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Calculate Double Integrals with Singular Derivatives and Singular Integrands Using MSu Method Numerically

Safaa M. Aljassas

Iraq\ University of Kufa. \ College of Education for girls \ Mathematics Dep.
safaam.musa@uokufa.edu.iq

Abstract: *The main aim of this research is computation Double integrals with improper derivatives integrands and improper numerically in one point of integration region using the method MSu ,we have provided theorem to find the correction terms accompanying for its , by depending on the correction terms that we found it we have improved the results by using the method of Romberg's acceleration where we obtained high precision in returns with relatively few subintervals and short time.*

Keywords: Double integrals, continuous integrand, improper derivatives, Romberg's acceleration.

1. Introduction:

Frequently there is a need to find the value to the definite Integral for a function that does not have an inverse explicit differentiation or it has inverse differentiation that cannot be easily found. The basic method involving approximation for the integral $\int_a^b f(x)dx$ is called numerical quadrature which uses the sum of the type $\sum_{1=0}^{n} a_i f(x)$ to approximate $\int_a^b f(x)dx$. Thus, some numerical methods must be obtained to find the value of the definite integral for such a function such as Newton-Coates methods.

In 2009, Dheyaa [3] presented four numerical methods composed of Romberg's acceleration with the midpoint method and Romberg's acceleration with Simpson method, RM(RS), RS(RS), RS(RM) and RM(RM), to calculate double integrals values that have continues integrands and improper or with improper derivatives. The best method among the previous methods was tested is RM(RM) regarding to the accuracy and speed to approach the real values of integrations.

In 2011, [6] discussed three numerical methods composed of Romberg's acceleration with two formulation of Newton-Coats (Simpson and Midpoint), RSS, RMS, and RSM when the number of partial periods on two axis X and Y is equal, to calculate double integrals values that have integrands with improper derivatives or only improper. RSS had been tested to be the best method in terms of accuracy and speed to approach to the real value of integrations with a few partial periods. For more information in this area, see [1].

In our research, we have derived the general form for the correction terms of the method, MSu, obtained from applying the midpoint rule to the external dimension and the suggested method on the internal dimension in case that the integrand is continuous with improper derivative or only improper in one point of integration region.

2. Deriving the error formula using the method MSu to calculate the double integral with continuous integrands and improper derivative from the upper end:

Theorem

Let the function f(x,y) be continuous and differentiable at each point of the integration region except at the point $(x,y)=(x_n,y_n)$, the approximate value of the double integral $J=\int_{y_0}^{y_n}\int_{x_0}^{x_n} f(x,y)dxdy$ can be evaluated using the following MSu method:

$$MSu=\int_{y_0}^{y_n}\int_{x_0}^{x_n} f(x,y)dxdy=\frac{h^2}{4}\sum_{j=1}^{n}\Big[f(x_0,y_j)+f(x_n,y_j)+2(f(x_0+(n-0.5)h,y_j)+2\sum_{i=1}^{n-1}\big(f(x_0+(i-0.5)h,y_j) f(x_0+ih,y_i)\Big]$$

And the error formula is :

$$\left[b_1h^4(D_x^2+D_y^2)+b_2h^5(D_x^3+D_x^2D_y+D_xD_y^2+D_y^3)+b_3h^6(D_x^4+D_y^4+\cdots)\right]f(x_{n-1},y_{n-1})$$ Where

$$+\beta_1h^2+\beta_2h^4+\beta_3h^6+\cdots$$

$b_1,b_2,b_3,\cdots$ and $\beta_1,\beta_2,\beta_3,\cdots$ are constants.

Proof: Suppose that the function $f(x,y)$ is defined at each point of the integration region $[x_0,x_n]\times[y_0,y_n]$ and it is not improper and the partial derivatives of the function are not defined at the point (x_n,y_n).

Write the double integral J by:

$$J=\int_{y_0}^{y_n}\int_{x_0}^{x_n}f(x,y)dxdy=\int_{y_0}^{y_{n-1}}\int_{x_0}^{x_{n-1}}f(x,y)dxdy+\sum_{s=0}^{n-2}\int_{y_s}^{y_{s+1}}\int_{x_{n-1}}^{x_n}f(x,y)dxdy+\int_{y_{n-1}}^{y_n}\sum_{r=0}^{n-2}\int_{x_r}^{x_{r+1}}f(x,y)dxdy+\int_{y_{n-1}}^{y_n}\int_{x_{n-1}}^{x_n}f(x,y)dxdy$$ .....(1)

