α-translation Fuzzy q-ideals of KK-algebra

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Abstract. *In this paper, we introduce the notion of fuzzy translation, fuzzy extensions and fuzzy α-translation of fuzzy KK-subalgebra and fuzzy ideal on KK-algebras and investigate some of their properties.*

Keywords: KK-algebra, fuzzy KK-subalgebra, fuzzy ideal, α-translation fuzzy KK-subalgebra, α-translation fuzzy q-ideal, fuzzy extension.
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1. Introduction

A BCK-algebra is an important class of logical algebras introduced by K. Is´eki [17] and was extensively investigated by several researchers. The class of all BCK-algebras is a quasivariety. K. Is´eki posed an interesting problem whether the class of BCK-algebras is a variety. In ([21,22]), C. Prabpayak and U. Leerawat the notion of KU-algebras. They gave the concept of homomorphism of KU-algebras and investigated some related properties. The concept of fuzzy subset and various operations on it were first introduced by L.A. Zadeh in [25], then fuzzy subsets have been applied to diverse field. The study of fuzzy subsets and their application to mathematical contexts has reached to what is now commonly called fuzzy mathematics. Fuzzy algebra is an important branch of fuzzy mathematics. They introduced of the concept of fuzzy subgroups in 1971 by A. Rosenfeld [23]. Since these ideas have been applied to other algebraic structures such as semi-groups, rings, ideals, modules and vector spaces. O.G. Xi, ([24]) applied this concept to BCK-algebra, and he introduced the notion of fuzzy sub-algebras (ideals) of the BCK-algebras with respect to minimum, and since then Y. B. Jun and E. H. Roh ([18]), studied fuzzy ideals and moreover several fuzzy structures in BCK-algebras are considered. S.M. Mostafa and et al [20] have introduced the notion of KUS-algebras, KUS-ideals, KUS-subalgebras and investigates the relations among them. S. Asawasamrit and A. Sudprasert [1-4] have introduced the notion of KK-algebras, ideals, KK-subalgebras and studied the relations among them and gave the concept of homomorphism of KK-algebras and investigated some related properties. A.T. Hameed and et al [6] have introduced the notion of KK-algebras, ideals, KK-subalgebras and investigates the relations among them. In this paper, we define a α-translation fuzzy KK-subalgebras and a α-translation fuzzy q-ideals of KK-algebras and look for some of their properties accurately by using the concepts of fuzzy KK-subalgebra and fuzzy q-ideal. We prove to that if μ and δ are fuzzy ideals of KK-algebras X, then $\mu^M_\gamma \times \delta^M_\alpha$ is a fuzzy q-ideal of X×X. Also, we show that if $\mu^M_\gamma \times \delta^M_\alpha$ is a fuzzy q-ideal of X×X, either μ or δ is a fuzzy q-ideal of X.

2. Preliminaries

Now, we give some definitions and preliminary results needed in the later sections.

Definition 2.1([1,2]).

Let $(X; *, 0)$ be an algebra with a binary operation $*$ and a nullary operation 0. Then X is called a **KK-algebra** if it satisfies the following: for all $, y, z \in X$,

(KK$_1$) : $(x * y) * ((y * z) * (x * z)) = 0$;

(KK$_2$) : $0 * x = x$;

(KK$_3$) : $x * y = 0$ *and* $y * x = 0$ if and only if $x = y$.

Definition 2.2 ([1,2]).

Define a binary relation ≤ on KK-algebra $(X; *, 0)$ by letting

$x \leq y$ if and only if $y * x = 0$.

Examples 2.3([1,2]).

Let X = {0, 1} and let ∗ be defined by:

∗	0	1
0	0	1
1	1	0

Then $(X; *, 0)$ is a KK-algebra.

Proposition 2.4 ([3,4]).

In any KK-algebra (X; ∗,0), the following properties hold: for all $x, y, z \in X$

(P$_1$) $x * ((x * y) * y) = 0$;

(P$_2$) $x * x = 0$;

(P$_3$) $x * (y * z) = y * (x * z)$;

(P_4) $((x * y) * y) * y = x * y$;
(P_5) $(x * y) * 0 = (x * 0) * (y * 0)$;
(P_6) $(x * y) * ((z * x) * (z * y)) = 0$;
(P_7) $If\ x \leq y,\ then\ \ y * z \leq x * z$;
(P_8) $If\ x \leq y,\ then\ \ z * x \leq z * y$.

Definition 2.5([1,2]).

Let $(X; *, 0)$ be a KK-algebra and let S be a nonempty subset of X. S is called a **KK-subalgebra of** X if $x * y \in S$ whenever $x \in S\ and\ y \in S$.

Definition 2.6([3,4]).

A nonempty subset I of a KK-algebra $(X; *, 0)$ is called **an ideal of X** if it satisfies the following conditions: for any $x, y \in X$,

(I_1) $0 \in I$;
(I_2) $x * y \in I\ and\ x \in I\ \ imply\ \ y \in I$.

Examples 2.7 ([3,4]).

Let X = {0, 1, 2, 3} and let $*$ be defined by the table

*	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	3	3	0	0
3	3	2	1	0

Thus, it can be easily shown that $(X; *, 0)$ is a KK-algebra. And we see that I = {0, 1} and J = {0, 3} are ideals of X.

Proposition 2.8 ([6]). Every ideal of KK-algebra is a KK-subalgebra.

Proposition 2.9 ([1,2]). Let $\{I_i \mid i \in \Lambda\}$ be a family of ideals of KK-algebra X. The intersection of any set of ideals of KK-algebra X is also an ideal.

Definition 2.10 ([8]). Let $(X ; *, 0)$ and $(Y; *`, 0`)$ be nonempty sets. The mapping $f: (X; *, 0) \rightarrow (Y; *`, 0`)$ is called **a homomorphism** if it satisfies:
$f\ (x * y) = f\ (x) *` f\ (y)$, for all $x , y \in X$. The set $\{x \in X \mid f\ (x) = 0'\}$ is called **the kernel of** f denoted by ker f.

Theorem 2.11 ([1,2]). Let $f: (X; *, 0) \rightarrow (Y; *`, 0`)$ be a homomorphism of a KK-algebra X into a KK-algebra Y, then :

A. f (0) = 0'.
B. f is injective if and only if ker f ={0}.
C. x ≤ y implies f (x) ≤ f (y).

Theorem 2.12 ([1,2]). Let $f: (X; *, 0) \rightarrow (Y; *`, 0`)$ be a homomorphism of a KK-algebra X into a KK-algebra Y, then:

(F_1) If S is a KK-subalgebra of X, then f (S) is a KK-subalgebra of Y , where f is onto.
(F_2) If I is ideal of X, then f (I) is ideal of Y , where f is onto.
(F_3) If H is a KK-subalgebra of Y, then f^{-1} (H) is a KK-subalgebra of X.
(F_4) If J is ideal of Y, then f^{-1} (J) is ideal of X .
(F_5) ker f is ideal of X.
(F_6) Im(f) is a KK-subalgebra of Y.

Definition 2.13([25]). Let $(X; *, 0)$ be a nonempty set, a fuzzy subset μ of X is a function $\mu: X \rightarrow [0,1]$.

Definition 2.14 ([25]). Let X be a nonempty set and μ be a fuzzy subset of $(X; *, 0)$, for $t \in [0,1]$, the set $\mu_t = \{ x \in X \mid \mu(x) \geq t\}$ is called **a level subset of μ** .

