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Bipolar hyper Fuzzy AT-ideals of AT-Algebra

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Abstract: *In this paper, using the notion of bipolar-valued fuzzy set, we establish the bipolar fuzzification the notion of (strong, weak) hyper AT-ideals in hyper AT-algebras, and investigate some of their properties.*

Keywords: AT-algebra, hyper AT-algebra, bipolar hyper AT-algebra, bipolar fuzzy hyper AT-ideal, bipolar weak fuzzy hyper AT-ideal, bipolar strong fuzzy hyper AT-ideal.

1. Introduction

Prabpayak and Leerawat [12,13] introduced a new algebraic structure which is called KU-algebras. They studied ideals and congruences in KU-algebras and investigated some related properties. Jun and Xin [7] considered the fuzzification of the notion of a (weak, strong, reflexive) hyper BCK-ideal, and investigated the relations among them. Mostafa et. al. [10] applied the hyper structures to KU- algebras and introduced the concept of a hyper KU-algebra which is a generalization of a KU-algebra, and investigated some related properties. In 1956, Zadeh [14] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches. Mostafa et. al. [9] introduced the notion of fuzzy KU-ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU-ideals. Mostafa et al.[11], stated and proved more several theorems of hyper KU-algebras and studied fuzzy set theory to the hyper KU-subalgebras (ideals). Jun et. al. [8] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets. The notion of AT-algebras was introduced by Areej Tawfeeq Hameed [3,4] introduced AT-algebra, she had studied a few properties of these algebras, the notion of AT-ideals on AT-algebras was formulated and some of its properties are investigated. She introduced the notion of fuzzy AT-ideals of AT-algebras and then they investigated several basic properties which are related to fuzzy AT-ideals in [2,4]. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [-1,1]. The authors in [1], introduced bipolar-valued fuzzy set on different algebraic structures. In [3], they also introduced the notion of a hyper AT-ideal, a weak hyper AT-ideal and gave relations between hyper AT-ideals and weak hyper AT-ideals. In this paper, the bipolar fuzzy set theory to the (s-weak-strong) hyper AT-ideals in hyper AT-algebras are applied and discussed.

2.1. Preliminaries

In this section, we give some basic definitions and preliminaries lemmas of AT-ideals and fuzzy AT-ideals of AT-algebra

Definition 2.1[4]. An **AT-algebra** is a nonempty set X with a constant (0) and a binary operation ($*$) satisfying the following axioms: for all $x, y, z \in X$,

(i) $(x*y)*((y*z)*(x*z))=0$,
(ii) $0*x=x$,
(iii) $x*0=0$.

In X we can define a binary relation ($\le$) by : $x \le y$ if and only if , $y*x=0$.

In AT-algebra (X ;*, 0), the following properties are satisfied: for all $x, y, z \in X$,
(i') $(y*z)*(x*z) \le (x*y)$,
(ii') $0 \le x$.

Definition 2.2 [4]. A nonempty subset S of an AT-algebra X is called **an AT-subalgebra of AT-algebra X** if $x*y \in S$, whenever $x, y \in S$.

Definition 2.3 [2,4]. A nonempty subset I of an AT-algebra X is called **an AT-ideal of AT-algebra X** if it satisfies the following conditions: for all $x, y, z \in X$;
AT_1) $0 \in I$; AT_2) $x*(y*z) \in I$ and $y \in I$ imply $x*z \in I$.

Definition 2.4 [2,4]. Let X be an AT-algebra. A fuzzy set μ in X is called **a fuzzy AT-subalgebra of X** if , for all $x, y \in X$, $\mu(x*y) \ge \min\{\mu(x), \mu(y)\}$.

Definition 2.5 [2,4]. Let X be an AT-algebra. A fuzzy set μ in X is called **a fuzzy AT-ideal of X** if it satisfies the following conditions: for all x, y and $z \in X$,
(AT_1) $\mu(0) \ge \mu(x)$. (AT_2) $\mu(x*z) \ge \min\{\mu(x*(y*z)), \mu(y)\}$.

Remark 2.6 [1]. Let X be the universe of discourse. A bipolar-valued fuzzy set μ in **X** is an object having the form $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) | x \in X\}$, where $\mu_\Phi^P : X \to [0, 1]$ and $\mu_\Phi^N : X \to [-1, 0]$ are mappings. The positive membership degree $\mu_\Phi^P(x)$ denoted the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy $\Phi = \{ (x, \mu_\Phi^P(x), \mu_\Phi^N(x)) | x \in X\}$, and the negative membership degree $\mu_\Phi^N(x)$ denotes the satisfaction degree of x to some implicit counter-property of $\Phi = \{ (x, \mu_\Phi^P(x), \mu_\Phi^N(x)) | x \in X\}$.

For the sake of simplicity, we shall use the symbol $\Phi = (X: \mu_\Phi^P, \mu_\Phi^N)$, for the bipolar-valued fuzzy set $\Phi = \{ (x, \mu_\Phi^P(x), \mu_\Phi^N(x)) | x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

Definition 2.7 [1]. A bipolar fuzzy set $\Phi = (X: \mu_\Phi^P, \mu_\Phi^N)$ in X is called **a bipolar fuzzy AT-subalgebra of X** if it satisfies the following properties: for any x, y ∈ X,

(1) $\mu_\Phi^P(x * y) \geq \min\{\mu_\Phi^P(x), \mu_\Phi^P(y)\}$, and (2) $\mu_\Phi^N(x * y) \leq \max\{\mu_\Phi^N(x), \mu_\Phi^N(y)\}$.

Remark 2.8 [1]. If $\Phi = (X: \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy AT-subalgebra of X, then $\mu_\Phi^P(0) \geq \mu_\Phi^P(x)$ and $\mu_\Phi^N(0) \leq \mu_\Phi^N(x)$, for all x ∈ X.

