a}	{b}
b	{0}	{b}	{0,b}

I=$\{0, b\}$ is strong hyper ideal but is not weak hyper ideal.

$(0 \circ b) \cap I \neq \emptyset \Longrightarrow \{(0 \circ b) \cap I \neq \emptyset$ & $0 \in I \Longrightarrow b \in I$, but $0 \circ b = \{0, a\} \nsubseteq I$.

Definition 4.11.

For a hyper SA-algebra $(H;\circ,*,0)$. A nonempty subset $I \subseteq H$, containing 0 is said **a weak hyper *SA*-ideal of *H*,** if $\forall\, a, b, c \in H$,

1) $\big((a \circ b) \circ c\big) \subseteq I$ and $b \in I$ imply $a \circ c \subseteq I$,
2) $((a * b) * c) \subseteq I$ and $b \in I$ imply that $a * c \subseteq I$.

Definition 4.12.

For a hyper SA-algebra $(H;\circ,*,0)$. A nonempty subset $I \subseteq H$, containing 0 is said **a strong hyper *SA*-ideal of *H*,** if $\forall\, a, b, c \in H$,

1) $((a \circ b) \circ c) \cap I \neq \emptyset)$ and $b \in I$ imply that $a \circ c \subseteq I$,

2) $((a * b) * c) \cap I \neq \emptyset)$ and $b \in I$ imply that $a * c \subseteq I$.

Theorem 4.13.

Every weak hyper *SA*-ideal of a hyper *SA*-algebra $(H;\circ,*,0)$ is strong hyper *SA*-ideal.

Proof:

Since I is weak hyper *SA*-ideal, then $((x \circ y) \circ z) \subseteq I$ *and* $y \in I$ imply $x \circ z \subseteq I$. Thus $((x \circ y) \circ z) \cap I \neq \emptyset$ *and* $y \in I$ imply $x \circ z \in I$. And $\left((x * y) * z\right) \subseteq I$ *and* $y \in I$ imply $x * z \subseteq I$. Thus $\left((x * y) * z\right) \cap I \neq \emptyset$ and $y \in I$ imply $x \circ z \in I$.

Hence I is strong hyper *SA*-ideal. ■

Remark 4.14:

Every strong hyper *SA*-ideal is not a weak hyper *SA*-ideal, as the following example:

Example 4.15:

Let $H = \{0, a, b\}$ be a set. Define a hyper operations $\circ,*$ on H as following table:

∘	0	1	2
0	{0}	{0,1}	{1,2}
1	{0}	{0,1}	{2}
2	{0}	{1,2}	{0,2}

∗	0	1	2
0	{0}	{0,1}	{1,2}
1	{0}	{0,1}	{2}
2	{0}	{1,2}	{0,2}

I={0,2} is a strong hyper *SA*-ideal, but is not weak hyper *SA*-ideal.

$((1 \circ 0) \circ 2) \cap I \neq \emptyset$ & $0 \in I \Longrightarrow 1 \circ 2 = \{2\} \subseteq I$, but $((1 \circ 0) \circ 2) = \{0,1,2\} \nsubseteq I$.

Theorem 4.16.

Every weak hyper *SA*-ideal of a hyper *SA*-algebra $(H;\circ,*,0)$ is a weak hyper ideal.

Proof:

Let I be a weak hyper *SA*-ideal of $H, \forall\, a, b, c \in H$, then $((a \circ b) \circ c) \ll I$ and $b \in I$ imply $a \circ c \subseteq I$. Putting $a = 0$, we get $((0 \circ b) \circ c) \ll I$ and $b \in I$ imply $0 \circ c = c \in I$.
Hence $(b \circ c) \ll I$ and $b \in I$ imply $c \in I$. And $((a * b) * c) \ll I$ and $b \in I$ imply $a * c \subseteq I$. Putting $a = 0$, we get $((0 * b) * c) \ll I$ and $b \in I$ imply $0 * c = c \in I$.

Hence $(b * c) \ll I$ and $b \in I$ imply $c \in I$. Hence I is weak hyper ideal. ■

Remark 4.17.

Generally,every weak hyper ideal is not a weak hyper *SA*-ideal. It can be observed with the help of examples given below:

Example 4.18.