Definition 2.15([1-4]). Let $(X; *, 0)$be a KK-algebra , a fuzzy subset μ of X is called a **fuzzy KK-subalgebra of X** if for all $x, y \in X,\ \mu\,(x * y) \geq\ min\,\{\mu\,(x), \mu\,(y)\}$.

Definition 2.16([1-4]). Let $(X; *, 0)$ be a KK-algebra, a fuzzy subset μ of X is called **a fuzzy ideal of X** if it satisfies the following conditions, for all $x, y \in X$,

(FKK_1) $\mu\,(0) \geq \mu\,(x)$,

(FKK2) $\mu(y) \geq min\{\mu(x * y), \mu(x)\}$.

Example 2.17([17]).

Let X ={0, 1, 2, 3} in which ($*$) is defined by the following table:

$*$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	3	3	0	0
3	3	2	1	0

Then $(X; *, 0)$ is a KK-algebra. Define a fuzzy subset $\mu: X \to [0,1]$ by

$$\mu(x) = \begin{cases} 0.7 & if \quad x \in \{0,1\} \\ 0.3 & otherwise \end{cases}$$

I = {0, 1} is ideal of X. Routine calculation gives that μ is a fuzzy ideal of KK-algebras X.

Lemma 2.18([1-4]). Let μ be a fuzzy ideal of KK-algebra $(X; *, 0)$and if $x \leq y$, then $\mu(x) \geq \mu(y)$, for all $x, y \in X$.

Proposition 2.19([1-4]).

1- Let μ be a fuzzy subset of KK-algebra $(X; *, 0)$. μ is a fuzzy KK-subalgebra of X if and only if for every t ∈ [0,1], μ_t is a KK-subalgebra of X .

2- Let μ be a fuzzy subset of KK-algebra $(X;*,0)$, μ is a fuzzy ideal of X if and only if for every t ∈ [0,1], μ_t is an ideal of X .

3- Let A be a nonempty subset of a KK-algebra $(X; *, 0)$ and μ be a fuzzy subset of X such that μ is into {0, 1}, so that μ is the characteristic function of A. Then μ is a fuzzy ideal of X if and only if A is an ideal of .

Proposition 2.20([1-4]).

1- The intersection of any set of fuzzy ideals of KK-algebra is also fuzzy ideal.

2- The union of any set of fuzzy ideals of KK-algebra is also fuzzy ideal where is chain.

Proposition 2.21([1-4]). Every fuzzy ideal of KK-algebra is a fuzzy KK-subalgebra.

Definition 2.22 ([22]). Let $f: (X; *, 0) \to (Y; *`, 0`)$be a mapping nonempty sets X and Y respectively. If μ is a fuzzy subset of X, then the fuzzy subset β of Y defined by: $f(\mu)(y) = \begin{cases} sup\{\mu(x): x \in f^{-1}(y)\} & if \ f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ 0 & otherwise \end{cases}$

is said to be **the image of μ under f.**

Similarly if β is a fuzzy subset of , then the fuzzy subset $\mu = (\beta \circ f)$ of X (i.e the fuzzy subset defined by $\mu(x) = \beta(f(x))$, for all $x \in X$) is called **the pre-image of β under f** .

Definition 2.23 ([22]). A fuzzy subset μ of a set X has sup property if for any subset T of X , there exist t_0 ∈T such that $\mu(t0) = sup\{\mu(t)/t \in T\}$.

Proposition 2.24([5,6]). Let $f: (X; *, 0) \to (Y; *`, 0`)$ be a homomorphism between KK-algebras X and Y respectively.

1- For every fuzzy KK-subalgebra β of Y, $f^{-1}(\beta)$ is a fuzzy KK-subalgebra of X.

2- For every fuzzy KK-subalgebra μ of X , $f(\mu)$ is a fuzzy KK-subalgebra of Y, where f is onto.

3- For every fuzzy ideal β of Y, $f^{-1}(\beta)$ is a fuzzy ideal of X.

4- For every fuzzy ideal μ of X with sup property, $f(\mu)$ is a fuzzy ideal of Y, where f is onto.

Definition 2.25[1,18]:

Let X be a nonempty set and μ be a fuzzy subset of X and let $\alpha \in [0, T]$. A mapping $\mu_\alpha^T : X \to [0,1]$ is called a **α-translation fuzzy subset of μ** if it satisfies:

$\mu_\alpha^T(x) = \mu(x) + \alpha$, for all $x \in X$, where$T = 1 - \sup\{\mu(x): x \in X\}$.

Definition 2.26([6]). Let $(X; *, 0)$ be a KK-algebra , a fuzzy subset μ *in* X is called **a fuzzy q-ideal of X** if it satisfies the following conditions: , for all $x, y, z \in X$,

(1) $\mu(0) \geq \mu(x)$,

(2) $\mu(x * z) \geq min\{\mu((x * y) * z), \mu(y)\}$.

Lemma 2.27([6]). Let μ be a fuzzy q-ideal of KK-algebra $(X; *, 0)$ and if $x \leq y$, then μ (x) ≥ μ (y), for all x, y∈ X.

Theorem 2.28([6]). Let A be a nonempty subset of a KK-algebra $(X; *, 0)$ and μ be a fuzzy subset of X such that μ is into {0, 1} , so that μ is the characteristic function of A . Then μ is a fuzzy q-ideal in X if and only if , A is a q-ideal of X .

Theorem 2.29([6]). Let μ be a fuzzy subset of KK-algebra $(X; *, 0)$. μ is a fuzzy q-ideal of X if and only if , for every t ∈[0,1] , μ_t is a q-ideal of X .

Proposition 2.30([6]). Every fuzzy q-ideal of KK-algebra $(X; *, 0)$ is a fuzzy ideal of X.

Theorem 2.31([6]). Let A be a q-ideal of KK-algebra $(X; *, 0)$. Then for any fixed number (t) in an open interval (0,1), there exists a fuzzy q-ideal μ of X such that $\mu_t = A$.

Theorem 2.32([6]).

A homomorphic pre-image of a fuzzy q-ideal is also a fuzzy q-ideal.

Theorem 2.33([6]).

Let f : (X; ∗,0) →(Y; ∗',0') be a homomorphism between KK-algebras X and Y respectively. For every fuzzy q-ideal μ of X with sup property, f (μ) is a fuzzy q-ideal of Y.

3. Fuzzy α-translations KK-subalgebras of KK-algebra:

In this section, we discuss α-translation on KK-algebras and we get some of relations, theorems, propositions and give examples of α-translation of fuzzy KK-subalgebra. We show the notion of α-translation fuzzy KK-subalgebras of KK-algebra and investigate some of their properties.

In what follows, let $(X;*,0)$ denote a KK-algebra, and for any fuzzy subset μ of X, we denote $T = 1 - sup\{\mu(x) \mid x \in X\}$.

Definition 3.1. Let μ be a fuzzy subset of a KK-algebra $(X;*,0)$ and let $\in [0,T]$. A mapping $\mu_\alpha^T : X \to [0,1]$ is called an **α-translation fuzzy subset of μ** if it satisfies: $\mu_{\alpha'}^T(x) = \mu(x) + \alpha$, for all $x \in X$.

Definition 3.2. Let $(X;*,0)$ be a KK-algebra, a fuzzy subset μ of X is called a **α-translation fuzzy KK-subalgebra** of X , if for all $x, y \in X$, $\mu_\alpha^T(x * y) \geq \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}$.