Indeed, for all x ∈ X, $\mu_\Phi^P(0) = \mu_\Phi^P(x * x) \geq \min\{\mu_\Phi^P(x), \mu_\Phi^P(x)\} = \mu_\Phi^P(x)$, and $\mu_\Phi^N(0) = \mu_\Phi^N(x * x) \leq \max\{\mu_\Phi^N(x), \mu_\Phi^N(x)\} = \mu_\Phi^N(x)$.

Definition 2.9 [1]. A bipolar fuzzy set $\Phi = (X: \mu_\Phi^P, \mu_\Phi^N)$ in X is called **a bipolar fuzzy AT-ideal of X** if it satisfies the following properties: for any x, y, z ∈ X,

(1) $\mu_\Phi^P(0) \geq \mu_\Phi^P(x)$, and $\mu_\Phi^N(0) \leq \mu_\Phi^N(x)$, (2) $\mu_\Phi^P(x * z) \geq \min\{\mu_\Phi^P(x * (y * z)), \mu_\Phi^P(y)\}$, and

(3) $\mu_\Phi^N(x * z) \leq \max\{\mu_\Phi^N(x * (y * z)), \mu_\Phi^N(y)\}$.

Definition 2.10 [1]. A bipolar fuzzy set $\Phi = (X: \mu_\Phi^P, \mu_\Phi^N)$ in X is called **a bipolar strongly fuzzy AT-ideal of X** if it satisfies the following properties: for any x, y, z ∈ X,

(1) $\mu_\Phi^P(0) \geq \mu_\Phi^P(x)$, and $\mu_\Phi^N(0) \leq \mu_\Phi^N(x)$, (2) $\mu_\Phi^P(z) \geq \min\{\mu_\Phi^P((x * y) * (x * z)), \mu_\Phi^P(y)\}$, and

(3) $\mu_\Phi^N(z) \leq \max\{\mu_\Phi^N((x * y) * (x * z)), \mu_\Phi^N(y)\}$.

Remark 2.11 [3]. Some properties of hyper AT-algebra are discussed. Let H be a nonempty set and $P^*(H) = P(H) \setminus \{\phi\}$ the family of the nonempty subsets of H. A multi valued operation (said also hyper operation) " ∘ " on H is a function, which associates with every pair $(x, y) \in H \times H = H^2$ a nonempty subset of H denoted $x \circ y$. An algebraic hyper structure or simply a hyper structure is a nonempty set H endowed with one or more hyper operations.

Definition 2.12 [3]. Let H be a nonempty set and " ∘ " a hyper operation on H, such that $\circ : H \times H \to P^*(H)$. Then H is called a hyper AT-algebra, if it contains a constant "0 " and satisfies the following axioms: for all $x, y, z \in H$

(HAT_1) $((y \circ z) \circ (x \circ z)) << x \circ y$, (HAT_2) $0 \circ x = \{x\}$, (HAT_3) $x \circ 0 = \{0\}$.

where x << y is defined by $0 \in y \circ x$ and for every $A, B \subseteq H$, $A << B$ is defined by $\forall a \in A, \exists b \in B$ $such\ that$ $a << b$. In such case, we call "<<" the hyper order in H.

We shall use the $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$ or $\{x\} \circ \{y\}$.

Note that:- if $A, B \subseteq H$, then by $A \circ B$ we mean the subset $\bigcup_{a \in A, b \in B} a \circ b$ of H.

Definition 2.13 [3]. Let S be a non-empty subset of a hyper AT-algebra (*H*; °,0). Then S is said to be a **hyper AT-subalgebra** of H if $S_2 : x \circ y << S, \forall x, y \in S$

Proposition 2.14 [3]. Let S be a non-empty subset of a hyper AT-algebra (*H*; °,0). If $x \circ y << S, \forall x, y \in S$, then $0 \in S$.

Definition 2.15 [3]. Let A be a nonempty subset of a hyper AT-algebra (*H*; ° ,0). Then A is said to be a **hyper AT-ideal** of H if

(HI_1) $0 \in A$; (HI_2) $x \circ (y \circ z) << A$ and $y \in A$ imply $x \circ z \in A$, for all $x, y, z \in H$.

Definition 2.16[3]. Let A be a nonempty subset of a hyper AT-algebra (*H*; ° ,0). Then A is said to be a **hyper ideal** of H if (HI_1) $0 \in A$; (HI_2) $x \circ y << A$ and $x \in A$ imply $y \in A$, for all $x, y \in H$.

Definition 2.17 [3]. Let *I* be a nonempty subset of hyper AT-algebra $(H, \circ, 0)$ and $0 \in I$. Then for all $x, y, z \in H$.

(1) I is said to be a **weak hyper AT-ideal** of H if $x \circ (y \circ z) \subseteq I$ and $y \in I$ imply $x \circ z \in I$,

(2) I is said a **strong hyper AT-ideal** of H if $x \circ (y \circ z) \cap I \neq \phi$ and $y \in I$ imply $x \circ z \in I$.

Definition 2.18 [3]. Let I be a nonempty subset of hyper AT-algebra $(H,\circ,0)$ and $0\in I$. Then for all $x,y\in H$.

(1) I is said to be a **weak hyper ideal** of H if $x\circ y\subseteq I$ and $x\in I$ imply $y\in I$,

(2) I is said a **strong hyper ideal** of H if $(x\circ y)\cap I\neq\phi$ and $x\in I$ imply $y\in I$.

Theorem 2.19 [3]. Every (strong, weak) hyper AT-ideal is a (strong, weak) hyper ideal.

