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Finding the error formula using method MSu to calculate the double integral with continuous integrands and improper derivative from the lower end

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Abstract: *The main goal for this research is to finding the general formula to the correction terms of the method MSu, when the integrand is continuous and the derivatives are improper or its just improper in one point of integration region. This research includes theorem about finding the error formulas for these kind of double integrals from lower end .Two examples are discussed to illustrate different cases in addition to attached tables which show all the details.*

Keywords: Double integrals, continuous integrand, improper derivatives, Romberg's acceleration.

1. Introduction:

Double integrals are important for finding surface area, evaluating intermediate centres and moments of inertia for plane surfaces as well as computing the volume under the double integral surface. As an example for this, the resulting volume from the rotation of heart curve $\rho = 2(1-\cos\theta)$ around the polar axis. In addition, the importance of double integrals as Frank Eyers in [2] explained, lies in evaluating the area of the piece of ball $x^2 + y^2 + z^2 = 36$ inside the cylinder $y^2 + z^2 = 6y$. Many researchers in the double integrals field, Schjear-Jacobsen H [7] in 1973, highlighted the computation of double integrals with continues integrands of the expression $f(x,y) = f_1(x) f_2(y)$ whereas, Davies and Rabinowitz [9] in 1975, worked with integrals that including improper integrands.

In 2009, Dheyaa [3] presented four numerical methods composed of Romberg's acceleration with the midpoint method and Romberg's acceleration with Simpson method, RM(RS), RS(RS), RS(RM) and RM(RM), to calculate double integrals values that have continues integrands and improper or with improper derivatives. The best method among these four methods was tested is RM(RM) regarding to the accuracy and speed to approach the real values of integrations.

In 2011, [6] discussed three numerical methods composed of Romberg's acceleration with two formulation of Newton-Coats (Simpson and Midpoint) when the number of partial periods on two axis X and Y is equal, to calculate double integrals values that have integrands with improper derivatives or only improper. RSS had been tested to be the best method in terms of accuracy and speed of approach to the real value of integrations with a few partial periods. For more information in this area, see [1,6].

In our research, we have derived the general form for the correction terms of the method MSu, in case that the integrand is continuous with improper derivative or only improper in one point of integration region.

2. Finding the error formula using method MSu to calculate the double integral with continuous integrands and improper derivative from the lower end:

Theorem

Let the function *f(x,y)* be continuous and differentiable at each point of the integration region except at the point $(x,y) = (x_0, y_0)$, the approximate value of the double integral $J = \int_{y_0}^{y_n}\int_{x_0}^{x_n} f(x,y)\,dx\,dy$ can be evaluated using the following SuSu method:

$$\boxed{\begin{aligned} MSu=\int_{y_0}^{y_n}\int_{x_0}^{x_n} f(x,y)dxdy &=\frac{h^2}{4}\sum_{j=1}^{n}\Big[f(x_0,y_j)+f(x_n,y_j)+2(f(x_0+(n-0.5)h,y_j)+2\sum_{i=1}^{n-1}\big(f(x_0+(i-0.5)h,y_j)\big) \\ &\quad f(x_0+ih,y_i)\Big]\end{aligned}}$$

And the error formula is :

$$\left[a_1h^4(D_x^2+D_y^2)+a_2h^5(D_x^3+D_x^2D_y+D_xD_y^2+D_y^3)+a_3h^6(D_x^4+D_y^4+\cdots)\right]f(x_1,y_1)$$
$$+\alpha_1h^2+\alpha_2h^4+\alpha_3h^6+\cdots$$

Where $a_1,a_2,a_3,\cdots$ and $\alpha_1,\alpha_2,\alpha_2,\cdots$ are constants.

Proof: Suppose that the function $f\ (x,y)$ is defined at each point of the integration region $[x_0,x_n]\times[y_0,y_n]$ and it is not improper and partial derivatives of the function are not defined at the point (x_0,y_0). This means that the Taylor's series of two-variable functions exists at each point of the integration region except at (x_0,y_0), [7].