Example 3.3. Let $X = \{0,1,2,3\}$ in which * be a defined by the following table:

∗	0	1	2	3
0	0	1	2	3
1	0	0	0	0
2	0	0	0	0
3	0	1	1	0

Then $(X;*,0)$ is a KK-algebra. It is easy to show that $S_1=\{0,1\}$, $S_2=\{0,2\}$, $S_3=\{0,3\}$ and $S_4=\{0,1,2,3\}$ are KK-subalgebras of X .

Define a fuzzy subset $\mu: X \to [0,1]$ such that $\mu(0) = t_1$, $\mu(1) = \mu(2) = \mu(3) = t_2$, where $t_1, t_2 \in [0, 1]$ and $t_1 > t_2$.

Routine calculation gives that μ is a fuzzy KK-subalgebra of .

Theorem 3.4. Let μ be a fuzzy KK-subalgebra of KK-algebra $(X;*,0)$ and α ∈ [0,T]. Then μ_α^T is a fuzzy KK-subalgebra of .

Proof:

Assume that μ is a fuzzy KK-subalgebra of X, and α ∈ [0,T]. Let $, y \in X$, then

$\mu(x * y) \geq min\{\mu(x), \mu(y)\}$. Thus

$$\mu(x * y) + \alpha \geq min\{\mu(x), \mu(y)\} + \alpha ,$$
$$= min\{\mu(x) + \alpha, \mu(y) + \alpha\} \ \ and\ so$$
$$\mu(x * y) + \alpha = \mu_\alpha^T(x * y) \geq min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}$$

Hence μ_α^T is a fuzzy KK-subalgebra of X.□

Theorem 3.5. Let μ be a fuzzy subset of KK-algebra $(X;*,0)$ such that μ_α^T of μ is a fuzzy KK-subalgebra of X, for some α ∈ [0,T]. Then μ is a fuzzy KK-subalgebra of X .

Proof.

Assume that μ_α^T is a fuzzy KK-subalgebra of X, for some α ∈ [0,T]. Let $x, y \in X$, then

$$\mu(x * y) + \alpha = \mu_\alpha^T(x * y) \geq \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}$$
$$= \min\{\mu(x) + \alpha, \mu(y) + \alpha\} = min\{\mu(x), \mu(y)\} + \alpha \quad \text{and} \quad so\ \mu(x * y) \geq min\{\mu(x), \mu(y)\}.$$

Hence μ is a fuzzy KK-subalgebra of X. □

Definition 3.6. For a fuzzy subset μ of a KK-algebra $(X;*,0)$, α ∈ [0,T] and t∈ Im(μ) with t ≥ α, let $U_\alpha(\mu; t) := \{x \in X \mid \mu(x) \geq t - \alpha\}$.

Remark 3.7. If μ is a fuzzy KK-subalgebra of KK-algebra $(X;*,0)$, then it is that $U_\alpha(\mu; t)$ is a KK-subalgebra of X, for all t ∈ Im(μ) with t ≥ α.

Let $x, y \in U_\alpha(\mu; t)$, then $\mu(x) \geq t - \alpha$, and $\mu(y) \geq t - \alpha$, then

$min\{\mu(x), \mu(y)\} \geq t - \alpha$, since μ is a fuzzy KK-subalgebra, then

$\mu(x * y) \geq min\{\mu(x), \mu(y)\} \geq t - \alpha$, therefore $x * y \in U_\alpha(\mu; t)\}$.

But if we do not give a condition that μ is a fuzzy KK-subalgebra of X, then $U_\alpha(\mu; t)$ is not a KK-subalgebra of X as seen in the following example.

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Example 3.8. Let $X = \{0,1,2,3\}$ in which * be a defined by the following table:

$*$	0	1	2	3
0	0	1	2	3
1	0	0	3	0
2	0	0	0	0
3	0	3	3	0

Then $(X;*,0)$ is a KK-algebra. Define a fuzzy subset λ of X:

X	0	1	2	3
λ	0.7	0.6	0.4	0.3

Then λ is not a fuzzy KK-subalgebra of X.
Since $\lambda(1*2) = 0.3 < 0.4 = \min\{\lambda(1), \lambda(2)\}$. For $\alpha = 0.2$ and $t = 0.6$, we obtain $U_\alpha(\lambda; t) = \{0, 1, 2\}$ which is not KK-subalgebra of X, since $1_*2 = 3 \notin U_\alpha(\lambda; t)$.

Proposition 3.9. Let μ be a fuzzy subset of a KK-algebra $(X;*,0)$ and $\alpha \in [0,T]$. μ_α^T is a fuzzy KK-subalgebra of X if and only if, $U_\alpha(\mu; t)$ is a KK-subalgebra of X, for all $t \in \mathrm{Im}(\mu)$ with $t \geq \alpha$.

Proof:
Necessity is clear {assume that μ_α^T is a fuzzy KK-subalgebra by Theorem (3.5), then μ is a fuzzy KK-subalgebra, by Remark (3.7), then $U_\alpha(\mu; t)$ is a KK-subalgebra}.
To prove the conversely, assume that $x, y \in U_\alpha(\mu; t)$ and μ_α^T of μ is not a fuzzy KK-subalgebra of X, therefore $\mu_\alpha^T(x * y) < t \leq \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}$. Then $\mu(x) \geq t - \alpha$ and $\mu(y) \geq t - \alpha$, but $\mu(x * y) < t - \alpha$. This shows that $x * y \notin U_\alpha(\mu; t)$. This is a contradiction, and so $\mu_\alpha^T(x * y) \geq \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}$, for all $x, y \in X$.
Hence μ_α^T is a fuzzy KK-subalgebra of X.□

Proposition 3.10. Let $f: (X;*,0) \to (Y;*`, 0`)$be an epimorphism between KK-algebras X and Y respectively. For every α-translation fuzzy KK-subalgebra μ_α^T of X, $f(\mu_\alpha^T)$ is a α-translation fuzzy KK-subalgebra of Y.

Proof:
By Definition (2.21), $\lambda_\alpha^T(y') = f(\mu_\alpha^T)(y') = sup_{x \in f^{-1}(y')} \mu(x) + \alpha$, for all $y' \in Y$ (sup ∅ =0).
By Theorem (2.12(1)). Hence $f(\mu_\alpha^T)$ is a fuzzy KK-subalgebra of Y.□

Proposition 3.11. An homomorphic pre-image of a α-translation fuzzy KK-subalgebra of KK-algebra is also a α-translation fuzzy KK-subalgebra of KK-algebra.

Proof:
Let $f: (X;*,0) \to (Y;*`, 0`)$be a homomorphism between KK-algebras X and Y respectively. λ the α-translation fuzzy KK-subalgebra of Y and μ the pre-image of λ under f, then $\mu_\alpha^T(x) = \lambda_\alpha^T(f(x))$, for all $x \in X$.
By Theorem (2.12(3)). Hence $\mu_\alpha^T$is a fuzzy KK-subalgebra of X. □

Definition 3.12. Let $(X;*,0)$ be a KK-algebra, μ_1 and μ_2 be fuzzy subsets of X. Then μ_2 is called **a fuzzy extension** of μ_1. If $\mu_2(x) \geq \mu_1(x)$, for all $x \in X$.