Definition 2.20 [5]. A fuzzy set μ in hyper AT-algebra (H; °,0) is said to be a **fuzzy hyper AT-subalgebra** of H if it satisfies the inequality: $\inf_{z\in x\circ y}\mu(z)\geq\min\{\mu(x),\mu(y)\},\ \forall x,y,z\in H$.

Definition 2.21 [5]. Let $(H,\circ,0)$ be a hyper AT-algebra, the map $\mu: H\to[0,1]$ is a fuzzy subset and $0\in I$. Then μ is said to be a **fuzzy hyper AT-ideal** of H, if for all $x,y,z,u\in H$, $x<<y$ *implies* $\mu(y)\geq\mu(x)$ and
$\mu(x\circ z)\geq\min\{\inf_{u\in(x\circ(y\circ z))}\mu(u),\mu(y)\}$.

Definition 2.22 [2]. Let μ be a fuzzy set in X. For any t,s ∈ [0, 1], the set Φ_t^P = U(μ;t) = {x ∈ X | μ (x) ≥ t} iscalled**anuppert-levelsubsetof μ.** The set Φ_s^N = L(μ; t) = {x ∈ X | μ(x) ≤ s} is called **a lower t-level subset of μ**.

3. Bipolar Fuzzy hyper AT-subalgebras (AT-ideals)

Now some fuzzy logic concepts are reviewed. A fuzzy set μ in a set H is a function μ: $H\to[0,1]$. A fuzzy set μ in a set H is said to satisfy the inf (resp. sup) property if for any subset T of H there exists $x_0\in T$ such that $\mu(x_0)=inf_{x\in T}\,\mu(x)$ (resp. $\mu(x_0)=sup_{x\in T}\,\mu(x)$).

Definition 3.1. A bipolar fuzzy set $\Phi=(H:\mu_\Phi^P,\mu_\Phi^N)$ in X is called **a bipolar fuzzy hyper AT-subalgebra of** H if it satisfies the following properties: for any x, y ∈ H,

(1) $inf_{z\in x^\circ y}\,\mu_\Phi^P(z)\geq\min\{\mu_\Phi^P(x),\mu_\Phi^P(y)\}$, and (2) $sup_{z\in x^\circ y}\,\mu_\Phi^N(z)\leq\max\{\mu_\Phi^N(x),\mu_\Phi^N(y)\}$.

Proposition 3.2. If $\Phi=(H:\mu_\Phi^P,\mu_\Phi^N)$ is a bipolar hyper fuzzy AT-subalgebra of H, then $\mu_\Phi^P(0)\geq\mu_\Phi^P(x)$ and $\mu_\Phi^N(0)\leq\mu_\Phi^N(x)$, for all x ∈ H.

Example 3.3. Consider an AT-algebra H = {0, 1, 2, 3} with the following Cayley table:

°	0	1	2	3
0	{0}	{1}	{2}	{3}
1	{0}	{0}	{0}	{3}
2	{0}	{1}	{0}	{3}
3	{0}	{1}	{0}	{0}

Then $(H;{}^\circ,0)$ is a hyper AT-algebra. Define $\mu_\Phi^N: H\to[-1,0]$ and $\mu_\Phi^P: H\to[0,1]$ are mappings by

H	0	1	2	3
μ_Φ^N	-0.7	-0.7	-0.6	-0.4
μ_Φ^P	0.6	0.5	0.3	0.3

By routine calculations, we know that $\Phi=(H:\mu_\Phi^P,\mu_\Phi^N)$ is a bipolar hyper fuzzy AT-subalgebra in X.

Definition 3.4. A bipolar fuzzy set $\Phi=(H:\mu_\Phi^P,\mu_\Phi^N)$ in H is called **a bipolar hyper fuzzy AT-ideal of X** if it satisfies the following properties: for any x, y, z ∈ H,

(1) $x\ll y$ implies $\mu_\Phi^P(x)\geq\mu_\Phi^P(y)$, and $\mu_\Phi^N(x)\leq\mu_\Phi^N(y)$, (2) $\mu_\Phi^P(x^\circ z)\geq\min\{inf_{u\in(x^\circ(y^\circ z))}\,\mu_\Phi^P(u),\mu_\Phi^P(y)\}$, and (3) $\mu_\Phi^N(x^\circ z)\leq\max\{sup_{w\in(x^\circ(y^\circ z))}\mu_\Phi^N(w),\mu_\Phi^N(y)\}$.

Definition 3.5. A bipolar fuzzy set $\Phi=(H:\mu_\Phi^P,\mu_\Phi^N)$ in H is called **a bipolar weak hyper fuzzy AT-ideal of** H if it satisfies the following properties: for any x, y, z ∈ H,

(1) $x\ll y$ implies $\mu_\Phi^P(x)\geq\mu_\Phi^P(y)$, and $\mu_\Phi^N(x)\leq\mu_\Phi^N(y)$, (2) $\mu_\Phi^P(0)\geq\mu_\Phi^P(x^\circ z)\geq\min\{inf_{u\in(x^\circ(y^\circ z))}\,\mu_\Phi^P(u),\mu_\Phi^P(y)\}$, and (3) $\mu_\Phi^N(0)\leq\mu_\Phi^N(x^\circ z)\leq\max\{sup_{w\in(x^\circ(y^\circ z))}\mu_\Phi^N(w),\mu_\Phi^N(y)\}$.