Let $H = \{0,1,2,3\}$ be a set with the following cayley table:

∘	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0}	{0,2}	{2,3}
2	{2}	{0,1}	{0}	{1}
3	{3}	{0,2}	{0}	{0}

∗	0	1	2	3
0	{0}	{0}	{0}	{0}
1	{1}	{0}	{0,2}	{2,3}
2	{2}	{0,1}	{0}	{1}
3	{3}	{0,2}	{0}	{0}

Then $(H,\circ,*,0)$ is a hyper *SA*-algebra.Take $I = \{0,1\}$, then I is a weak hyper ideal, however, I is not a weak hyper *SA*-ideal of H as $((3 \circ 1) \circ 1) \subseteq I, 1 \in I$, but $3 \circ 1 = 2 \notin I$.

Theorem 4.19.

Every strong hyper *SA*-ideal of a hyper *SA*-algebra $(H,\circ,*,0)$ is a strong hyper ideal.

Proof:

Let I be a strong hyper *SA*-ideal of $H, \forall\, a, b, c \in H$, then $((a \circ b) \circ c) \cap I \neq \emptyset$ and $b \in I$ imply $a \circ c \subseteq I$. Putting $a = 0$, we get $\left((0 \circ b) \circ c\right) = (b \circ c)$ and $b \in I$ imply $0 \circ c = c \in I$.

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Thus $(b \circ c) \cap I \neq \emptyset$ and $b \in I$ imply $c \in I$. And $((a * b) * c) \cap I \neq \emptyset)$ and $b \in I$ imply $a * c \subseteq I$. Putting $a = 0$, we get $\left((0 * b) * c\right) = (b * c)$ and $b \in I$ imply $0 * c = c \in I$. Thus $(b * c) \cap I \neq \emptyset$ and $b \in I$ imply $c \in I$. Hence I is strong hyper ideal. ■

Remark 4.20.

Generally,every strong hyper ideal is not a strong hyper SA-ideal. It can be observed with the help of examples given below:

Example 4.21.

Let $H = \{0, a, b\}$ be a set with the following table:

∘	0	a	b
0	{0}	{a}	{0,a}
a	{0}	{0,a}	{b}
b	{0}	{b}	{0,b}

*	0	a	b
0	{0}	{a}	{0,a}
a	{0}	{0,a}	{b}
b	{0}	{b}	{0,b}

Then $(H,\circ,*,0)$ is a hyper SA-algebra.Take $I = \{0, b\}$,Then I is a hyper ideal, but not a hyper SA-ideal of H, since $((0 \circ b) \circ a) \ll I, b \in I$, but $0 \circ a = a \notin I$.

Here $I = \{0, b\}$ is also a strong hyper ideal, but it is not a strong hyper SA-ideal of H, since $((0 \circ b) \circ a) = \{b\} \cap I \neq \emptyset$ and $b \in I$, but $0 \circ a = a \notin I$.

Definition 4.22.

A subset I of a hyper SA-algebra $(H;\circ,*,0)$ such that $0 \in I$ is called the following: $\forall\, x, y, z \in H$

(1) **A hyper SA-ideal of type 1**, if $\left((x \circ y) \circ z\right) \ll I, y \in I$, then $x \circ z \subseteq I$, and if $\left((x * y) * z\right) \ll I, y \in I$, then $x * z \subseteq I$.

(2) **A hyper SA-ideal of type 2**, if $\left((x \circ y) \circ z\right) \subseteq I, y \in I$, then $x \circ z \subseteq I$, and if $\left((x * y) * z\right) \subseteq I, y \in I$, then $x * z \subseteq I$.

(3) **A hyper SA-ideal of type 3**, if $\left((x \circ y) \circ z\right) \ll I, y \in I$, then $x \circ z \ll I$, and if $\left((x * y) * z\right) \ll I, y \in I$, then $x * z \ll I$.

(4) **A hyper SA-ideal of type 4**, if $\left((x \circ y) \circ z\right) \subseteq I, y \in I$, then $x \circ z \ll I$, and if $\left((x * y) * z\right) \subseteq I, y \in I$, then $x * z \ll I$.

Theorem 4.23.

In any hyper SA-algebra $(H;\circ,*,0)$, the following statements are valid.

(1) Any hyper SA-ideal of type 1 is a hyper SA-ideal of type 2 and 3.

(2) Any hyper SA-ideal of type 2 is a hyper SA-ideal of type 4.

(3) Any hyper SA-ideal of type 3 is a hyper SA-ideal of type 4.

(4) Any hyper SA-ideal of type 1 is a hyper ideal.

(5) Any hyper SA-ideal of type 2 is a weak hyper ideal.