Definition 3.13. Let $(X;*,0)$ be a KK-algebra, μ_1 and μ_2 be fuzzy subsets of X. Then μ_2 is called **a fuzzy S-extension** of μ_1 if the following assertions are valid:
(S_i) $\mu_2$is a fuzzy extension of μ_1.
(S_{ii}) If μ_1 is a fuzzy KK-subalgebra of X, then μ_2 is a fuzzy KK-subalgebra of X.

Proposition 3.14. Let μ be a fuzzy KK-subalgebra of a KK-algebra X and $\alpha \in [0,T]$. Then the α-translation fuzzy subset μ_α^T of μ is a fuzzy S-extension of μ.

Proof:
Since $\mu_\alpha^T(x) = \mu(x) + \alpha \geq \mu(x)$, then $\mu_\alpha^T(x)$ is a fuzzy extension of $\mu(x)$, for all $x \in X$and since μ is a fuzzy KK-subalgebra of X, then μ_α^T of μ is a fuzzy KK-subalgebra by Theorem (3.4).□
In general, the converse of Proposition (3.14) is not true as seen in the following example.

Example 3.15. Let X={0, 1, 2, 3} be an KK-algebra which is given in Example (3.3).
Define a fuzzy KK-subalgebra μ of X by:

X	0	1	2	3
μ	0.8	0.5	0.7	0.5

Then μ is a fuzzy KK-subalgebra of X. Let μ_α^T be fuzzy subsets of X where α =0.1 given by:

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X	0	1	2	3
μ_α^T	0.9	0.6	0.8	0.6

Then μ_α^T is a fuzzy S-extension of μ, but the μ is not a fuzzy S-extension of μ_α^T.

Proposition 3.16. The intersection of fuzzy S-extensions of a fuzzy subset μ of X is a fuzzy S-extension of μ.

Proof:

Let $\{\mu_i | i\in\Lambda\}$ be a family of fuzzy S-extensions of μ of KK-algebra $(X;*,0)$, then for any $x, y\in X$, $i\in\Lambda$,
$(\cap_{i\in\Lambda}\mu_i)(x*y) = \inf(\mu_i(x*y)) \geq \inf(\min\{\mu_i(x), \mu_i(y)\})$
$= \min\{\inf(\mu_i(x)), \inf(\mu_i(y)\} = \min\{(\cap_{i\in\Lambda}\mu_i)(x), (\cap_{i\in\Lambda}\mu_i)(y)\}$. □

Remark 3.17. The union of fuzzy S-extensions of a fuzzy subset μ of X, is not a fuzzy S-extension of μ as seen in the following example.

Example 3.18. Let X = {0, 1, 2, 3} be a KK-algebra which is given in Example (3.3). Define a fuzzy subset μ of X by:

X	0	1	2	3
μ	0.8	0.5	0.6	0.5

Then $\mu$is a fuzzy KK-subalgebra of X. Let ν and δ be fuzzy subsets of X given by:

X	0	1	2	3
ν	0.8	0.6	0.6	0.7
δ	0.9	0.5	0.8	0.5
$\nu\cup\delta$	0.9	0.6	0.8	0.7

Then ν and δ are fuzzy S-extension KK-subalgebras of μ. But the union $\nu\cup\delta$ is not a fuzzy S-extension of μ since $(\nu\cup\delta)(3*2) = 0.6 < 0.7 = \min\{(\nu\cup\delta)(3), (\nu\cup\delta)(2)\}$.

Proposition 3.19. Let μ be a fuzzy KK-subalgebra of KK-algebra $(X;*,0)$ and α, $\lambda\in[0,T]$. If $\alpha\geq\lambda$, then the α-translation fuzzy KK-subalgebra μ_α^T of μ is a fuzzy S-extension of the λ-translation fuzzy KK-subalgebra μ_λ^T of μ.

Proof:

For every $x\in X$and $\alpha, \lambda\in[0,T]$ and $\alpha\geq\lambda$,we have
$\mu_\alpha^T(x) = \mu(x)+\alpha \geq \mu(x)+\lambda = \mu_\lambda^T(x)$, then $\mu_\alpha^T(x) \geq \mu_\lambda^T(x)$, therefore $\mu_\alpha^T(x)$ is a fuzzy extension of $\mu_\lambda^T(x)$. Since μ is a fuzzy KK-subalgebra of X, then μ_α^T and μ_λ^T of μ are a fuzzy KK-subalgebras by Theorem (3.4).

Hence μ_α^T of μ is a fuzzy S-extension of the translation fuzzy KK-subalgebra μ_λ^T of μ. □

Proposition 3.20. Let μ be a fuzzy KK-subalgebra of an KK-algebra X and $\lambda\in[0,T]$. For every fuzzy S-extension ν of the translation fuzzy KK-subalgebra μ_λ^T of μ, there exists $\alpha\in[0,T]$ such that $\alpha\geq\lambda$ and ν is a fuzzy S-extension of the translation fuzzy KK-subalgebra μ_α^T of μ.

Proof:

Since μ is a fuzzy KK-subalgebra of a KK-algebra $(X;*,0)$ and $\lambda\in[0,T]$, the λ-translation fuzzy subset μ_λ^T of μ is a fuzzy KK-subalgebra of X. If ν is a fuzzy S-extension of μ_λ^T, then there exists $\alpha\in[0,T]$ such that $\alpha\geq\lambda$ and $\nu(x)\geq\mu_\alpha^T(x)$, for all $x\in X$, hence ν is a fuzzy S-extension of the translation fuzzy KK-subalgebra μ_α^T of μ. □

The following example illustrates Proposition (3.20).

Example 3.21.

Let X = {0, 1, 2, 3 } be a KK-algebra which is given in Example (3.3). Define a fuzzy subset μ of X by:

X	0	1	2	3
μ	0.7	0.5	0.4	0.4

Then μ is a fuzzy KK-subalgebra of X and T=0.3. If we take λ = 0.2, then the λ-translation fuzzy KK-subalgebra μ_λ^T of μ is given by :

X	0	1	2	3
μ_λ^T	0.9	0.7	0.6	0.6

Let ν be a fuzzy subset of X defined by:

X	0	1	2	3

ν	0.94	0.76	0.64	0.64

Then ν is clearly a fuzzy KK-subalgebra of X which is fuzzy extension of μ_λ^T and hence ν is a fuzzy S-extension. λ-translation fuzzy subset μ_λ^T of μ. Take $\alpha = 0.23$, then α = 0.23 > 0.2 = λ, and the α-translation fuzzy KK-subalgebra μ_α^T of μ .
Note that ν(x) ≥ μ_α^T (x), for all $x \in X$, and hence ν is a fuzzy S-extension of the α-translation fuzzy KK-subalgebra μ_α^T of μ.

4. α-translation Fuzzy q-ideals of KK-algebra

In this section, we shall define the notion of α-translation of fuzzy q-ideals, and we study some of the relations, theorems, propositions and examples of α-translation of fuzzy q-ideals of KK-algebra.

Definition 4.1[6]. Let $(X;*,0)$ be a KK-algebra and I be a nonempty subset of X. Then I is called **an q-ideal of X** if it satisfies:

i. 0∈I,

ii. $(x*y)*z$∈I and y∈I imply $x*z$∈I, for all $x, y, z \in X$.

