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Calculation Triple Integrals with Continuous Integrands Using SMS and Improve Results Using Aitken's Acceleration

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Abstract: *The main goal of this article to calculate the triple integrals with continuous integrand numerically by using the composite rule SMS . To improve results, we used the mentioned rule with Aitken's acceleration we called the method by $AI(SMS)$ we obtained high accuracy results in relatively few subintervals and short time.*

Keywords: Triple integrals, continuous integrand, improper derivatives, Aitken's acceleration

1-Introduction

The triple integrals are very useful to find volumes, middle centres and the inertia of volumes for instance the volume inside $x^2+y^2=4x$, up $z=0$ and under $x^2+y^2=4z$. And the volume within the cylinder $\rho=4\cos(\theta)$ that determines by the sphere $p^2+z^2=16$ from the top and by z=0 down. In addition, calculate the middle centre for volume under $z^2=xy$ and up the tringle $y=x, y=0, x=4$. As well as calculating the moment of inertia for volume which is located inside $x^2+y^2=9$, up z=0 and down the plane $x+z=4$. The importance of triple integral stands out in finding the mass with unsteady density, Frank Ayres [6].

There are many researchers was interested in evaluating the triple integrals such as Dheyaa [3], in 2009, he used numerical composite method ($RMRM(RS)$, $RMRM(RM)$, $RMRS(RM)$ and $RMRS(RS)$). These methods have obtained from Romberg acceleration method with Midpoint method (RM) on the exterior dimension (Z). $RS(RS), RS(RM), RM(RM), RM(RS)$ on the middle dimension (Y) and interior dimension (X). He reached that the composite method from Simpson's rule with Romberg acceleration on interior and middle dimension and Midpoint method with Romberg acceleration on exterior dimension RMRS(RS) was better method for evaluating the triple integrals with continuous integrand in terms of accuracy, the number of sub intervals used and time.

In 2010, Eghaar [5], introduced numerical method to calculate the value of triple integrals by Romberg acceleration method on the resulting values from applying Midpoint method on three dimensions X , Y and Z when the number of sub intervals which obtained from divided the interior dimension interval equals to the number of sub intervals that obtained from divided the middle dimension interval and also equals to the number of sub intervals which obtained from divided the exterior dimension interval. She got good results in terms of accuracy and a relatively few sub intervals.

Mohammed et al. [1] presented in 2013 numerical method to evaluate the value of triple integrals with continuous integrands by RSSS method that obtained from Romberg acceleration with Simpson's rule on three dimensions X,Y and Z as the same approach of Eghaar [5].

In 2015, Aljassas [11] introduced a numerical method $RM(RMM)$ to calculating triple integrals with continuous integrands by using Romberg acceleration with Mid-point rule on the three dimensions when the number of divisions on the interior dimension is equal to the number of divisions on the middle dimension, but both of them are deferent from the number of divisions on the exterior dimension and she got a high accuracy in the results in a little sub-intervals relatively and a short time. Also in 2015, Sarada et al. [9] use the generalized Gaussian Quadrature to evaluate triple integral and got a good results.
Additionally, in 2018 Safaa et al [10]. introduced two numerical methods R(MSM) and R(SMM) to evaluate the value of triple integral with continuous integrands, these methods obtained from Romberg acceleration with two rules from Newton–Cotes

formulas (Midpoint and Simpson) and they got good results in terms of accuracy and the access of approximate values to the real values was fast in a relatively few sub intervals.

In this paper, we will compute the triple integrals with continuous integrand numerically by using the composite rule SMS **[2]**. To improve results, we used the mentioned rule with Aitken's acceleration. (denoted it by AI(SMS))

2. Newton-Cotes Formulas

The Newton-Coates formulas are the most important methods of numerical integration, we review two rules of them Mid-point and Simpson and their correction terms if the function of integration continuous.