Definition 3.6. A bipolar fuzzy set $\Phi = (H: \mu_\Phi^P, \mu_\Phi^N)$ in H is called **a bipolar strong hyper fuzzy AT-ideal of** H if it satisfies the following properties: for any x, y, z ∈ H,

(1) $x \ll y$ implies μ_Φ^P (x) ≥ μ_Φ^P (y), and μ_Φ^N (x) ≤ μ_Φ^N (y),

(2) $inf_{u\in(x^\circ(y^\circ z))}\mu_\Phi^P$ (u) ≥ μ_Φ^P (x°z) ≥ min {$sup_{u\in(x^\circ(y^\circ z))}$ μ_Φ^P (u), μ_Φ^P (y)}, and

(3) $sup_{w\in(x^\circ(y^\circ z))}\mu_\Phi^N$ (w) ≤ μ_Φ^N (x°z) ≤ max {$inf_{w\in(x^\circ(y^\circ z))}\mu_\Phi^N$ (w), μ_Φ^N (y)}.

Definition 3.7. A bipolar fuzzy set $\Phi = (X: \mu_\Phi^P, \mu_\Phi^N)$ in X is called **a bipolar hyper fuzzy ideal of X** if it satisfies the following properties: for any x, y, z ∈ H,

(1) $x \ll y$ implies μ_Φ^P (x) ≥ μ_Φ^P (y), and μ_Φ^N (x) ≤ μ_Φ^N (y),

(2) μ_Φ^P (y) ≥ min {$inf_{u\in(x^\circ y)}$ μ_Φ^P (u), μ_Φ^P (x)}, and (3) μ_Φ^N (y) ≤ max {$sup_{w\in(x^\circ y)}\mu_\Phi^N$ (w), μ_Φ^N (x)}.

Definition 3.8. A bipolar fuzzy set $\Phi = (H: \mu_\Phi^P, \mu_\Phi^N)$ in H is called **a bipolar weak hyper fuzzy ideal of** H if it satisfies the following properties: for any x, y, z ∈ H,

(1) $x \ll y$ implies μ_Φ^P (x) ≥ μ_Φ^P (y), and μ_Φ^N (x) ≤ μ_Φ^N (y),

(2) μ_Φ^P (0) ≥ μ_Φ^P (y) ≥ min {$inf_{u\in(x^\circ y)}$ μ_Φ^P (u), μ_Φ^P (x)}, and (3) μ_Φ^N (0) ≤ μ_Φ^N (y) ≤ max {$sup_{w\in(x^\circ y)}\mu_\Phi^N$ (w), μ_Φ^N (x)}.

Definition 3.9. A bipolar fuzzy set $\Phi = (H: \mu_\Phi^P, \mu_\Phi^N)$ in H is called **a bipolar strong hyper fuzzy ideal of** H if it satisfies the following properties: for any x, y, z ∈ H,

(1) $x \ll y$ implies μ_Φ^P (x) ≥ μ_Φ^P (y), and μ_Φ^N (x) ≤ μ_Φ^N (y),

(2) $inf_{u\in(x^\circ y)}\mu_\Phi^P$ (u) ≥ μ_Φ^P (y) ≥ min {$sup_{u\in(x^\circ y)}$ μ_Φ^P (u), μ_Φ^P (x)}, and

(3) $sup_{w\in(x^\circ y)}\mu_\Phi^N$ (w) ≤ μ_Φ^N (y) ≤ max {$inf_{w\in(x^\circ y)}$ μ_Φ^N (w), μ_Φ^N (x)}.

Example 3.10. Consider an AT-algebra H = {0, 1, 2} with the following Cayley table:

°	0	1	2
0	{0}	{1}	{2}
1	{0}	{0}	{1,2}
2	{0}	{0,1}	{0}

Then $(H; \circ, 0)$is a hyper AT-algebra. Define μ_Φ^N and μ_Φ^P are mappings by

X	0	1	2
μ_Φ^N	-0.7	-0.7	-0.6
μ_Φ^P	1	0.5	0

By routine calculations, we know that $\Phi = (H: \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar (weak) hyper fuzzy AT-ideal in H .

Example 3.11. Consider an AT-algebra H = {0, 1, 2} with the following Cayley table:

°	0	1	2
0	{0}	{1}	{2}
1	{0}	{0}	{1,2}
2	{0}	{0,1}	{0}

Then $(H; \circ, 0)$ is a hyper AT-algebra. Define $\mu_\Phi^N$and μ_Φ^P are mappings by

H	0	1	2
μ_Φ^N	-0.8	-0.6	-0.2
μ_Φ^P	0.9	0.5	0.3

By routine calculations, we know that $\Phi = (H: \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar strong hyper fuzzy AT-ideal in H .

Theorem 3.12. Any bipolar (weak, strong) hyper fuzzy AT-ideal is a bipolar (weak, strong) hyper fuzzy ideal.

Proof :

Let $\Phi = (H: \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar weak hyper fuzzy AT-ideal of H, we get for any x, y, z ∈ H,

(1) μ_Φ^P (0) ≥ μ_Φ^P (x°z) ≥ min {$inf_{z\in(x^\circ(y^\circ z))}$ μ_Φ^P (u), μ_Φ^P (y)}, and

(2) μ_Φ^N (0) $\leq$ μ_Φ^N (x°z) ≤ max $\{sup_{w\in(x^\circ(y^\circ z))}\mu_\Phi^N$ (w), μ_Φ^N (y)}.

Put x = 0 in (1) and (2), we get

μ_Φ^P (0) $\geq$ μ_Φ^P (0°z) ≥ min $\{inf_{u\in(0^\circ(y^\circ z))}$ μ_Φ^P (u), μ_Φ^P (y)},⟹ μ_Φ^P (0) $\geq$ μ_Φ^P (z) ≥ min $\{inf_{u\in(y^\circ z)}$ μ_Φ^P (u), μ_Φ^P (y)}, and

μ_Φ^N (0) $\leq$ μ_Φ^N (0°z) ≤ max $\{sup_{w\in(0^\circ(y^\circ z))}\mu_\Phi^N$ (w), μ_Φ^N (y)} ⟹μ_Φ^N (0) $\leq$ μ_Φ^N (z) ≤ max $\{sup_{w\in(y^\circ z)}\mu_\Phi^N$ (w), μ_Φ^N (y)} .