Example 4.2. Let X ={0 ,a, b, c} in which ($*$) be defined by the following table:

∗	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	0	0	0
c	0	a	a	0

Then $(X;*,0)$ is a KK-algebra. It is easy to show that I_1 ={0,a} and I_2 ={0, a, b, c} are q-ideals of X .

Definition 4.3. Let $(X;*,0)$ be a KK-algebra, a α-translation fuzzy subset μ of X is called a **α-translation fuzzy q-ideal of X** if it satisfies the following conditions: for all $x, y, z \in X$,

(FKK_1) $\mu_\alpha^T(0) \geq \mu_\alpha^T(x)$,

(FKK_2) $\mu_\alpha^T(x*z) \geq min\{\mu_\alpha^T((x*y)*z), \mu_\alpha^T(y)\}$.

Example 4.4. Let $X = \{0,1,2,3\}$ in which * be a defined by the following table:

∗	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	0	0	0
3	0	1	1	0

Then $(X;*,0)$ is a KK-algebra.

1) Define a fuzzy subset $\mu : X \to [0,1]$ by $\mu(x) = \begin{cases} 0.7 & if\ x \in \{0,1\} \\ 0.3 & otherwise \end{cases}$

I = {0, 1} is an q-ideal of X. Routine calculation gives that μ is a fuzzy q-ideal of X.

2) Define a fuzzy subset μ: $X \to [0,1]$ such that μ(0) = t_1 , μ(1) = μ(2) = μ(3) = t_2 , where t_1, t_2∈[0, 1] and $t_1 > t_2$.
Routine calculation gives that μ is a fuzzy q-ideal of .

Proposition 4.5. Let μ is a fuzzy q-ideal of a KK-algebra $(X;*,0)$, then μ_α^T is a fuzzy q-ideal of X, for all α ∈ [0, T].

Proof :

Assume that μ is a fuzzy q-ideal of X and let α ∈ [0, T]. Then for all $x, y, z \in X$.

1-

2- $\mu_\alpha^T(x*z) = \mu(x*z) + \alpha \geq min\{\mu((x*y)*z), \mu(y)\} + \alpha$
$= min\{\mu((x*y)*z) + \alpha, \mu(y) + \alpha\}$
$= min\{\mu_\alpha^T((x*y)*z), \mu_\alpha^T(y)\}$.

Hence μ_α^T is a fuzzy q-ideal of X . □

Proposition 4.6. Let μ be a fuzzy subset of KK-algebra $(X;*,0)$ such that μ_α^T is a fuzzy q-ideal of X , for some α ∈ [0,T]. Then μ is a fuzzy q-ideal of X.

Proof :

Assume that μ_α^T is a α-translation fuzzy q-ideal of X for some α∈ [0, T].

Let $x, y, z \in X$, we have $\mu(0) + \alpha = \mu_\alpha^T(0) \geq \mu_\alpha^T(x) = \mu(x) + \alpha$
and so $\mu(0) \geq \mu(x)$.

$\mu(x * z) + \alpha = \mu_\alpha^T(x * z) \geq min\{\mu_\alpha^T((x * y) * z), \mu_\alpha^T(y)\}$,
$= min\{\mu((x * y) * z) + \alpha, \mu(y) + \alpha\}$,
$= min\{\mu((x * y) * z), \mu(y)\} + \alpha$, then
$\mu(x * z) \geq min\{\mu((x * y) * z), \mu(y)\}$.

Hence μ is a fuzzy q-ideal of X. □

Theorem 4.7. For $\alpha \in [0,T]$, let μ_α^T be the α-translation fuzzy subset μ of KK-algebra $(X;*,0)$. Then μ_α^T is a fuzzy q-ideal of X if and only if

$\forall\, t \in Im(\mu), t > \alpha \Rightarrow U_\alpha(\mu; t)$ is an q-ideal of X.

Proof:

Assume that μ_α^T is a α-translation fuzzy q-ideal of X and let $t \in Im(\mu)$ be such that $t > \alpha$.

Since $\mu_\alpha^T(0) \geq \mu_\alpha^T(x)$, for all $x \in X$, we have
$\mu(0) + \alpha = \mu_\alpha^T(0) \geq \mu_\alpha^T(x) = \mu(x) + \alpha \geq t$, for all $x \in U_\alpha(\mu; t)$.

Hence $0 \in U_\alpha(\mu; t)$.

Let $x, y, z \in X$, such that $((x * y) * z) \in U_\alpha(\mu; t)$ and $y \in U_\alpha(\mu; t)$. Then $\mu((x * y) * z) \geq t - \alpha$ and $\mu(y) \geq t - \alpha$, i.e.,
$\mu_\alpha^T((x * y) * z) = \mu((x * y) * z) + \alpha \geq t$ and $\mu_\alpha^T(y) = \mu(y) + \alpha \geq t$.

Since μ_α^T is a α-translation fuzzy q-ideal of X, it follows that
$\mu(x * z) + \alpha = \mu_\alpha^T(x * z) \geq min\{\mu_\alpha^T((x * y) * z), \mu_\alpha^T(y)\} \geq t$, that is
$\mu(x * z) \geq t - \alpha$, so that $x * z \in U_\alpha(\mu; t)$, therefore $U_\alpha(\mu; t)$ is an q-ideal of X.

Conversely, suppose that $U_\alpha(\mu; t)$ is an q-ideal of X, for every $t \in Im(\mu)$ with $t > \alpha$. If there exists $x \in X$ such that
$\mu_\alpha^T(0) < t \leq \mu_\alpha^T(x)$, then $\mu(x) \geq t - \alpha$, but $\mu(0) < t - \alpha$. This shows that
$x \in U_\alpha(\mu; t)$ and $0 \notin U_\alpha(\mu; t)$This is a contradiction, and so $\mu_\alpha^T(0) \geq \mu_\alpha^T(x)$, for all $x \in X$.

Now, assume that there exist $x, y, z \in X$ such that
$\mu_\alpha^T(x * z) < t \leq min\{\mu_\alpha^T((x * y) * z), \mu_\alpha^T(y)\}$.

Then $\mu((x * y) * z) \geq t - \alpha$ and $\mu(y) \geq t - \alpha$, but $\mu(x * z) < t - \alpha$.

Hence $((x * y) * z) \in U_\alpha(\mu; t)$ and $\in U_\alpha(\mu; t)$, but $x * z \notin U_\alpha(\mu; t)$ and this is a contradiction.

Therefore as $\mu_\alpha^T(x * z) \geq min\{\mu_\alpha^T((x * y) * z), \mu_\alpha^T(y)\}$, for all $x, y, z \in X$.

Hence μ_α^T is a α-translation fuzzy q-ideal of X. □

Theorem 4.8. Let $f: (X;*,0) \to (Y;*`,0`)$ be an epimorphism between KK-algebras X and Y respectively. For every α-translation fuzzy q-ideal μ of X with sup property, $f(\mu)$ is a α-translation fuzzy q-ideal of Y.

Proof:

Let $f: (X;*,0) \to (Y;*`,0`)$ be an epimorphism of KK-algebras, μ_α^T is a α-translation fuzzy q-ideal of X and λ_α^T the image of μ_α^T under f. Since μ_α^T is a α-translation fuzzy q-ideal of X, we have $\mu_\alpha^T(0) \geq \mu_\alpha^T(x)$, for all $\in X$.

Note that $0 \in f(0`)$, where 0 and 0' are the zero elements of X and Y respectively.