Proof:

(1) Let I be a hyper SA-ideal of type 1, then $\left((x \circ y) \circ z\right) \ll I, y \in I$, imply $x \circ z \subseteq I$, and $\left((x * y) * z\right) \ll I, y \in I$, imply $x * z \subseteq I$. By Definition of $\ll$, we gate $\left((x \circ y) \circ z\right) \subseteq I, y \in I$, then $x \circ z \subseteq I$, and $\left((x * y) * z\right) \subseteq I, y \in I$, then $x * z \subseteq I$. And $\left((x \circ y) \circ z\right) \ll I, y \in I$, then $x \circ z \ll I$, and $\left((x * y) * z\right) \ll I, y \in I$, then $x * z \ll I$. Hence hyper SA-ideal of type 1 is a hyper SA-ideal of type 2 and 3.

(2) Let I be a hyper SA-ideal of type 2, then $\left((x \circ y) \circ z\right) \subseteq I, y \in I$, then $x \circ z \subseteq I$, and if $\left((x * y) * z\right) \subseteq I, y \in I$, then $x * z \subseteq I$.
By Definition of $\ll$, we gate $\left((x \circ y) \circ z\right) \subseteq I, y \in I$, then $x \circ z \ll I$, and $\left((x * y) * z\right) \subseteq I, y \in I$, then $x * z \ll I$. Hence hyper SA-ideal of type 2 is a hyper SA-ideal of type 4.

(3) Let I be a hyper SA-ideal of type 3, then $\left((x \circ y) \circ z\right) \ll I, y \in I$, then

$x \circ z \ll I$, and $\big((x * y) * z\big) \ll I, y \in I$, then $x * z \ll I$. By Definition of $\ll$, we gate $\big((x \circ y) \circ z\big) \subseteq I, y \in I$, then $x \circ z \ll I$, and $\big((x * y) * z\big) \subseteq I, y \in I$, then $x * z \ll I$. Hence hyper *SA*-ideal of type 3 is a hyper *SA*-ideal of type 4.

(4) Let I be a hyper *SA*-ideal of type 1, $(y \circ z) \ll I$ and $(y * z) \ll I)$, and $y \in I$. Hence, by proposition (p_7). We obtain $((0 \circ y) \circ z) \ll I$ $((0 * y) * z) \ll I)$. But $y \in I$ so applying the hypothesis and proposition (p_5), we get $\{z\} = 0 \circ z \subseteq I$ $(\{z\} = 0 * z \subseteq I)$. This shows that I is a hyper-ideal of H.

(5) Let I be a hyper *SA*-ideal of type 2, $y \circ z \subseteq I$ $(y * z \subseteq I)$, and $y \in I$. Hence, by proposition (p_7), we obtain $0 \circ (y \circ z) \ll I, (0 * (y * z) \subseteq I)$. But $y \in I$, so applying the hypothesis and proposition (p_5), we get $\{z\} = 0 \circ z \subseteq I, (\{z\} = 0 * z \subseteq I)$. This shows that I is a hyper-ideal of H. ■

Theorem 4.24.

In any hyper *SA*-algebra $(H;\circ,*,0)$, then

(1) A hyper *SA*-ideal of type 1&2&3&4 is weak hyper *SA*-ideal .

(2) A hyper *SA*-ideal of type 1&2&3&4 is weak hyper ideal.

Proof:

We prove only (1) and the type (2) , (3) , (4) are similary.

(1) Let I be a hyper *SA*-ideal of type 1, then $\big((x \circ y) \circ z\big) \ll I, y \in I$, imply $x \circ z \subseteq I$, and $\big((x * y) * z\big) \ll I, y \in I$, imply $x * z \subseteq I$. Since $\big((x \circ y) \circ z\big) \ll I$ is $\big((x \circ y) \circ z\big) \subseteq I$ and $\big((x * y) * z\big) \ll I$ is $\big((x * y) * z\big) \subseteq I$ by Definition of $\ll$, then $\big((x \circ y) \circ z\big) \subseteq I, y \in I$, imply $x \circ z \subseteq I$, and $\big((x * y) * z\big) \subseteq I, y \in I$, imply $x * z \subseteq I$. Hence I is weak hyper *SA*-ideal.

(2) By Theorem (4.16) and (1) above. ■

Theorem 4.25.

In any hyper *SA*-algebra $(H;\circ,*,0)$, then

(1) A hyper *SA*-ideal of type 1&2&3&4 is strong hyper *SA*-ideal .

(2) A hyper *SA*-ideal of type 1&2&3&4 is strong hyper ideal.

Proof:

We prove only (1) and the type (2) , (3) , (4) are similary.

