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Special Functions and Their Applications

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Abstract *— A branch of mathematics of utmost importance to scientists and engineers concerned with real mathematical calculations is addressed in this study. The reader will find a systematic treatment here of the fundamental theory of the most important specific functions, as well as applications of theory to specific physics and engineering problems.*

Keywords— Functions, Special Functions , Gamma Function , Beta Function, Orthogonal Function, Bessel Function, Legendre Function, Hyper geometric Function.

1.Introduction

This research provides an introduction to the well-known classical special functions that play a role in mathematical physics, especially in major problems of boundary value. This branch of mathematics has a respectable history with great names, including Gauss, Euler, Fourier, Legendre, Bessel, and Riemann. All of them spent a lot of time on this topic. A good portion of their work was inspired by physics and the resulting differential equations. These activities culminated in the standard work of Whittaker and Watson about 70 years ago, A Course of Modern Analysis, which has had a great influence and is still important. As well as in applied fields such as electric current, fluid dynamics, heat conduction, wave equation, and quantum mechanics, special functions have extensive applications for more details in pure mathematics see([5],[8],[18],[21],[23]).

2.**Gamma Function** [8]

The Gamma function usually denoted by $\Gamma(s)$ is seen as a generalization of the factorial. It's was Euler (1707-1783) , a Swiss mathematician, who first worked on the curve of the function in 1729. The name and the notation (1752-1833) in 1809. This function is also called Euler Gamma function or the Euler Ian Integral of the second kind. Gamma function is defined by:

$$\Gamma(s) = \int_0^\infty e^{-x}\, x^{s-1} dx \quad \dots (1)$$

The variable s may be complex. The integral is absolutely convergent for $Re(s) > 0$.

Gamma function shows up in various applications and areas such as definite integrals, asymptotic series , Zeta function, Number Theory, Bessel function, etc….Gauss (1777-1855) gave an alternate notation called the Pi-function as:

$$\pi(s) = \Gamma(s+1)$$

Gamma function has numerous properties as following :

Property 1: $\Gamma(1) = 1$.

Property 2: $\Gamma(s+1) = s\Gamma(s) \quad , \quad s > 0$.

Property 3: If s is a positive integer, then : $\Gamma(s+1) = n!$.

Property 4: $\Gamma(s) = 2\int_0^\infty e^{-m^2} m^{2s-1} dm$.

Property 5: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Property 6: $\Gamma(s) = \int_0^1 \left(ln\left(\frac{1}{Q}\right)\right)^{s-1} dQ$.

Property 7: $\Gamma(s) = \alpha^s \int_0^\infty e^{-\alpha\kappa} \kappa^{s-1} d\kappa$.

Property 8: $\Gamma(s+1) = \int_0^\infty e^{-Q^{\frac{1}{s}}} dQ$.

Property 9: If s is zero or a negative integer, then $\Gamma(s) = \infty$.

Property 10: $\Gamma(s) = \frac{1}{s}\prod_{Z=1}^{\infty}\left[\left(1+\frac{1}{Z}\right)^{s}\left(1+\frac{s}{Z}\right)^{-1}\right]$.

2.1- Definite Integrals Related to The Gamma function [7]

In terms of the Gamma function, the class of integrals that can be expressed are very large. Our first result is the formula:

$$\int_0^\infty e^{-pm} m^{q-1} dm = \frac{\Gamma(q)}{p^q} \quad , Re\, p > 0 \quad , \quad Re\, q > 0 \qquad \dots (1)$$

Next consider the integral

$$\beta(\zeta,\eta)=\int_0^1 \mathfrak{m}^{\zeta-1}(1-\mathfrak{m})^{\eta-1}d\mathfrak{m} \quad ,Re\,\zeta>0\,, \qquad Re\,\eta>0 \qquad \dots(2)$$

As the beta function is Know. It is easy to see that (2) represents an analytical function rather than a complex variable ζ and η. If we implement the new variable of integration $\mathfrak{U}=\frac{\mathfrak{m}}{1-\mathfrak{m}}$, then (2) becomes:

$$\beta(\zeta,\eta)=\int_0^\infty \frac{\mathfrak{U}^{\zeta-1}}{(1+\mathfrak{U})^{\zeta+\eta}}d\mathfrak{U}\,, \quad Re\,\zeta>0\,,\ Re\,\eta>0 \qquad \dots(3)$$

Setting $\mathfrak{p}=1+\mathfrak{U}$, $\mathfrak{q}=\zeta+\eta$ in (1). We find that :

$$\frac{1}{(1+\mathfrak{U})\zeta+\eta}=\frac{1}{\Gamma(\zeta+\eta)}\int_0^\infty e^{-(1+\mathfrak{U})^{\mathfrak{m}}\mathfrak{m}^{\zeta+\eta-1}}d\mathfrak{m} \qquad \dots(4)$$

And substituting the overall result for the end result (3), We are obtaining

$$\beta(\zeta,\eta)=\frac{1}{\Gamma(\zeta+\eta)}\int_0^\infty e^{-\mathfrak{m}}\,\mathfrak{m}^{\zeta+\eta-1}\,d\mathfrak{m}=\frac{\Gamma(\zeta)\Gamma(\eta)}{\Gamma(\zeta+\eta)} \quad \dots(5)$$

Thus we have derived the formula

$$\beta(\zeta,\eta)=\frac{\Gamma(\zeta)\Gamma(\eta)}{\Gamma(\zeta+\eta)} \qquad \dots(6)$$

3- Bessel Functions of Nonnegative Integral Order [8],[10]

In many applied problems, one need only consider class of cylinder functions, corresponding to the case where the parameter $\mathfrak{V}$ in equation:

$$\mathfrak{U}''+\frac{1}{2}\mathfrak{U}'+\left(1-\frac{\mathfrak{V}^2}{\mathfrak{Z}^2}\right)\mathfrak{U}=0$$

Where $\mathfrak{Z}$ is a complex variable and $\mathfrak{V}$ is a parameter which can take arbitrary real or complex values, this equation called Bessel's equation of order $\mathfrak{V}$, is a nonnegative integer $\mathfrak{n}$.