Thus $\lambda_\alpha^T(0') = f(\mu_\alpha^T)(0') = sup_{t \in f^{-1}(0')}\mu(t) + \alpha = \mu_\alpha^T(0) \geq \mu_\alpha^T(x)$, for all $x \in X$, which implies that $\lambda_\alpha^T(0') \geq sup_{t \in f^{-1}(x')}\mu(t) + \alpha = \lambda_\alpha^T(x')$.

For any $x', y', z' \in Y$, let $x_0 \in f^{-1}(x')$, $y_0 \in f^{-1}(y')$ and $z_0 \in f^{-1}(z')$ be such that
$f(\mu_\alpha^T((x' *' y') *' z')) = sup_{t \in f^{-1}((x' *' y') *' z')}\mu(t) + \alpha$, $f(\mu_\alpha^T)(y') = sup_{t \in f^{-1}(y')}\mu(t) + \alpha$.

Then $f(\mu_\alpha^T)((x' *' y') *' z') = \lambda_\alpha^T((x' *' y') *' z') = sup_{((x_0 * y_0) * z_0) \in f^{-1}((x' *' y') *' z')}\mu((x_0 * y_0) * z_0) + \alpha$

$= sup_{t \in f^{-1}((x' *' y') *' z')}\mu(t) + \alpha$

Then $\lambda_\alpha^T(x' *' z') = sup_{t \in f^{-1}(x' *' z')}\mu(t) + \alpha = \mu_\alpha^T(x_0 * z_0)$
$\geq min\{\mu_\alpha^T((x_0 * y_0) * z_0), \mu_\alpha^T(y_0)\}$
$= min\{sup_{t \in f^{-1}((x' *' y') *' z')}\mu(t) + \alpha, sup_{t \in f^{-1}(y')}\mu(t) + \alpha\}$.
$= min\{\lambda_\alpha^T((x' *' y') *' z'), \lambda_\alpha^T(y')\}$.

Hence $\lambda_\alpha^T = f(\mu_\alpha^T)$ is a α-translation fuzzy q-ideal of Y. □

Proposition 4.9. A homomorphic pre-image of a α-translation fuzzy q-ideal of KK-algebra is also a α-translation fuzzy q-ideal.

Proof:

Let $f: (X;*,0) \to (Y;*`,0`)$ be a homomorphism of KK-algebras, μ_α^T is a α-translation fuzzy q-ideal of Y and λ_α^T the pre-image of μ_α^T under f such that $\lambda_\alpha^T(x) = \mu_\alpha^T(f(x))$, for all $x \in X$.

Since μ_α^T is a α-translation fuzzy q-ideal of Y, we have $\mu_\alpha^T(0) \geq \mu_\alpha^T(y)$, for all $y \in Y$.

Note that $0' \in f^{-1}(0)$, where 0 and 0' are the zero elements of X and Y respectively.

Thus $\lambda_\alpha^T(0) = \mu_\alpha^T(f(0)) = sup_{t\in f^{-1}(0)}\mu(t)+\alpha \geq sup_{t\in f^{-1}(x)}\mu(t)+\alpha = \mu_\alpha^T(f(x)) = \lambda_\alpha^T(x)$, for all $x \in X$, which implies that $\lambda_\alpha^T(0) \geq \lambda_\alpha^T(x)$.

Now, let x, y, z ∈ X, then we get

$$\lambda_\alpha^T(x * z) = \mu_\alpha^T(f(x * z))$$
$$\geq min\{\mu_\alpha^T(f((x * y) * z), \mu_\alpha^T(f(y))\}$$
$$= min\{\lambda_\alpha^T((x * y) * z), \lambda_\alpha^T(y)\}$$

$i.e.,\ \lambda_\alpha^T(x * z) \geq min\{\lambda_\alpha^T((x * y) * z), \lambda_\alpha^T(y)\}$. This completed the proof. ⌂

Definition 4.10. Let μ_1 and μ_2 be fuzzy subsets of a KK-algebra $(X;*,0)$. Then μ_2 is called a **fuzzy extension q-ideal** of μ_1 if the following assertions are valid:

(I_i) μ_2 is a fuzzy extension of μ_1.

(I_{ii}) If μ_1 is a fuzzy q-ideal of X, then μ_2 is a fuzzy q-ideal of X.

Proposition 4.11. Let μ be a fuzzy q-ideal of X and let α, γ ∈ [0,T]. If α ≥ γ, then the fuzzy subset α-translation μ_α^T of μ is a fuzzy extension q-ideal of the fuzzy q-ideal γ-translation μ_γ^T of μ.

Proposition 4.12. Let μ be a fuzzy q-ideal of a KK-algebra $(X;*,0)$ and γ ∈ [0,T]. For every fuzzy extension q-ideal ν of the fuzzy q-ideal γ-translation μ_γ^T of μ, there exists α ∈ [0,T] such that α ≥ γ and ν is a fuzzy extension q-ideal of the fuzzy q-ideal α-translation μ_α^T of μ.

The following example illustrates Proposition (4.12).

Example 4.13. Let X = {0, 1,2} in which (∗) be given by:

∗	0	1	2
0	0	1	2
1	0	0	2
2	0	0	0

Then (X; ∗ ,0) is a KK-algebra. Define a fuzzy subset μ of X by:

X	0	1	2
μ	0.8	0.7	0.6

Then μ is a fuzzy q-ideal of X and T = 0.2. If we take γ = 0.12, then the γ-translation fuzzy q-ideal μ_γ^T of μ is given by :

X	0	1	2
μ_γ^T	0.92	0.82	0.72

Let ν be a fuzzy subset of X defined by:

X	0	1	2
ν	0.98	0.89	0.81

Then ν is clearly a fuzzy extension q-ideal of the γ-translation fuzzy q-ideal μ_γ^T of . But ν is a α-translation fuzzy q-ideal μ_α^T of μ for all α ∈[0,T]. Take α = 0.17, then α = 0.17 > 0.12 = γ , and the α-translation fuzzy q-ideal $\mu_\alpha^T$of μ is given as follows:

X	0	1	2
μ_α^T	0.97	0.87	0.77

Note that ν(x) ≥ μ_α^T (x) for all $x \in X$, and hence ν is a fuzzy extension q-ideal of the α-translation fuzzy q-ideal μ_α^T of μ.

Proposition 4.14. Let μ be a fuzzy q-ideal of a KK-algebra $(X;*,0)$ and α∈ [0,T]. Then the α-translation fuzzy subset μ_α^T of μ is a fuzzy extension q-ideal of μ.

A fuzzy extension q-ideal of a fuzzy q-ideal μ may not be represented as a α-translation fuzzy q-ideal μ_α^T of μ , that is , the converse of Proposition (4.14) is not true in general , as shown by the following example.

Example 4.15. Let X = {0, 1, 2, 3} be a KK-algebra with the following table:

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$*$	0	1	2	3
0	0	1	2	3
1	0	0	0	0
2	0	1	0	3
3	0	1	0	0

Define a fuzzy subset μ of X by:

X	0	1	2	3
μ	0.8	0.4	0.5	0.4

Then μ is a fuzzy q-ideal of X and T = 0.2. If we take $\alpha = 0.03$, then the α-translation fuzzy q-ideal μ_α^T of μ is given by

X	0	1	2	3
μ_α^T	0.83	0.43	0.53	0.43

Let ν be a fuzzy subset of X defined by:

X	0	1	2	3
ν	0.82	0.46	0.59	0.46

Then ν is a fuzzy extension q-ideal of μ . But ν is not α-translation fuzzy q-ideal μ_α^T of μ, for all $\alpha \in [0,T]$.