We can write the double integral J by :

$$J=\int_{y_0}^{y_n}\int_{x_0}^{x_n} f(x,y)dxdy=\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)dxdy+\int_{y_0}^{y_1}\sum_{r=1}^{n-1}\int_{x_r}^{x_{r+1}} f(x,y)dxdy+\sum_{s=1}^{n-1}\int_{y_s}^{y_{s+1}}\int_{x_0}^{x_1} f(x,y)dxdy+\int_{y_1}^{y_n}\int_{x_1}^{x_n} f(x,y)dxdy \qquad .....(1)$$

For the first integral on the partial integration region $[x_0,x_1]\times[y_0,y_1]$, use Taylor's series for $f\ (x,y)$ about (x_1,y_1)

,

$$\begin{aligned} f(x,y)=\Big[&1+(x-x_1)D_x+(y-y_1)D_y+\frac{(x-x_1)^2}{2!}D_x^2+(x-x_1)(y-y_1)D_xD_y+\frac{(y-y_1)^2}{2!}D_y^2 \\ &+\frac{(x-x_1)^3}{3!}D_x^3+\frac{(x-x_1)^2(y-y_1)}{2!}D_x^2D_y+\frac{(x-x_1)(y-y_1)^2}{2!}D_xD_y^2+\frac{(y-y_1)^3}{3!}D_y^3+\frac{(x-x_1)^4}{4!} \\ &D_x^4+\frac{(x-x_1)^3(y-y_1)}{3!}D_x^3D_y+\frac{(x-x_1)^2(y-y_1)^2}{2!2!}D_x^2D_y^2+\frac{(x-x_1)(y-y_1)^3}{3!}D_xD_y^3+\frac{(y-y_1)^4}{4!}\ \text{By} \\ &D_y^4+\frac{(x-x_1)^5}{5!}D_x^5+\frac{(x-x_1)^4(y-y_1)}{4!}D_x^4D_y+\frac{(x-x_1)^3(y-y_1)^2}{3!2!}D_x^3D_y^2+\frac{(x-x_1)^2(y-y_1)^3}{2!3!} \\ &D_x^2D_y^3+\frac{(x-x_1)(y-y_1)^4}{4!}D_xD_y^4+\frac{(y-y_1)^5}{5!}D_y^5+\cdots\Big]f(x_1,y_1) \qquad \cdots(2)\end{aligned}$$

integrate equation (2) on $(x_0,x_1)\times(y_0,y_1)$ we get:

$$\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)dxdy = \Big[h^2 - \frac{h^3}{2}D_x - \frac{h^3}{2}D_y + \frac{h^4}{6}D_x^2 + \frac{h^4}{4}D_xD_y + \frac{h^4}{6}D_y^2 - \frac{h^5}{24}D_x^3 - \frac{h^5}{12}D_x^2D_y - \frac{h^5}{12}D_xD_y^2$$
$$- \frac{h^5}{24}D_y^3 + \frac{h^6}{120}D_x^4 + \frac{h^6}{48}D_x^3D_y + \frac{h^6}{36}D_x^2D_y^2 + \frac{h^6}{48}D_xD_y^3 + \frac{h^6}{120}D_y^4 - \frac{h^7}{720}D_x^5 - \frac{h^7}{240}D_x^4D_y - \frac{h^7}{144}D_x^3D_y^2$$
$$- \frac{h^7}{144}D_x^2D_y^3 - \frac{h^7}{240}D_xD_y^4 - \frac{h^7}{720}D_y^5 + \cdots \Big] f(x_1,y_1) \qquad \cdots(3)$$

Substituting the points:

$$(x_0, y_0+0.5h), (x_1, y_0+0.5h), (x_0+0.5h, y_0+0.5h)$$

in equation (2) and adding the result to equation (3) to obtain:

$$\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)dxdy = \frac{h^2}{2}[f(x_0,y_0+0.5h)+f(x_1,y_0+0.5h)+f(x_0+0.5h,y_0+0.5h)$$
$$+\left[a_1h^4(D_x^2+D_y^2)+a_2h^5(D_x^3+D_x^2D_y+D_xD_y^2+D_y^3)+a_3h^6(D_x^4+D_y^4+\cdots)\right]f(x_1,y_1) \qquad ...(4)$$