Similarly, we can prove that , every bipolar strong hyper fuzzy AT-ideal of H is bipolar strong hyper fuzzy ideal of H. Ending the proof.

Theorem 3.13. Let $\Phi = (H{:}\,\mu_\Phi^P,\mu_\Phi^N)$ be bipolar strong hyper fuzzy AT-ideal of H and for any x, y, z ∈ H, then

i. μ_Φ^P (0) ≥ μ_Φ^P (x), and μ_Φ^N (0) ≤ μ_Φ^N (x),

ii. $x \ll y$ implies μ_Φ^P (x) ≥ μ_Φ^P (y), and μ_Φ^N (x) ≤ μ_Φ^N (y),

iii. μ_Φ^P (x°z) ≥ min { μ_Φ^P (u), μ_Φ^P (y)}, ∀ u ∈(x°(y°z)) , and
μ_Φ^N (x°z) ≤ max {μ_Φ^N (w), μ_Φ^N (y)}, ∀ w ∈(x°(y°z)) .

Proof.

(i) Since 0∈x ° x, ∀x∈H ,we have μ_Φ^P (0) $\geq \{inf_{u\in(x^\circ x)}$ μ_Φ^P (u), μ_Φ^P (x)}, μ_Φ^N (0) $\leq \{sup_{u\in(x^\circ x)}$ μ_Φ^N (u), μ_Φ^N (x)}.

(ii) Let x, y∈H be such that ,we have $x \ll y$. Then 0∈x ° y, ∀ x, y∈H and so
$sup_{u\in(x^\circ y)}$ μ_Φ^P (u) ≥ μ_Φ^P (0), $inf_{w\in(x^\circ y)}$ μ_Φ^N (w) ≤ μ_Φ^N (0). It follows from (i) that
μ_Φ^P (0°y) $=$ μ_Φ^P (y) ≥ min $\{sup_{u\in(x^\circ y)}$ μ_Φ^P (u), μ_Φ^P (x)} ≥ min { μ_Φ^P (0), μ_Φ^P (x)} = μ_Φ^P(x). And
μ_Φ^N (0°y) $=$ μ_Φ^N (y) ≤ max$\{inf_{u\in(x^\circ y)}$ μ_Φ^N (u), μ_Φ^N (x)} ≤ max { μ_Φ^N (0), μ_Φ^N (x)} = μ_Φ^N(x).

(iii) Let x, y, z ∈H be such that , we get μ_Φ^P (x°z) ≥ min $\{sup_{u\in(x^\circ(y^\circ z))}$ μ_Φ^P (u), μ_Φ^P (y)} ≥ min { μ_Φ^P (u), μ_Φ^P (y)}, ∀ u ∈(x°(y°z)).

And μ_Φ^N $(x^\circ z)$ ≤ max$\{inf_{w\in(x^\circ(y^\circ z))}$ μ_Φ^N (w), μ_Φ^N (y)} ≤ max { μ_Φ^N (w), μ_Φ^N (y)}, ∀w∈(x°(y°z)). We conclude that (iii) is true.

Note that, in a finite hyper AT-algebra, every bipolar fuzzy set satisfies inf-sup property. Hence the concept of bipolar fuzzy weak hyper AT-ideals coincide in a finite hyper AT-algebra.

Proposition 3.14. Let $\Phi = (H{:}\,\mu_\Phi^P,\mu_\Phi^N)$ be a bipolar fuzzy hyper AT-ideal of H, then:
μ_Φ^P (0) ≥ μ_Φ^P (x), and μ_Φ^N (0) ≤ μ_Φ^N (x). If $\Phi = (H{:}\,\mu_\Phi^P,\mu_\Phi^N)$ satisfies the inf-sup property, then μ_Φ^P (x°z) ≥ min { μ_Φ^P (u), μ_Φ^P (y)}, ∀ u ∈(x°(y°z)) , and μ_Φ^N (x°z) ≤ max {μ_Φ^N (w), μ_Φ^N (y)}, ∀ w ∈(x°(y°z)).

Proof.

Since 0<<x, ∀x∈H , it follows from Definition (3.4), μ_Φ^P (0) ≥ μ_Φ^P (x), and μ_Φ^N (0) ≤ μ_Φ^N (x)؛

Since$\Phi = (H{:}\,\mu_\Phi^P,\mu_\Phi^N)$ satisfies the inf-sup property there exists u_0, w_0∈ (x°(y°z)) such that μ_Φ^P (u_0) = $inf_{u\in(x^\circ(y^\circ z))}$ μ_Φ^P (u), and μ_Φ^N (w_0) $=$ $sup_{w\in(x^\circ(y^\circ z))}$ μ_Φ^N (w). Hence
μ_Φ^P (x°z) ≥ min $\{inf_{u\in(x^\circ(y^\circ z))}$ μ_Φ^P (u), μ_Φ^P (y)} ≥ min { μ_Φ^P (u_0), μ_Φ^P (y)}, and
μ_Φ^N $(x^\circ z)$ ≤ max$\{sup_{w\in(x^\circ(y^\circ z))}$ μ_Φ^N (w), μ_Φ^N (y)} ≤ max { μ_Φ^N (w_0), μ_Φ^N (y)}.

Corollary 3.15. Every bipolar hyper fuzzy AT-ideal is a bipolar weak hyper fuzzy AT-ideal.