Proposition 4.16:
The intersection of any set of α-translation fuzzy q-ideals of KK-algebra $(X;*,0)$ is also α-translation fuzzy q-ideal of X.
Proof:
Let $\{\mu_i | i \in \Lambda.\}$ be a family of α-translation fuzzy q-ideals of KK-algebra X , then for any $x, y, z \in X$, $i \in \Lambda$,

$$(\bigcap_{i\in\Lambda}(\mu_\alpha^T)_i)\ (0) = inf\ ((\mu_\alpha^T)i\ (0)) = inf\ (\mu\ i\ (0) + \alpha)$$

$$\geq inf\ (\mu\ i(x) + \alpha) = inf\ ((\mu_\alpha^T)i\ (x)) = (\cap_{i\in\Lambda}(\mu_\alpha^T)_i\ (x),$$ and

$$(\bigcap_{i\in\Lambda}(\mu_\alpha^T)_i)(x*z) = inf\ ((\mu_\alpha^T)i)(x*z)) = inf\ (\mu\ i(x*z) + \alpha)$$

$$\geq inf\ (min\{\mu_i\ ((x*y)*z), \mu_i(y)\}) + \alpha$$
$$= inf\ (min\{\mu_i\ ((x*y)*z) + \alpha, \mu_i(y) + \alpha\})$$
$$= min\{inf\ (\mu_i((x*y)*z) + \alpha)\ , inf\ (\mu_i\ (y) + \alpha\}$$
$$= min\{(\bigcap_{i\in\Lambda}\mu_i\ ((x*y)*z) + \alpha)\ , (\bigcap_{i\in\Lambda}\mu_i\ (y) + \alpha\}$$
$$= min\{(\cap_{i\in\Lambda}(\mu_\alpha^T)_i)((x*y)*z)\ , (\cap_{i\in\Lambda}(\mu_\alpha^T)_i)(y)\}.\ \square$$

Clearly, the union of fuzzy extensions of a α-translation fuzzy subset of KK-algebra X is not a fuzzy extension of μ as seen in the following example.

Example 4.17. Let $X = \{0,1,2,3\}$ in which * be a defined by the following table:

$*$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	3	3	0	0
3	3	2	1	0

Then $(X;*,0)$ is a KK-algebra. Define a fuzzy subset μ of X by:

X	0	1	2	3

μ	0.7	0.7	0.6	0.6

Let α = 0.1, then μ is a α-translation fuzzy q-ideal of X.

X	0	1	2	3
μ_α^T	0.8	0.8	0.7	0.7

Define a fuzzy subset v of X by:

X	0	1	2	3
v	0.7	0.4	0.4	0.7

Let α = 0.1, then v is a α-translation fuzzy q-ideal of X.

X	0	1	2	3
v_α^T	0.8	0.5	0.5	0.8

Let ν and δ be α-translation fuzzy subsets of X given by:

X	0	1	2	3
μ_α^T	0.8	0.8	0.7	0.7
v_α^T	0.8	0.5	0.5	0.8
$\mu_\alpha^T \cup v_\alpha^T$	0.8	0.8	0.7	0.8

Then the union $\mu_\alpha^T \cup v_\alpha^T$ is not a fuzzy extension of μ since

$(\mu_\alpha^T \cup v_\alpha^T)(0 * 2) = 0.6 < 0.7 = \min\{(\mu_\alpha^T \cup v_\alpha^T)((0*1)*2), (\mu_\alpha^T \cup v_\alpha^T)(1)\}$

$= \min\{(\mu_\alpha^T \cup v_\alpha^T)(3), (\mu_\alpha^T \cup v_\alpha^T)(1)\}$.

Proposition 4.18. Let μ be a fuzzy q-ideal of a KK-algebra $(X;*,0)$ and let α ∈ [0,T], then the α-translation fuzzy subset μ_α^T of μ is a fuzzy KK-subalgebra of X.

Proof:

Since μ be a fuzzy q-ideal of a KK-algebra X, then by Proposition (2.30), μ be a fuzzy ideal of a KK-algebra X and by Proposition (2.21) μ be a fuzzy KK-subalgebra of a KK-algebra X and let α ∈ [0,T], then by Proposition (3.4), the α-translation fuzzy subset μ_α^T of μ is a α-translation fuzzy KK-subalgebra of X. □

In general, the converse of the Proposition (4.18) is not true.

Example 4.19. Consider a KK-algebra $X = \{0, 1, 2\}$ with the example (4.13). Define a fuzzy subset μ of X by:

X	0	1	2
μ	0.7	0.5	0.6

Then μ is not fuzzy q-ideal of X. Since

$\mu(1)=0.5 < 0.6 = \min\{\mu(2 * 1), \mu(2)\} = \min\{\mu(0), \mu(2)\}$, and T=0.3.

But if we take α=0.2 the α-translation fuzzy subalgebra μ_α^T of μ is given as follows

:

X	0	1	2
μ_α^T	0.9	0.7	0.8

5- Cartesian Product on Fuzzy q-ideal of KK-algebra

In this section, we discuss the Cartesian product of α-translation fuzzy of KK-algebras and establish some of its properties in detail on the basis of fuzzy q-ideal as [11].

Definition 5.1. Let μ_α^{T} and $\delta_\alpha^{\mathrm{T}}$ be α-translations fuzzy of a KK-algebra $(X;*,0)$. The Cartesian product $\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}}$:X×X→[0,1] is defined by $(\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}})(x,y) = min\{\mu_\alpha^{\mathrm{T}}(x), \delta_\alpha^{\mathrm{T}}(y)\}$, for all $x, y \in X$.

Proposition 5.2. Let μ_α^{T} and ν_α^{T} be α-translations fuzzy of a KK-algebra$(X;*,0)$. Then $\mu_\alpha^{\mathrm{T}} \times \nu_\alpha^{\mathrm{T}}(x,y) = (\mu \times \nu)_\alpha^{\mathrm{T}}(x,y)$, for all $(x,y) \in$ X × X.

Proof : Let (x, y)∈X×X.

$$(\mu \times \nu)_\alpha^{\mathrm{T}}(x,y) = (\mu \times \nu)(x,y) + \alpha$$
$$= min\{\mu(x), \nu(y)\} + \alpha$$
$$= min\{\mu(x) + \alpha, \nu(y) + \alpha\}$$
$$= min\{\mu_\alpha^{T}(x), \nu_\alpha^{T}(y)\}$$
$$= (\mu_\alpha^{\mathrm{T}} \times \nu_\alpha^{\mathrm{T}})(x,y).$$

Theorem 5.3. Let μ and ν be two fuzzy q-ideals of a KK-algebra $(X;*,0)$. Let $\mathrm{T} = min\{\mathrm{T}_\mu, \mathrm{T}_\nu\}$ where $\mathrm{T}_\mu = 1 - sup\{\mu(x) : x \in \mathrm{X}\}$and $\mathrm{T}_\nu = 1 - sup\{\nu(x) : x \in \mathrm{X}\}$ and α∈[0,T]. then the α-translation fuzzy subset of Cartesian product $\mu \times \nu$ is a fuzzy q-ideal of X×X.

Proof:

Let μ and ν be two fuzzy q-ideals of a KK-algebra X. let α∈[0,T].