For the other three integrals in equation (1), the derivative of the function is continuous, so we can calculate their values and add them to equation (4) to get:

$$MSu = \int_{y_0}^{y_n}\int_{x_0}^{x_n} f(x,y)dxdy = \frac{h^2}{4}\sum_{j=1}^{n} \left[f(x_0,y_j)+f(x_n,y_j)+2(f(x_0+(n-0.5)h,y_j)+2\sum_{i=1}^{n-1}\left(f(x_0+(i-0.5)h,y_j)\right)\right.$$
$$\left[a_1h^4(D_x^2+D_y^2)+a_2h^5(D_x^3+D_x^2D_y+D_xD_y^2+D_y^3)+a_3h^6(D_x^4+D_y^4+\cdots)\right]f(x_1,y_1)$$
$$+\alpha_1h^2+\alpha_2h^4+\alpha_3h^6+\cdots$$

…(5)

Where $a_1, a_2, a_3, \cdots$ and $\alpha_1, \alpha_2, \alpha_2, \cdots$ are constants.

5. Integrals with improper integrands in one of the integration:

Suppose that $J = \int_{y_0}^{y_n}\int_{x_0}^{x_n} f(x,y)dxdy$, *f(x,y)* is continuous on the integration region $[x_0, x_n]\times[y_0 \times y_n]$ but it is not defined at the points (x_0, y_0) and (x_n, y_n) or on one of them. Thus we cannot apply any of the above three theorems, so to evaluate the improper integral, we will ignore the value of the function on impropriety point as Phillip and Rabinowitz suggested in [9].

6. Examples:

6.1: The integral $\int_0^1\int_0^1 y\sqrt{x+y}\,dxdy$ that its function had been shown in Fig:1, has analytic value 0.53221191422770 which approximates to fourteen decimal digits.

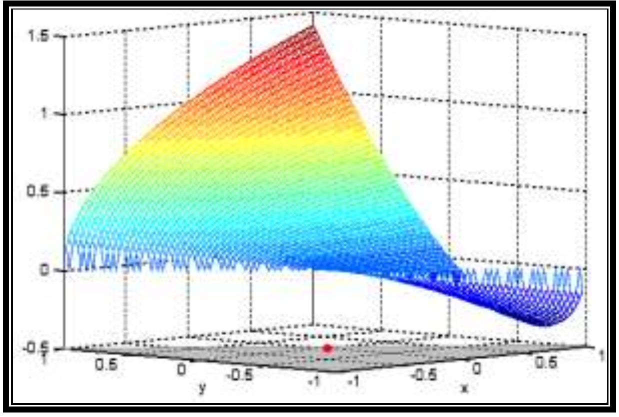

Fig:(1)

In this example, the integrand is continuous in integration region but the partial derivatives are improper at the point *(x,y) = (0,0)* such that the impropriety is radical. By Theorem, the suitable correction terms are

$$E_{MSu}(h) = \alpha_1 h^2 + \beta h^{3.5} + \alpha_2 h^4 + \alpha_3 h^6 + \alpha_4 h^8 + \cdots$$

where $a, \alpha_{Su}, \beta_{Su}, \cdots$ are constants.

Applying MSu method, we obtained four correct decimal digits at n=64. Moreover, when we used Romberg's acceleration to improve these results with the above correct terms in (5),

6.3: The integral $J = \int_{-1}^{0}\int_{0}^{1} \frac{(1+y)}{(1+x+y)^2} dxdy$ that its function had been shown in Fig:3, has analytic value 0.69314718055995 which approximates to fourteen decimal digits.