Proposition 3.16 . Let $\Phi = (H;\mu_\Phi^P,\mu_\Phi^N)$ be a bipolar fuzzy set, then $\Phi = (H;\mu_\Phi^P,\mu_\Phi^N)$ is bipolar weak hyper fuzzy AT-ideal of H, if and only if, the positive level set Φ_t^P and negative level set Φ_s^N , for every (α,β) ∈[0,1]×[-1,0] are weak hyper AT-ideals of H.

Proof.

Assume that $\Phi = (H;\mu_\Phi^P,\mu_\Phi^N)$ is bipolar weak hyper fuzzy AT-ideal of H and $\Phi_t^P \neq \Phi \neq \Phi_s^N$, for every (α,β) ∈[0,1]×[-1,0]. It clear from

μ_Φ^P (0) $\geq$ μ_Φ^P (x°z) ≥ min { $inf_{u\in(x^\circ(y^\circ z))}\mu_\Phi^P$ (u), μ_Φ^P (y)},

μ_Φ^N (0) ≤ μ_Φ^N (x°z) ≤ max $\{sup_{w\in(x^\circ(y^\circ z))}\mu_\Phi^N$ (w), μ_Φ^N (y)}.

That 0∈ $\Phi_t^P \cap \Phi_s^N$, Let x, y, z∈H be such that (x°(y°z)) ⊆ Φ_t^P,y∈ Φ_t^P. Then for any u ∈ (x°(y°z)), u ∈ Φ_t^P. It follows that $\mu_\Phi^P(u) \geq \alpha$ so that $inf_{u\in(x^\circ(y^\circ z))}$ $\mu_\Phi^P(u) \geq \alpha$, thus μ_Φ^P(x°z) ≥ min{ $inf_{u\in(x^\circ(y^\circ z))}$ $\mu_\Phi^P(u)$,$\mu_\Phi^P(y)\} \geq \alpha$ and so x°z ∈ Φ_t^P, there for Φ_t^P is weak hyper AT-ideal of H .

Now, Let x, y, z∈H be such that (x°(y°z)) ⊆ Φ_s^N,y∈ Φ_s^N. Then for any w∈ (x°(y°z)), w ∈ Φ_s^N. It follows that $\mu_\Phi^N(u) \leq \beta$ so that $sup_{w\in(x^\circ(y^\circ z))}$ $\mu_\Phi^N(w) \leq \beta$, thus μ_Φ^N(x°z) ≤ max{ $sup_{w\in(x^\circ(y^\circ z))}$ $\mu_\Phi^P(w)$,$\mu_\Phi^P(y)\} \leq \beta$ and so x°z ∈ Φ_s^N, there for Φ_s^N is weak hyper AT-ideal of H .

Conversely, suppose that the nonempty positive and negative level sets Φ_t^P, Φ_s^N are is weak hyper AT-ideals of H for every (α,β)∈[0,1]×[-1,0]. Let $\mu_\Phi^P(x) = \alpha$, $\mu_\Phi^N(x) = \beta$, for x ∈H, then by 0 ∈Φ_t^P, 0 ∈Φ_s^N . It follows that $\mu_\Phi^P(x) \geq \alpha$, $\mu_\Phi^N(x) \leq \beta$ and so μ_Φ^P (0) ≥ μ_Φ^P (x), and μ_Φ^N (0) ≤ μ_Φ^N (x). Now, let min { $inf_{u\in(x^\circ(y^\circ z))}\mu_\Phi^P$ (u), μ_Φ^P (y)} = α, max $\{sup_{w\in(x^\circ(y^\circ z))}\mu_\Phi^N$ (w), μ_Φ^N (y)} = β.

Theorem 3.17 . Let $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set, then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is bipolar hyper fuzzy AT-ideal of *H*, if and only if, the positive level set Φ_t^P and negative level set Φ_s^N , for every (α,β) ∈[0,1]×[-1,0] are hyper AT-ideals of *H*.

Proof.

Assume that $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is bipolar hyper fuzzy AT-ideal of H and $\Phi_t^P \neq \Phi \neq \Phi_s^N$, for every (α,β) ∈[0,1]×[-1,0]. It clear that 0∈ $\Phi_t^P \cap \Phi_s^N$.

Let x, y, z∈H be such that (x°(y°z)) ⊆ Φ_t^P,y∈ Φ_t^P. Then for any u ∈ (x°(y°z)), u ∈ Φ_t^P. It follows that $\mu_\Phi^P(u) \geq \alpha$ so that $inf_{u\in(x^\circ(y^\circ z))}\ \mu_\Phi^P(u) \geq \alpha$, thus μ_Φ^P(x°z) ≥ min{ $inf_{u\in(x^\circ(y^\circ z))}\ \mu_\Phi^P(u)\ , \mu_\Phi^P(y)\} \geq \alpha$ and so x°z ∈ Φ_t^P, there for Φ_t^P is hyper AT-ideal of H .

Now, Let x, y, z∈H be such that (x°(y°z)) ⊆ Φ_s^N,y∈ Φ_s^N. Then for any w∈ (x°(y°z)), w ∈ Φ_s^N. It follows that $\mu_\Phi^N(u) \leq \beta$ so that $sup_{w\in(x^\circ(y^\circ z))}\ \mu_\Phi^N(w) \leq \beta$, thus μ_Φ^N(x°z) ≤ max{ $sup_{w\in(x^\circ(y^\circ z))}\ \mu_\Phi^N(w)\ , \mu_\Phi^N(y)\} \leq \beta$ and so x°z ∈ Φ_s^N, there for Φ_s^N is hyper AT-ideal of H .