Let the integral J defined by the formula $J=\int_{x_0}^{x_n} g(x)dx = \beta(k)+E_\beta(k)+R_\beta$ … (1)

Fox [7], where $\beta(k)$ is the numerical rule to evaluate the integral J, β denoted to a type of rule, $E_\beta(k)$ is the correction terms for $\beta(k)$ and R_β is the remainder which is related to truncation from $E_\beta(k)$ after using a serval terms of $E_\beta(k)$. The general formulas of Mid-point rule $M(k)$ and Simpson's rule $S(k)$ are :

$$M = k\sum_{i=1}^{n} g(x_0+(i-0.5)h) \quad \text{…(2)}$$

$$S(k)=\frac{k}{3}\left[g(x_0)+g(x_n)+2\sum_{i=1}^{\frac{n}{2}-1} g(x_{2i})+4\sum_{i=1}^{\frac{n}{2}} g(x_{2i+1})\right] \quad \text{…(3)}$$

Where $k=\frac{x_n-x_0}{n}$

3-The numerical rule SMS ,[2]

The integral I can be written as

$$I=\int_{z_0}^{z_n}\int_{y_0}^{y_n}\int_{x_0}^{x_n} g(x,y,z)dxdydz = SMS(k)+\lambda_{SMS}(k)$$

Such that $SMS(k)$ is the approximate integral value that is calculated numerically by Simpson's rule on exterior dimension Z and interior dimension X. And the midpoint rule on middle dimension Y, $\lambda_{SMS}(k)$ is the correction terms series that can be add to the $SMS(k)$ values and if $(n=n_1=n_2)$ then $k=\frac{z_n-z_0}{n_2}=\frac{y_n-y_0}{n_1}=\frac{x_n-x_0}{n}$, where n_2, n_1, n are the number of divided on Z, Y, X respectively.

And the general formula for the SMS method is :-

$$SMS=\frac{k^3}{9}\sum_{j=1}^{n}\Big[\ g(x_0,y_j,z_0)+g(x_0,y_j,z_n)+g(x_n,y_j,z_0)+g(x_n,y_j,z_n)+4\sum_{r=1}^{\frac{n}{2}}\Big(g\left(x_0,y_j,z_{(2r-1)}\right)$$

$$+g\left(x_n,y_j,z_{(2r-1)}\right)\Big)+2\sum_{r=1}^{\frac{n}{2}-1}\Big(g\left(x_0,y_j,z_{(2r)}\right)+g\left(x_n,y_j,z_{(2r)}\right)\Big)+4\sum_{i=1}^{\frac{n}{2}}\Big(\ g(x_{(2i-1)},y_j,z_0)+g(x_{(2i-1)},y_j,z_n)$$

$$+4\sum_{r=1}^{\frac{n}{2}}g(x_{(2i-1)},y_j,z_{(2r-1)})+2\sum_{r=1}^{\frac{n}{2}-1}g(x_{(2i-1)},y_j,z_{(2r)})\ \Big)+2\sum_{i=1}^{\frac{n}{2}-1}\Big(\ g(x_{(2i)},y_j,z_0)+g(x_{(2i)},y_j,z_n)$$

$$+4\sum_{r=1}^{\frac{n}{2}}g(x_{(2i)},y_j,z_{(2r-1)})+2\sum_{r=1}^{\frac{n}{2}-1}g(x_{(2i)},y_j,z_{(2r)})\ \Big)\Big] \qquad ...(4)$$

$$x_{(2i-1)}=x_0+(2i-1)h\ ,\ i=1,2,\ldots,\frac{n}{2} \quad \text{and} \quad x_{(2i)}=x_0+(2i)h\ ,\ i=1,2,\ldots,\frac{n}{2}-1$$

$$z_{(2k-1)}=z_0+(2k-1)h\ ,\ \mathrm{k}=1,2,\ldots,\frac{n}{2} \quad \text{and} \quad z_{(2k)}=z_0+(2k)h\ ,\ \mathrm{k}=1,2,\ldots,\frac{n}{2}-1$$

$$y_j=y_0+\frac{(2j-1)}{2}h\ ,\ \mathrm{j}=1,2,\ldots,n$$

4-Aitken's delta – Squared Process

In 1926, Alexander Aitken (1985-1926) found a new approach to accelerate the sequence convergence rate. To explain this method, we suppose the sequence $\{x_n\}$ such that $\{x_n\}=\{x_1,x_2,...x_k...\}$ linearly convergence to a certain final value β, so $\beta-x_{i+1}=C_i(\beta-x_i)$, Ralston [4], such that $|C_i|<1$ and $C_i\to C$.
We can see that C_i will be approximately steady and we can write