(1) Let I be a hyper *SA*-ideal of type 1, then $\big((x \circ y) \circ z\big) \ll I, y \in I$, imply $x \circ z \subseteq I$, and $\big((x * y) * z\big) \ll I, y \in I$, imply $x * z \subseteq I$. Since $\big((x \circ y) \circ z\big) \ll I$ is $\big((x \circ y) \circ z\big) \subseteq I$ and $\big((x * y) * z\big) \ll I$ is $\big((x * y) * z\big) \subseteq I$ by Definition of $\ll$, then $\big((x \circ y) \circ z\big) \cap I \neq \emptyset$, $y \in I$, imply $x \circ z \subseteq I$, and $\big((x * y) * z\big) \cap I \neq \emptyset$, $y \in I$, imply $x * z \subseteq I$. Hence I is strong hyper *SA*-ideal.

(2) By Theorem (4.19) and (1) above. ■

5. *Homomorphism on Hyper SA-algebras*.

In this section, we introduce some result on images and pre-images of homomorphism on hyper *SA*-algebra and investigate some related theorems.

Definition 5.1.

Let $(H,\circ,*,0)$, $(K,\grave{\circ},\grave{*},\grave{0})$ be hyper *SA*-algebras. A mapping $f: H \longrightarrow K$ is called **a hyper homomorphism** if $for\ all\ x, y \in H$, then

(HH_1) $f(0) = \grave{0}$,

(HH_2) $f(x \circ y) = f(x) \grave{\circ} f(y)$,

(HH_3) $f(x * y) = f(x) \grave{*} f(y)$.

Theorem 5.2.

Let $f: (H,\circ,*,0) \longrightarrow (K,\grave{\circ},\grave{*},\grave{0})$ is a hyper homomorphism of hyper *SA*-algebras, if $x \ll y$ in H, then $f(x) \ll f(y)$ in K.

Proof:

Let $x, y \in H$ be such that $x \ll y$. Then, $0 \in x * y \wedge 0 \in x \circ y$, and so by (HH_1), $\grave{0} = f(0)$, by (HH_2) $f(0) = f(x \circ y) = f(x) \grave{\circ} f(y)$ and by (HH_3) $f(0) = f(x * y) = f(x) \grave{*} f(y)$. Hence $f(x) \ll f(y)$ in K. ■

Theorem 5.3.

Let $f:(H,\circ,*,0) \longrightarrow (K,\grave{\circ},\grave{*},\grave{0})$ be a hyper homomorphism of hyper *SA*-algebra, if I is a hyper ideal of K, then $f^{-1}(I)$ is a hyper ideal of H.

Proof:

By Theorem (5.2(1)), $f(0) = \grave{0}$. Since I is hyper ideal of K, then $\grave{0} \in I$ and $f(0) = \grave{0}$, then $f(0) \in I$ $\Rightarrow f^{-1}(f(0)) \in f^{-1}(I) \Rightarrow 0 \in f^{-1}(I)$.

Let $x, y \in H$ such that $x \circ y \ll f^{-1}(I) \wedge x * y \ll f^{-1}(I)$ and $x \in f^{-1}(I)$, then $f(x) \in I$, and for every $z \in x \circ y \wedge z \in x * y$, there exists $w \in f^{-1}(I)$ such that $z \ll w$, that is , $0 \in z \circ w \wedge$ $0 \in z * w$. It follows that by (HH_1) $\grave{0} = f(0)$ and by (HH_2)
$f(0) \in f(z \circ w) = f(z) \grave{\circ} f(w) \subseteq f(x \circ y) \grave{\circ} I = f(x) \grave{\circ} f(y) \grave{\circ} I$, and by (HH_3)
$f(0) \in f(z * w) = f(z) \grave{*} f(w) \subseteq f(x * y) \grave{*} I = f(x) \grave{*} f(y) \grave{*} I$,
so that $f(x) \grave{\circ} f(y) \ll I \wedge f(x) \grave{*} f(y) \ll I$. Since I is a hyper ideal of K.

It follows that $f(y) \in I$, that is , $y \in f^{-1}(I)$ by (HI_1). Hence $f^{-1}(I)$ is a hyper ideal of H. ■

Definition 5.4.

Let $f:(H,\circ,*,0) \rightarrow (K,\grave{\circ},\grave{*},\grave{0})$ be a hyper homomorphism of hyper *SA*-algebras.
$ker(f) = \{x \in H : f(x) = \grave{0}\}$, called **the kernel of f** .

Theorem 5.5.