Conversely, suppose that the nonempty positive and negative level sets Φ_t^P, Φ_s^N are is hyper AT-ideals of *H* for every (α,β)∈[0,1]×[-1,0]. Let $\mu_\Phi^P(x) = \alpha,\ \mu_\Phi^N(x) = \beta$, for x ∈*H*, then by 0 ∈Φ_t^P, 0 ∈Φ_s^N . It follows that $\mu_\Phi^P(x) \geq \alpha,\ \mu_\Phi^N(x) \leq \beta$ and so μ_Φ^P (0) ≥ μ_Φ^P (x), and μ_Φ^N (0) ≤ μ_Φ^N (x). Now, let min { $inf_{u\in(x^\circ(y^\circ z))}\mu_\Phi^P$ (u), μ_Φ^P (y)} = α, max {$sup_{w\in(x^\circ(y^\circ z))}\mu_\Phi^N$ (w), μ_Φ^N (y)} = β.

Note that, in a finite hyper AT-algebra, every bipolar fuzzy set satisfies inf -sup property. Hence the concept of bipolar weak hyper fuzzy AT-ideals and bipolar strong hyper fuzzy AT-ideals coincide in a finite hyper AT-algebra.

Corollary 3.18. Every bipolar strong hyper fuzzy AT-ideal is bipolar hyper fuzzy AT-ideal.

Proposition 3.19. Let $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ be bipolar hyper fuzzy AT-ideal of *H* and let *x*, *y*, *z* ∈ *H* . Then

(i) μ_Φ^P (0) ≥ μ_Φ^P (x), and μ_Φ^N (0) ≤ μ_Φ^N (x).

(ii) If $\Phi = (H: \mu_\Phi^P, \mu_\Phi^N)$ satisfies inf -sup property, then , μ_Φ^P (x°z) ≥ min { μ_Φ^P (u), μ_Φ^P (y)}, for some u ∈(x°(y°z)) , and μ_Φ^N (x°z) ≤ max {μ_Φ^N (w), μ_Φ^N (y)}, for some w ∈(x°(y°z)).

Proof.

(i) Since $0 \ll x$, ∀x∈H ,we have μ_Φ^P (0) ≥ μ_Φ^P (x), μ_Φ^N (0) ≤ μ_Φ^N (x), by Definition (3.4(1)) and hence (i) holds.

(ii) Since $\Phi = (H: \mu_\Phi^P, \mu_\Phi^N)$ satisfies the inf -sup property, there is exists u_0, w_0∈ (x°(y°z)) such that μ_Φ^P (u_0) = $inf_{u\in(x^\circ(y^\circ z))}\ \mu_\Phi^P$ (u), and μ_Φ^N (w_0) = $sup_{w\in(x^\circ(y^\circ z))}\ \mu_\Phi^N$ (w). Hence μ_Φ^P (x°z) ≥ min {$inf_{u\in(x^\circ(y^\circ z))}\ \mu_\Phi^P$ (u), μ_Φ^P (y)} ≥ min { μ_Φ^P (u_0), μ_Φ^P (y)}, and μ_Φ^N $(x^\circ z)$ ≤ max{$sup_{w\in(x^\circ(y^\circ z))}\ \mu_\Phi^N$ (w), μ_Φ^N (y)} ≤ max { μ_Φ^N (w_0), μ_Φ^N (y)}, which implies that (ii) is true. The proof is complete.

Corollary 3.20. Every bipolar hyper fuzzy AT-ideal of *H* is bipolar weak hyper fuzzy AT-ideal of *H*.

The following example shows that the converse of Corollary (3.18) and (3.20) may not be true.

Example 3.21.

(1) Consider the hyper AT-algebra *H* = {0, 1, 2} with the following table:

°	0	1	2
0	{0}	{1}	{2}
1	{0}	{0}	{1,2}
2	{0}	{0,1}	{0}

Then $(H; \circ, 0)$ is a hyper AT-algebra. Define μ_Φ^N and μ_Φ^P are mappings by

H	0	1	2
μ_Φ^N	-0.7	-0.7	-0.6
μ_Φ^P	1	0.5	0

By routine calculations, we know that $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar hyper fuzzy AT-ideal in *H* and hence it is also bipolar weak hyper fuzzy AT-ideal in *H*. But $\Phi = (H: \mu_\Phi^P, \mu_\Phi^N)$ is not bipolar strong hyper fuzzy AT-ideal in *H* since min {$sup_{u\in(0^\circ(1^\circ 2))}\ \mu_\Phi^P$ (u), μ_Φ^P (1)} ≥ min { μ_Φ^P (1), μ_Φ^P (1)}= 0.5 ≥ 0 = $\mu_\Phi^P(2)$, for all u∈(0°(1°2)).

(2) Consider the hyper AT-algebra *H* = {0, 1, 2} with the following Cayley table:

°	0	1	2
0	{0}	{1}	{2}
1	{0}	{0}	{1,2}
2	{0}	{0,1}	{0}

Then $(H;\circ,0)$ is a hyper AT-algebra. Define μ_Φ^N and μ_Φ^P are mappings by

H	0	1	2
μ_Φ^N	-0.7	-0.7	-0.6
μ_Φ^P	1	0	0.5

Then $\Phi = (H;\mu_\Phi^P,\mu_\Phi^N)$ is a bipolar weak hyper fuzzy AT-ideal in H , but it is not bipolar hyper fuzzy AT-ideal in H since $1 \ll 2$, but μ_Φ^P (1)$\ngeq$ μ_Φ^P (2).

Theorem 3.22. Let $\Phi = (H;\mu_\Phi^P,\mu_\Phi^N)$ be a bipolar fuzzy set. If $\Phi = (H;\mu_\Phi^P,\mu_\Phi^N)$ is bipolar strong hyper fuzzy AT-ideal of H, then the set the positive level set Φ_t^P and negative level set Φ_s^N , for every t ∈[0,1] and s ∈ [-1,0] are strong hyper AT-ideals of H.

Proof.