One Bessel's equation solutions:

$$\mathfrak{U}''+\frac{1}{2}\mathfrak{U}'+\left(1-\frac{\mathfrak{V}^2}{\mathfrak{Z}^2}\right)\mathfrak{U}=0\,, \qquad ,n=0,1,2,\dots(3.1)$$

Is the function $\mathfrak{U}_1=\mathfrak{J}_1(\mathfrak{Z})$, known as the Bessel function of the first type of order $\mathfrak{n}$, and defined for arbitrary $\mathfrak{Z}$ by the series

$$\mathfrak{J}_{\mathfrak{n}}(\mathfrak{Z})=\sum_{\mathfrak{K}=0}^{\infty}\frac{(-1)^{\mathfrak{K}}\left(\frac{\mathfrak{Z}}{2}\right)^{\mathfrak{n}+2\mathfrak{K}}}{\mathfrak{K}!\,(\mathfrak{n}+\mathfrak{K})!} \quad , \qquad |\mathfrak{Z}|<\infty \qquad (3.2)$$

4- Legendre Function [9],[7]

The simplest class of spherical harmonics consists of the Legendre polynomials which a solutions of equation:

$$(1-\mathfrak{H}'')\mathfrak{D}''-2\mathfrak{H}\mathfrak{D}'+\left[\mathfrak{V}(\mathfrak{V}+1)-\frac{\mathfrak{M}^2}{1-\mathfrak{H}^2}\right]\mathfrak{D}=0 \qquad (4.1)$$

Where $\mathfrak{H}$ ia a complex variable and $\mathfrak{M},\mathfrak{V}$ are parameters which can take arbitrary real or complex values, for $\mathfrak{M}=0$ and nonnegative integral $\mathfrak{V}=\mathfrak{n}(\mathfrak{n}=0,1,2,\dots)$.

The next class of spherical harmonics, in the order of increasing complexity, consists of the Legendre functions, which are solutions of (4.1) for $\mathfrak{M}=0$ and arbitrary real or complex $\mathfrak{V}$, i.e., solutions of the equation :

$$(1-\mathfrak{H}'')\mathfrak{D}''-2\mathfrak{H}\mathfrak{D}'+\mathfrak{V}(\mathfrak{V}+1)\mathfrak{D}=0 \qquad (4.2)$$

Known as Legendre's Equation.

5- The Hyper geometric Function [11],[12]

The hyper geometric function $\mathfrak{F}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{x})$ is defined by the series : $\sum_{\mathfrak{n}=0}^{\infty}\frac{(\mathfrak{a})_{\mathfrak{n}}(\mathfrak{b})_{\mathfrak{n}}}{(\mathfrak{c})_{\mathfrak{n}}\mathfrak{n}!}\mathfrak{X}^{\mathfrak{n}}$

For $|\mathfrak{X}|<1$, and by continuation elsewhere

6- Physics Techniques: Spherical harmonics [13],[7]

Associated Legendre polynomials occur in physics in many instances where spherical symmetry is involved in angle terms. The colatitude angle θ is the perspective in spherical coordinates. The longitude angle ϕ is displayed in the multiplying factor. Together, they form a set of functions entitled " spherical harmonics. Such functions define the symmetry of the two-sphere under the action of the Lie group. As such, to express Legendre polynomials, the symmetries of semi-simple Lie groups and Riemannian symmetric spaces can be generalized.

In randomly oriented coordinates θ(colatitude) and ϕ (longitude), the Laplacian is

$$\nabla^2\psi = \frac{\partial\psi}{\partial\theta^2} + \cot(\theta)\frac{\partial\psi}{\partial\theta} + csc^2(\theta)\frac{\partial\psi}{\partial\phi^2}$$

When the PDE.

$$\frac{\partial\psi}{\partial\theta^2} + cot\theta\frac{\partial\psi}{\partial\theta} + csc^2\theta\frac{\partial\psi}{\partial\phi^2} + \lambda\psi = 0$$

A ϕ-dependent part $sin(m\phi)$ or $cos(m\phi)$ for integer $\mathfrak{M} \geq 0$ is solved by the separation method and an equation for the θ-dependent part

$$\frac{d^2\mathcal{Z}}{d\theta^2} + cot\theta\frac{d\mathcal{Z}}{dx} + \left[\lambda - \frac{m^2}{sin^2\theta}\right]\mathcal{Z} = 0$$

Where are the solutions $\mathcal{P}_\ell^m\left(cos(\theta)\right)$ with $\ell \geq m$ and $\lambda = \ell(\ell+1)$therefore, the equation :

$$\nabla^2\psi + \lambda\psi = 0$$

has separated nonsingular solutions only if $\lambda = \ell(\ell+1)$ And those alternative solutions are directly proportionate to $\mathcal{P}_\ell^m\left(cos(\theta)\right)cos(m\phi) \qquad 0 \leq m \leq \ell$

and $\mathcal{P}_\ell^m\left(cos(\theta)\right)sin(m\phi) \qquad 0 \leq m \leq \ell$

There are 2 ℓ +1 functions for the different values of m and the selection of sine and cosine for each choice of ℓ. When integrated over the sphere's surface, they are all orthogonal in both ℓ and m.

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