For the fourth integral on the partial integration region $[x_{n-1},x_n]\times[y_{n-1},y_n]$, use Taylor's series for $f(x,y)$ about (x_{n-1},y_{n-1}),

$$f(x,y)=\Bigg[1+(x-x_{n-1})D_x+(y-y_{n-1})D_y+\frac{(x-x_{n-1})^2}{2!}D_x^2+(x-x_{n-1})(y-y_{n-1})D_xD_y+\frac{(y-y_{n-1})^2}{2!}D_y^2$$
$$+\frac{(x-x_{n-1})^3}{3!}D_x^3+\frac{(x-x_1)^2(y-y_{n-1})}{2!}D_x^2D_y+\frac{(x-x_{n-1})(y-y_{n-1})^2}{2!}D_xD_y^2+\frac{(y-y_{n-1})^3}{3!}D_y^3+\frac{(x-x_{n-1})^4}{4!}$$
$$D_x^4+\frac{(x-x_{n-1})^3(y-y_{n-1})}{3!}D_x^3D_y+\frac{(x-x_{n-1})^2(y-y_{n-1})^2}{2!2!}D_x^2D_y^2+\frac{(x-x_{n-1})(y-y_{n-1})^3}{3!}D_xD_y^3+\frac{(y-y_{n-1})^4}{4!}$$ By
$$D_y^4+\frac{(x-x_{n-1})^5}{5!}D_x^5+\frac{(x-x_{n-1})^4(y-y_{n-1})}{4!}D_x^4D_y+\frac{(x-x_{n-1})^3(y-y_{n-1})^2}{3!2!}D_x^3D_y^2+\frac{(x-x_{n-1})^2(y-y_{n-1})^3}{2!3!}$$
$$D_x^2D_y^3+\frac{(x-x_{n-1})(y-y_{n-1})^4}{4!}D_xD_y^4+\frac{(y-y_{n-1})^5}{5!}D_y^5+\cdots\Bigg]f(x_{n-1},y_{n-1})\qquad\cdots(2)$$

integrate equation (2) on $(x_{n-1},x_n)\times(y_{n-1},y_n)$ we get:

$$\int_{y_{n-1}}^{y_n}\int_{x_{n-1}}^{x_n}f(x,y)dxdy=\Bigg[h^2+\frac{h^3}{2}D_x+\frac{h^3}{2}D_y+\frac{h^4}{6}D_x^2+\frac{h^4}{4}D_xD_y+\frac{h^4}{6}D_y^2+\frac{h^5}{24}D_x^3+\frac{h^5}{12}D_x^2D_y+\frac{h^5}{12}D_xD_y^2$$
$$+\frac{h^5}{24}D_y^3+\frac{h^6}{120}D_x^4+\frac{h^6}{48}D_x^3D_y+\frac{h^6}{36}D_x^2D_y^2+\frac{h^6}{48}D_xD_y^3+\frac{h^6}{120}D_y^4+\frac{h^7}{720}D_x^5+\frac{h^7}{240}D_x^4D_y+\frac{h^7}{144}D_x^3D_y^2$$
$$+\frac{h^7}{144}D_x^2D_y^3+\frac{h^7}{240}D_xD_y^4+\frac{h^7}{720}D_y^5+\cdots\Bigg]f(x_{n-1},y_{n-1})\qquad\cdots(3)$$

Substituting the points:

$$(x_n,y_{n-1}+0.5h),(x_{n-1},y_{n-1}+0.5h),(x_{n-1}+0.5h,y_{n-1}+0.5h)$$

in equation (2) and adding the result to equation (3) to obtain:

$$\int_{y_{n-1}}^{y_n}\int_{x_{n-1}}^{x_n} f(x,y)dxdy = \frac{h^2}{2}[f(x_{n-1},y_{n-1}+0.5h)+f(x_n,y_{n-1}+0.5h))+2f(x_{n-1}+0.5h,y_{n-1}+0.5h)]$$
$$+\left[b_1h^4(D_x^2+D_y^2)+b_2h^5(D_x^3+D_x^2D_y+D_xD_y^2+D_y^3)+b_3h^6(D_x^4+D_y^4+\cdots)\right]f(x_{n-1},y_{n-1}) \quad ...(4)$$