Let $f:(H,\circ,*,0) \longrightarrow (K,\grave{\circ},\grave{*},\grave{0})$ be a hyper homomorphism of hyper *SA*-algebras. $ker(f)$ is a hyper ideal of H.

Proof:

By Definition (5.4), $0 \in \ker(f)$, let $x, y \in H$ such that $x \circ y \ll \ker(f) \wedge x * y \ll \ker(f)$ and $x \in \ker(f)$, then $f(x) = \grave{0}$, and for each $a \in x * y \wedge a \in x \circ y$, there exists $b \in \ker(f)$ such that $a \ll b$, it follows from (HH_2) and (p_5) that is, $a \in \ker(f)$ so that $\grave{0} = f(a) \in f(x \circ y) = f(x) \grave{\circ} f(y) = \grave{0} \grave{\circ} f(y) = f(y)$ and by (HH_3) $\grave{0} = f(a) \in f(x * y) = f(x) \grave{*} f(y) = \grave{0} \grave{*} f(y) = f(y)$ that is $y \in \ker(f)$. Hence $ker(f)$ is a hyper ideal of H. ■

Theorem 5.6.

Let $f:(H,\circ,*,0) \longrightarrow (K,\grave{\circ},\grave{*},\grave{0})$ be an epimorphism of hyper *SA*-algebra. If I is a hyper ideal of H contaning $ker(f)$, then $f(I)$ is a hyper ideal of K.

Proof:

By (HH_1), $\grave{0} = f(0) \in f(I)$. Let $x, y \in K$ such that $x \grave{\circ} y \ll f(I) \wedge x \grave{*} y \ll f(I)$ and $x \in f(I)$. Since f is onto, it follows that there exist $a, b \in H$ such that $f(a) = x$ and $f(b) = y$.

Thus $f(a \circ b) = f(a) \grave{\circ} f(b) = x \grave{\circ} y \ll f(I)$. $\wedge$ $f(a * b) = f(a) \grave{*} f(b) = x \grave{*} y \ll f(I)$.

Let $w \in a \circ b \wedge w \in a * b$ and for every $z \in f(a \circ b) \wedge z \in f(a * b)$, there exist $w \in f(I)$ such that $z \ll w$. Then $f(z) \ll f(w)$, that is $0 \in f(z) \grave{\circ} f(w) = f(z \circ w) \wedge 0 \in f(z) \grave{*} f(w) = f(z * w)$.

It follows that $z \circ w \subseteq \ker(f) \subseteq I \wedge z * w \subseteq \ker(f) \subseteq I$, so that $z \circ w \ll I \wedge z * w \ll I$ by (p_1).

Since I is a hyper ideal of H, it follows that $w \in I$, by (HI_2). Hence $a \circ b \subseteq I \wedge a * b \subseteq I$ and $a \circ b \ll I \wedge a * b \ll I$. Since $a \in I$, it follows from (HI_2) that $b \in I$, so that $y = f(b)$ in $f(I)$. Hence $f(I)$ is a hyper ideal of K. ■

Theorem 5.7.

Let $f:(H,\circ,*,0) \longrightarrow (K,\grave{\circ},\grave{*},\grave{0})$ and $g:(H,\circ,*,0) \longrightarrow (M,\ddot{\circ},\ddot{*},\ddot{0})$ be two homomorphism of hyper *SA*-algebras such that f is onto and $ker(f) \subseteq ker(g)$, then there exists a homomorphism $h:(K,\grave{\circ},\grave{*},\grave{0}) \longrightarrow (M,\ddot{\circ},\ddot{*},\ddot{0})$ such that $h \circ f = g$.

Proof:

Let $y \in K$, since f is onto, there exists $x \in H$ such that $y = f(x)$. Define $h : K \longrightarrow M$ by $h(y) = g(x)$, for all $y \in K$.

Now, we show that h is well-defined. Let $y_1, y_2 \in K$ and $y_1 = y_2$, since f is onto, then there are $x_1, x_2 \in H$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Hence $f(x_1) = f(x_2)$ and$0 \in f(x_1) \grave{\circ} f(x_2) = f(x_1 \circ x_2) \wedge$ $0 \in f(x_1) \grave{*} f(x_2) = f(x_1 * x_2)$.

It follows that there exists $z \in x_1 \circ x_2 \wedge z \in x_1 * x_2$ such that $f(z) = 0$. Thus $z \in ker(f) \subseteq ker(g)$ and $g(z) = 0$.