Assume that $\Phi = (H;\mu_\Phi^P,\mu_\Phi^N)$ is bipolar strong hyper fuzzy AT-ideal of H and $\Phi_t^P \neq \Phi \neq \Phi_s^N$, for every t ∈[0,1] and s ∈ [-1,0]. Let x∈H, by Theorem (3.13 (i)), μ_Φ^P (0) ≥ μ_Φ^P (x), and μ_Φ^N (0) ≤ μ_Φ^N (x). It clear that 0∈ Φ_t^P and 0∈ Φ_s^N.

Let x, y, z∈H be such that $(x\circ(y\circ z)) \cap \Phi_t^P \neq \Phi$, and y∈ Φ_t^P, also, $(x\circ(y\circ z)) \cap \Phi_s^N \neq \Phi$, and y∈ Φ_s^N. Then there exists u ∈$(x\circ(y\circ z)) \cap \Phi_t^P$, μ_Φ^P (u) ≥ t , also w ∈$(x\circ(y\circ z)) \cap \Phi_s^N$, μ_Φ^N (w) ≤ s. By Definition (3.6), we have
μ_Φ^P (x°z) ≥ min {$sup_{u\in(x\circ(y\circ z))}$ μ_Φ^P (u), μ_Φ^P (y)} ≥ min { μ_Φ^P (u), μ_Φ^P (y)}≥ min {t,t} = t, and
μ_Φ^N (x°z) ≤ max {$inf_{w\in(x\circ(y\circ z))}\mu_\Phi^N$ (w), μ_Φ^N (y)} ≤ max { μ_Φ^N (w), μ_Φ^N (y)}≤ max{s,s} =s.

So x°z ∈ Φ_t^P, and x°z ∈ Φ_s^N, there for Φ_t^P and Φ_s^N are strong hyper AT-ideals of H .

Theorem 3.23. Let $\Phi = (H;\mu_\Phi^P,\mu_\Phi^N)$ be a bipolar fuzzy set of H satisfies the inf -sup property. If the set the positive level set Φ_t^P and negative level set Φ_s^N , for every t ∈[0,1] and s ∈ [-1,0] are nonempty strong hyper AT-ideals of H, then $\Phi = (H;\mu_\Phi^P,\mu_\Phi^N)$ is bipolar strong hyper fuzzy AT-ideal of H.

Proof.

Assume that Φ_t^P and Φ_s^N are nonempty strong hyper AT-ideals of H, for every t ∈[0,1] and s ∈ [-1,0]. Then there x∈ μ_Φ^P and x∈ μ_Φ^N such that x°x ≪ x. Using Proposition (2.14), we have x°x ⊆ μ_Φ^P and x°x ⊆ μ_Φ^N. Thus for a, b ∈ x°x, we have a ∈μ_Φ^P and b ∈μ_Φ^N , hence μ_Φ^P (a) ≥ t and μ_Φ^N (b) ≤ s. It follows that
$inf_{a\in(x\circ(y\circ z))}\mu_\Phi^P$ (a) ≥ t = μ_Φ^P (x) and $sup_{b\in(x\circ(y\circ z))}\mu_\Phi^N$ (b) ≤ s = μ_Φ^N (x).

Moreover, let x, y, z∈H and such that α' = min {$sup_{a\in(x\circ(y\circ z))}$ μ_Φ^P (a), μ_Φ^P (y)}, β' = max {$inf_{b\in(x\circ(y\circ z))}\mu_\Phi^N$ (b), μ_Φ^N (y)}.

By hypothesis Φ_t^P and Φ_s^N are strong hyper AT-ideals of H and $\Phi = (H;\mu_\Phi^P,\mu_\Phi^N)$ satisfies the inf -sup property, there u, w ∈ $(x\circ(y\circ z))$ such that μ_Φ^P (u) =$sup_{a\in(x\circ(y\circ z))}$ μ_Φ^P (a), μ_Φ^N (w) =$inf_{b\in(x\circ(y\circ z))}\mu_\Phi^N$ (b). Thus
μ_Φ^P (u) =$sup_{a\in(x\circ(y\circ z))}$ μ_Φ^P (a) ≥ min {$sup_{a\in(x\circ(y\circ z))}$ μ_Φ^P (a), μ_Φ^P (y)}= α' , and
μ_Φ^N (w) =$inf_{b\in(x\circ(y\circ z))}\mu_\Phi^N$ (b) ≤ max {$inf_{b\in(x\circ(y\circ z))}\mu_\Phi^N$ (b), μ_Φ^N (y)}= β'.

This shows that u∈ μ_Φ^P and w∈ μ_Φ^N, and u∈ $(x\circ(y\circ z)) \cap \mu_\Phi^P$ and w∈ $(x\circ(y\circ z)) \cap \mu_\Phi^N$, hence $(x\circ(y\circ z)) \cap \mu_\Phi^P \neq \Phi$ and $(x\circ(y\circ z)) \cap \mu_\Phi^N \neq \Phi$.

Combining y∈ μ_Φ^P and y∈ μ_Φ^N, and noticing that any bipolar (weak, strong)hyper fuzzy AT-ideal is a bipolar (weak, strong)hyper fuzzy AT-ideal, we get x°z ∈ μ_Φ^P and x°z ∈ μ_Φ^N. Hence μ_Φ^P (x°z) ≥ min {$sup_{z\in(x\circ(y\circ z))}$ μ_Φ^P (u), μ_Φ^P (y)}, and μ_Φ^N (x°z) ≤ max {$inf_{w\in(x\circ(y\circ z))}\mu_\Phi^N$ (w), μ_Φ^N (y)}. Therefore, $\Phi = (H;\mu_\Phi^P,\mu_\Phi^N)$ is bipolar strong hyper fuzzy AT-ideal of H.

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