For the other three integrals in equation (1), the derivative of the function is continuous, so we can calculate their values and add them to equation (4) to get:

$$MSu = \int_{y_0}^{y_n}\int_{x_0}^{x_n} f(x,y)dxdy = \frac{h^2}{4}[f(x_0,y_0)+f(x_0,y_n)+f(x_n,y_0)+f(x_n,y_n)+2(f(x_0,y_0+(n-0.5)h)$$
$$+f(x_n,y_0+(n-0.5)h)+f(x_0+(n-0.5)h,y_0)+f(x_0+(n-0.5)h,y_n)+2f(x_0+(n-0.5)h,y_0+(n-0.5)h))$$
$$+2\sum_{j=1}^{n-1}\Big(f(x_0,y_j)+f(x_0,y_0+(j-0.5)h)+f(x_n,y_j)+f(x_n,y_0+(j-0.5)h)+2f(x_0+(n-0.5)h,y_j)$$
$$+2f(x_0+(n-0.5)h,y_0+(j-0.5)h)+f(x_j,y_0)+f(x_j,y_n)+f(x_0+(j-0.5)h,y_0)+f(x_0+(j-0.5)h,y_n)$$

such

$$+2f(x_j,y_0+(n-0.5)h)+2f(x_0+(j-0.5)h,y_0+(n-0.5)h)+2\sum_{i=1}^{n-1}\Big(f(x_i,y_j)+f(x_i,y_0+(j-0.5)h)$$
$$+f(x_0+(i-0.5)h,y_j)+f(x_0+(i-0.5)h,y_0+(j-0.5)h))\ \Big]+\Big[b_1h^4(D_x^2+D_y^2)+b_2h^5(D_x^3+D_x^2D_y$$
$$+D_xD_y^2+D_y^3)+b_3h^6(D_x^4+D_y^4+\cdots)\Big]f(x_{n-1},y_{n-1})+\beta_1h^2+\beta_2h^4+\delta_3h^6+\cdots \quad ...(5)$$

that $b_i, \beta_i, \ldots$, $i= 1,2,3$ are constant

and the proof is complete.

3. Integrals with improper integrands in one or both ends of the integration:

Suppose that $J = \int_{y_0}^{y_n}\int_{x_0}^{x_n} f(x,y)dxdy$, *f(x,y)* is continuous on the integration region $[x_0,x_n]\times[y_0\times y_n]$ but it is not defined at the point (x_n,y_n). Thus we cannot apply any of the above tre theorem, so to evaluate the improper integral, we will ignore the value of the function on impropriety point as Phillip and Rabinowitz suggested in [9].

4. Examples:

First example:- The integral $I = \int_0^1\int_0^1 y\sqrt{1-xy}\,dxdy$ that its function had been shown in Fig:1, has exact value 0.4

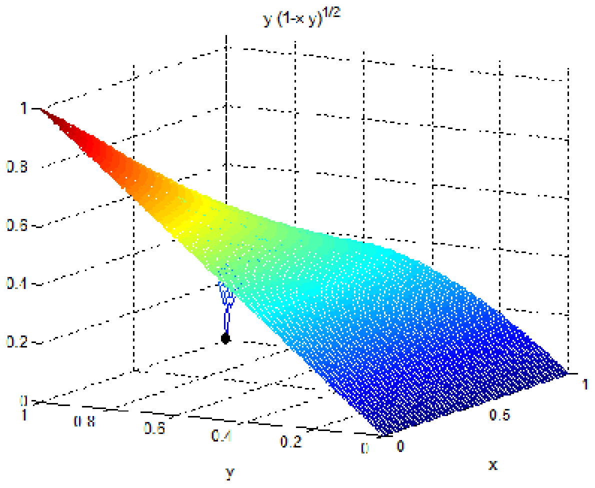

Fig:(1): Geometric shape for $y\sqrt{1-xy}$

In this example, the integrand is continuous in integration region but the partial derivatives are improper at the point (x,y) = (1,1) such that the impropriety is radical. By Theorem, the suitable correction terms are

$$E_{MSu} = \beta_1 h^2 + b_1 h^{5/2} + b_2 h^{7/2} + \beta_2 h^4 + b_3 h^{9/2} + \cdots$$

where $b_1, b_2, b_3, \cdots$, $\beta_1, \beta_2, \beta_3, \cdots$ are constants.

Applying MSu method, we obtained five correct decimal digits at n=128. Moreover, when we used Romberg's acceleration to improve these results with the above correct terms , we got fourteen correct decimal digits at n=128 with 2^{16} partial periods (which is equal to the analytic value).

Second example: The integral $\int_0^1\int_0^1 \frac{y}{\sqrt{1-xy}}dxdy$ that its function had been shown in Fig:2, has analytic value 0.66666666666667 which approximates to fourteen decimal digits.