Since $z \in x_1 \circ x_2 \wedge z \in x_1 * x_2$, then $0 = g(z) \in g(x_1 \circ x_2) = g(x_1) \ddot{\circ} g(x_2) \wedge 0 = g(z) \in g(x_1 * x_2) = g(x_1) \ddot{*} g(x_2)$ which implies that $g(x_1) \ll g(x_2)$.

On the other hand, since $0 \in f(x_1) \grave{\circ} f(x_2) = f(x_1 \circ x_2) \wedge 0 \in f(x_1) \grave{*} f(x_2) = f(x_1 * x_2)$, similarly we can conclude that $0 \in g(x_2 \circ x_1) = g(x_2) \ddot{\circ} g(x_1) \wedge 0 \in g(x_2 * x_1) = g(x_2) \ddot{*} g(x_1)$, then $g(x_2) \ll g(x_1)$.

Thus $g(x_1) = g(x_2)$ which shows that h is well-defined. Clearly, $h \circ f = g$.

Finally, we show that h is a homomorphism, let $y_1, y_2 \in K$. Since f is onto, there are $x_1, x_2 \in H$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then

$$\begin{aligned} h(y_1 \dot{*} y_2) &= h\big(f(x_1) \dot{*} f(x_2)\big) \\ &= h\big(f(x_1 * x_2)\big) \\ &= (h \circ f)(x_1 * x_2) \\ &= g(x_1 * x_2) \\ &= g(x_1) \ddot{*} g(x_2) \\ &= (h \circ f)(x_1) \ddot{*} (h \circ f)(x_2) \\ &= h\big(f(x_1)\big) \ddot{*} h\big(f(x_2)\big) \\ &= h(y_1) \ddot{*} h(y_2) \quad \text{and} \end{aligned}$$

$$\begin{aligned} h(y_1 \dot{\circ} y_2) &= h\big(f(x_1) \dot{\circ} f(x_2)\big) \\ &= h\big(f(x_1 \circ x_2)\big) \\ &= (h \circ f)(x_1 \circ x_2) \\ &= g(x_1 \circ x_2) \\ &= g(x_1) \ddot{\circ} g(x_2) \\ &= (h \circ f)(x_1) \ddot{\circ} (h \circ f)(x_2) \\ &= h\big(f(x_1)\big) \ddot{\circ} h\big(f(x_2)\big) \\ &= h(y_1) \ddot{\circ} h(y_2). \end{aligned}$$

Moreover, since $f(0) = \dot{0}$, and $g(0) = \ddot{0}$, then

$$h(0) = h\big(f(0)\big) = (h \circ f)(0) = g(0) = \ddot{0}.$$

Hence h is a homomorphism. ■

References

[1] A.T. Hameed, (2018), **AT-ideals & Fuzzy AT-ideals of AT-algebras**, Journal of Iraqi AL-Khwarizmi Society, vol.1, no.2, pp: 2521-2621.

[2] A.T. Hameed, 2016, **AT-ideals and Fuzzy AT-ideals of AT-algebra**, LAP LEMBRT Academic Publishing, Germany.

[3] A.T. Hameed, 2015, **Fuzzy ideal of some algebras** ,PH.D.SC. Thesis, Faculty of Science, Ain Shams University, Egypt.

[4] A.T. Hameed and N.J. Raheem and A.H. Abed, (2020), **Anti-Fuzzy *SA*-Ideals on *SA* -algebras,** Journal of Physics: Conference Series, vol.1530, pp: 1-15.

[5] A.T. Hameed, F. F. Kareem and S.H. Ali, (2020)**, Hyper AT-ideal on AT-algebra,** Journal of Physics: Conference Series, vol.1530, pp: 1-17.

[6] R. A. Borzooei and M. Bakhshi, **Some results on hyper BCK-algebras**, Quasigroups and Related Systems, vol. 11 (2004), pp: 9-24.

[7] S. M. Mostafa , F.Kareem, B. Davvaz, (2017), **Hyper structure theory applied to KU-algebras**, The Journal of Hyperstructures , vol.6, no. 2, pp: 82-95.

[8] S.M. Mostafa, M.A. Abd-Elnaby and M.M.M. Yousef, (2011), **Fuzzy ideals of KU-Algebras**, Int. Math. Forum, vol.63, no.6, pp: 3139-3149.

[9] Y. B. Jun, M. M. Zahedi, X. L. Xin, and R. A. Borzooei, (2000), **On hyper BCK-algebra**, Italian J. Pure App. Math., Oxford Ser. , vol.10, pp: 127-136.