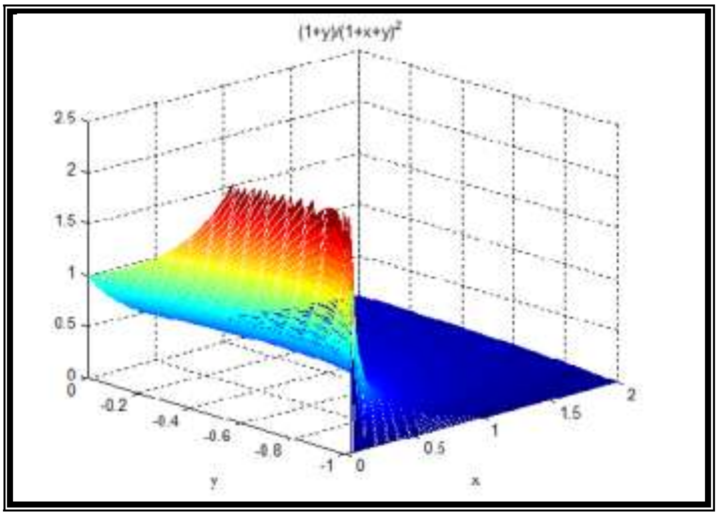

Fig (2)

In this example, the integrand is continuous in integration region except at the point *(x,y) = (0,0)*, so it is improper such that the impropriety is proportional. From Theorem, the suitable correct terms are

$$E_{MSu}(h) = ch + \alpha_1 h^2 + \alpha_2 h^4 + \alpha_3 h^6 \cdots$$

where $c, \alpha_1, \alpha_2, \alpha_3, \cdots$ are constants.

Applying MSu method, we obtained four correct decimal digits at n=64. Moreover, when we used Romberg's acceleration to improve these results with the above correct terms in (5),

Applying MSu method on axes, we obtained two correct decimal digits at n=128 Moreover, when we used Romberg's acceleration to improve these results with the above correct terms in (10),

7. Conclusion:

We conclude that the values of double integrals using MSu method give the correct values of several decimal digits compared with the exact values of the integrals using a number of partial periods without using a method of teasing. Moreover the tables show that the results will be better with a few relatively partial periods as well as the values are correct for several decimal digits which are between thirteen and fourteen correct decimal digits, when we use the Romberg's acceleration with the MSu method accompanying with the correction terms. In addition, the Romberg acceleration without ignore the impropriety will play an importance role to improve results in terms of accuracy and speed of approach to the real value of integrations with a few partial periods. Therefore we can use SuSu method to evaluate the double integral.

n	MSu method	k=2	k=3.5	k=4	k=6	k=8	k=10
1	0.49148145657227						
2	0.52168241714830	0.53174940400697					
4	0.52954082627501	0.53216029598391	0.53220013538571				
8	0.53154018464856	0.53220663743975	0.53221113063013	0.53221186364642			
16	0.53204359793144	0.53221140235907	0.53221186435771	0.53221191327288	0.53221191406060		
32	0.53216979908002	0.53221186612955	0.53221191109596	0.53221191421185	0.53221191422675	0.53221191422740	
64	0.53220138211831	0.53221190979774	0.53221191403173	0.53221191422745	0.53221191422770	0.53221191422770	0.53221191422770

Table(1): The value of $\int_0^1\int_0^1 y\sqrt{x+y}\,dxdy = 0.53221191422770$

n	Msu	k=1	k=2	k=4	k=6	k=8	k=10	k=12
1	0.80555555555556							
2	0.76586167800454	0.72616780045352						
4	0.73415607089350	0.70245046378247	0.69454468489212					
8	0.71485687131210	0.69555767173069	0.69326007438010	0.69317443367930				
16	0.70430621722599	0.69375556313989	0.69315486027628	0.69314784600270	0.69314742397608			
32	0.69880293100685	0.69329964478772	0.69314767200366	0.69314719278549	0.69314718241696	0.69314718146967		
64	0.69599412540051	0.69318531979416	0.69314721146298	0.69314718076026	0.69314718056939	0.69314718056214	0.69314718056125	
128	0.69457542110991	0.69315671681931	0.69314718249436	0.69314718056312	0.69314718055999	0.69314718055995	0.69314718055995	0.69314718055995

Table(2): The value of $I = \int_{-1}^{0}\int_0^1 \frac{(1+y)}{(1+x+y)^2}dxdy = 0.69314718055995$

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