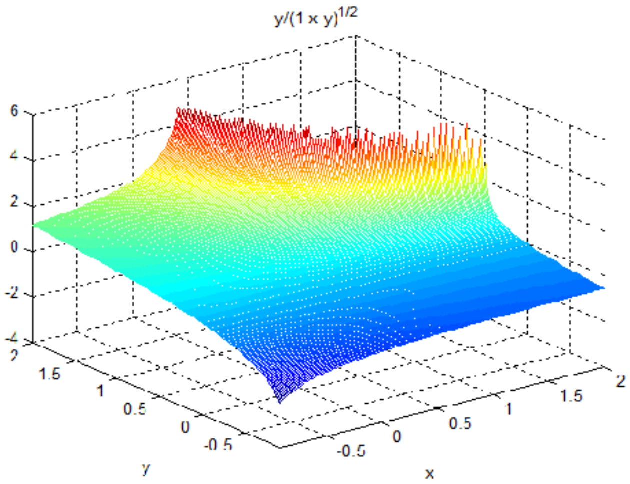

Fig:(2): Geometric shape for $\frac{y}{\sqrt{1-xy}}$

In this example, the integrand is continuous in integration region except at the point *(x,y) = (1,1)*, so it is improper such that the impropriety is radical. From Theorem 2, the suitable correction terms are

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$$E_{MSu} = b_1 h^{3/2} + \beta_1 h^2 + b_2 h^{5/2} + b_3 h^{7/2} + \beta_2 h^4 + \cdots$$

where $b_1, b_2, b_3, \cdots$, $\beta_1, \beta_2, \beta_3, \cdots$ are constants.

Applying MSu method on axes, we obtained four correct decimal digits at n=256 taken with Pillip-Rabinowitz in [9] suggestion (*ignoring the function value at impropriety point*).

Moreover, when we used Romberg's acceleration to improve these results with the above correct terms ,we got thirteen correct decimal digits at n=256 with 2^{18}.

7. Conclusion:

We conclude that the values of double integrals using MSu method give the correct values of several decimal digits compared with the exact values of the integrals using a number of partial periods without using a method of teasing. Moreover the tables show that the results will be better with a few relatively partial periods as well as the values are correct for several decimal digits which are between thirteen and fourteen correct decimal digits, when we use the Romberg's acceleration with the MSu method accompanying with the correction terms. In addition, the Romberg acceleration without ignore the impropriety will play an importance role to improve results in terms of accuracy and speed of approach to the real value of integrations with a few partial periods. Therefore we can use MSu method to evaluate the double integral.

n	MSu	k=2	k=2.5	k=3.5	k=4	k=4.5	k=5.5	k=6
1	0.42989469859443							
2	0.40775768403207	0.40037867917795						
4	0.40199080641736	0.40006851387913	0.40000190984082					
8	0.40050725788907	0.40001274171297	0.40000076535229	0.40000065438458				
16	0.40012856372056	0.40000233233106	0.40000009704918	0.40000003225163	0.39999999077610			
32	0.40003245631409	0.40000042051193	0.40000000997318	0.40000000153043	0.39999999948236	0.39999999988491		
64	0.40000817041531	0.40000007511572	0.40000000094629	0.40000000007106	0.39999999997376	0.39999999999649	0.39999999999901	
128	0.40000205261634	0.40000001335002	0.40000000008662	0.40000000000327	0.39999999999875	0.39999999999990	0.39999999999998	0.40000000000000

Table (2) :- The value of $\int_0^1 \int_0^1 \sqrt{1-xy}\, dxdy = 0.4$

n	MSu	k=1.5	k=2	k=2.5	k=3.5	k=4	k=4.5	k=5.5	k=6
1	0.59045182989145								
2	0.63897185106071	0.66550833179465							
4	0.65686721285896	0.66665451121833	0.66703657102622						
8	0.66323176811674	0.66671265897186	0.66673204155637	0.66666664774039					
16	0.66546549101634	0.66668715463605	0.66667865319077	0.66666718872083	0.66666724117339				
32	0.66624635811083	0.66667342850589	0.66666885312917	0.66666674869105	0.66666670602649	0.66666667035003			
64	0.66651934928672	0.66666865311851	0.66666706132272	0.66666667655516	0.66666666956099	0.66666666712995	0.66666666698106		
128	0.66661493739116	0.66666721626142	0.66666673730905	0.66666666773126	0.66666666687570	0.66666666669669	0.66666666667665	0.66666666666977	
256	0.66664847252427	0.66666681349759	0.66666667924299	0.66666666677404	0.66666666668123	0.66666666666826	0.66666666666695	0.66666666666673	0.66666666666668

Table (2) :- The value of $\int_0^1 \int_0^1 \frac{y}{\sqrt{1-xy}}\, dxdy = 0.66666666666667$

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