$$\beta-x_{i+1}\approx\bar{C}\ (\beta-x_i) \qquad \ldots(5)$$

Such that $|\bar{C}|=C$

We also can see that

$$\frac{\beta-x_{i+2}}{\beta-x_{i+1}}\approx\frac{\beta-x_{i+1}}{\beta-x_i} \qquad \ldots(6)$$

i.e, $\beta\cong\dfrac{x_i x_{i+2}-x_{i+1}^2}{x_{i+2}-2x_{i+1}+x_i}=x_{i+2}-\dfrac{(\Delta x_{i+1})^2}{\Delta^2 x_i}$ …(7)

such that $\Delta x_i = (x_{i+1} - x_i)$ and $\Delta^2 x_i = x_i - 2x_{i+1} - x_{i-2}$

when using u from elements of the sequence $\{x_u\}$, we can get u-2 of another sequence {S} Approaching faster than $\{x_u\}$

$$s_{i+2} = x_{i+2} - \frac{(\Delta x_{i+1})^2}{\Delta^2 x_i} \qquad \ldots(8)$$

where $i = 1, 2, \ldots, u-2$

This process is accelerating the convergence to the final value β.

6-Examples and results:

Example (1): the integral $I = \int_2^3\int_2^3\int_2^3 \sqrt{x+y+z}\,dxdydz$ which its analytical value is 2.7370857931 (Rounded to 10 decimal places) with integrand is defined for all $(x, y, z) \in [2,3]\times[2,3]\times[2,3]$, results that listed in tables (1), where $u = 64$ the value is correct for six decimal places using SMS. Then applying $AI\,(SMS)$ method, we got a correct value for 10 decimal places with (2^{18} sub intervals).

Example (2): The integral $\int_2^3\int_1^2\int_0^1 xe^{-x-y-z}\,dxdydz$ which its analytical value is 0.0052567435 (Rounded to 10 decimal places) with integrand is defined for all $(x, y, z) \in [0,1]\times[1,2]\times[2,3]$ We can conclude from table (2), where $n = 128$ the value is correct for five decimal places and using SMS. When applying $AI(SMS)$ method, we got a correct value for 10 decimal places with (2^{18} sub intervals).

Example (3): The integral $I = \int_1^2\int_1^2\int_1^2 \ln(x+y+z)\,dxdydz$ which its analytical value is 1.49780228858 (Rounded to 10 decimal places) with integrand is defined for all $(x, y, z) \in [1,2]\times[1,2]\times[1,2]$ We can conclude from table (3), where $n = 32$ the value is correct for five decimal places and using SMS. When applying $AI(SMS)$ method, we got a correct value for 10 decimal places with (2^{18} sub intervals).

7-Conclusion

It can be seen from the tables:

When we evaluated the approximate value of triple integral with continuous integrand by using composite rule SMS gives us correct value (for several decimal places) comparing with the real value for integrals by using several sub intervals without using any acceleration method, while we got a correct value for 10 decimal places for all examples if Aitken acceleration was used.

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u	SMS values	AI(SMS)	AI(SMS)
2	2.7372130127		
4	2.7371177178		
8	2.7370937818	2.7370857529	
16	2.7370877907	2.7370857905	
32	2.7370862925	2.7370857930	2.7370857934
64	2.7370859179	2.7370857921	2.7370857931

Table (1) :-

evaluating the triple integral $I = \int_2^3\int_2^3\int_2^3 \sqrt{x+y+z}\,dxdydz = 2.7370857931$

u	SMS values	AI(SMS)	AI(SMS)	AI(SMS)
2	0.0051893459			
4	0.0051985957			
8	0.0052339626	0.0051860698		
16	0.0052465504	0.0052535066		
32	0.0052519023	0.0052558607	0.0052559458	
64	0.0052543815	0.0052565209	0.0052567783	
128	0.0052555765	0.0052566885	0.0052567434	0.0052567435

Table (2) :-

evaluating the triple integral $I = \int_2^3\int_1^2\int_0^1 xe^{-(x+y+z)}dxdydz = 0.0052567435$

u	SMS values	AI(SMS)	AI(SMS)
2	1.4983244962		
4	1.4979351308		
8	1.4978356477	1.4978015066	
16	1.4978106377	1.4978022889	
32	1.4978043765	1.4978022886	1.4978022886

Table (3) :-

evaluating the triple integral $I = \int_1^2\int_1^2\int_1^2 \ln(x+y+z)dxdydz = 1.4978022886$

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