Now, by Proposition (4.5), μ_α^{T} and ν_α^{T} are fuzzy q-ideals of X and by [4,Theorem (6.9)], $\mu_\alpha^{\mathrm{T}} \times \nu_\alpha^{\mathrm{T}}$ is a fuzzy q-ideal of X×X, that mean $(\mu \times \nu)_\alpha^{\mathrm{T}}(0,0) = (\mu_\alpha^{\mathrm{T}} \times \nu_\alpha^{\mathrm{T}})(0,0) \geq (\mu_\alpha^{\mathrm{T}} \times \nu_\alpha^{\mathrm{T}})(x,y) = (\mu \times \nu)_\alpha^{\mathrm{T}}(x,y)$.

Also (x, y), (z,u), (h,k)∈X×X.

$$(\mu \times \nu)_\alpha^{\mathrm{T}}\big((x*h),(y*k)\big) = (\mu \times \nu)\big((x*h),(y*k)\big) + \alpha$$
$$= min\{\mu((x*h)), \nu((y*k))\} + \alpha$$
$$\geq min\big\{\min\{\mu\big((x*z)*h\big), \mu(z)\}, \min\{\nu\big((y*u)*k\big), \nu(u)\}\big\} + \alpha$$
$$= min\big\{\min\{\mu\big((x*z)*h\big), \nu\big((y*u)*k\big)\}, \min\{\mu(z), \nu(u)\}\big\} + \alpha$$
$$= min\{(\mu \times \nu)((x*z)*h, (y*u)*k), (\mu \times \nu)(z,u)\} + \alpha$$
$$= min\{(\mu \times \nu)((x*z)*h, (y*u)*k) + \alpha, (\mu \times \nu)(z,u) + \alpha\}$$
$$= min\{(\mu \times \nu)_\alpha^{\mathrm{T}}((x*z)*h, (y*u)*k), (\mu \times \nu)_\alpha^{\mathrm{T}}(z,u)\}$$

Hence $(\mu \times \nu)_\alpha^{\mathrm{T}}$ is a fuzzy q-ideal of X×X. □

Theorem 5.4. Let μ and δ be fuzzy subsets of a KK-algebra $(X;*,0)$ such that $\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}}$ is a fuzzy q-ideal of X×X. Then the following equivalent:

(i) Either $\mu_\alpha^{\mathrm{T}}(0) \geq \mu_\alpha^{\mathrm{T}}(x)$ or $\delta_\alpha^{\mathrm{T}}(0) \geq \delta_\alpha^{\mathrm{T}}(x)$ for all $x \in X$

(ii) If $\mu_\alpha^{\mathrm{T}}(0) \geq \mu_\alpha^{\mathrm{T}}(x)$ for all $x \in X$. then either $\delta_\alpha^{\mathrm{T}}(0) \geq \mu_\alpha^{\mathrm{T}}(x)$ or $\delta_\alpha^{\mathrm{T}}(0) \geq \delta_\alpha^{\mathrm{T}}(x)$

(iii) If $\delta_\alpha^{\mathrm{T}}(0) \geq \delta_\alpha^{\mathrm{T}}(x)$ for all $x \in X$. Then either $\mu_\alpha^{\mathrm{T}}(0) \geq \mu_\alpha^{\mathrm{T}}(x)$ or $\mu_\alpha^{\mathrm{T}}(0) \geq \delta_\alpha^{\mathrm{T}}(x)$

proof:

(i)→(ii) Let $\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}}$ be a fuzzy q-ideal of X×X. Suppose that $\mu_\alpha^{\mathrm{T}}(0) < \mu_\alpha^{\mathrm{T}}(x)$ and $\delta_\alpha^{\mathrm{T}}(0) < \delta_\alpha^{\mathrm{T}}(x)$, for some $x, y \in X$. Then $(\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}})(x,y) = min\{\mu_\alpha^{\mathrm{T}}(x), \delta_\alpha^{\mathrm{T}}(y)\}$ $> \min\{\mu_\alpha^{\mathrm{T}}(0), \delta_\alpha^{\mathrm{T}}(0)\} = (\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}})(0,0)$ which is a contradiction. Therefore $\mu_\alpha^{\mathrm{T}}(0) \geq \mu_\alpha^{\mathrm{T}}(x)$ or $\delta_\alpha^{\mathrm{T}}(0) \geq \delta_\alpha^{\mathrm{T}}(x)$ for all $x \in X$.

(ii) →(iii) Assume that there exists$x, y \in X$ such that $\delta_\alpha^{\mathrm{T}}(0) < \mu_\alpha^{\mathrm{T}}(x)$ and $\delta_\alpha^{\mathrm{T}}(0) < \delta_\alpha^{\mathrm{T}}(x)$. Then $(\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}})(0,0) = min\{\mu_\alpha^{\mathrm{T}}(0), \delta_\alpha^{\mathrm{T}}(0)\} = \delta_\alpha^{\mathrm{T}}(0)$ and hence $(\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}})(x,y) = min\{\mu_\alpha^{\mathrm{T}}(x), \delta_\alpha^{\mathrm{T}}(y)\} > \delta_\alpha^{\mathrm{T}}(0) = (\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}})(0,0)$ which is a contradiction. Hence ,if $\mu_\alpha^{\mathrm{T}}(0) \geq \mu_\alpha^{\mathrm{T}}(x)$ for all $x \in X$, then either $\delta_\alpha^{\mathrm{T}}(0) \geq \mu_\alpha^{\mathrm{T}}(x)$ or $\delta_\alpha^{\mathrm{T}}(0) \geq \delta_\alpha^{\mathrm{T}}(x)$.

(iii)→(i) it is clear. □

Theorem 5.5. Let μ and δ be fuzzy subsets of a KK-algebra $(X;*,0)$ such that $\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}}$ is a fuzzy q-ideal of X×X, then either μ or δ is a fuzzy q-ideal of X.

proof:

First, we prove that δ is a fuzzy q-ideal of X. Since by Theorem (5.4(i)) either $\mu_\alpha^{\mathrm{T}}(0) \geq \mu_\alpha^{\mathrm{T}}(x)$ or $\delta_\alpha^{\mathrm{T}}(0) \geq \delta_\alpha^{\mathrm{T}}(x)$ for all $x \in X$. Assume that $\delta_\alpha^{\mathrm{T}}(0) \geq \delta_\alpha^{\mathrm{T}}(x)$ for all $x \in X \Rightarrow \delta(0) + \alpha \geq \delta(x) + \alpha. \Rightarrow \delta(0) \geq \delta(x)$.

It follows from Theorem (5.4(iii)) that either $\mu_\alpha^{\mathrm{T}}(0) \geq \mu_\alpha^{\mathrm{T}}(x)$ or $\mu_\alpha^{\mathrm{T}}(0) \geq \delta_\alpha^{\mathrm{T}}(x)$.

If $\mu_\alpha^{\mathrm{T}}(0) \geq \delta_\alpha^{\mathrm{T}}(x)$, for any $x \in X$

$(\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}})(0,x) = min\{\mu_\alpha^{\mathrm{T}}(0), \delta_\alpha^{\mathrm{T}}(x)\} = \delta_\alpha^{\mathrm{T}}(x)$.

Let (x_1, x_2) ,(y_1,y_2) ,(z_1,z_2)∈X×X since $\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}}$ is a fuzzy q-ideal of X×X. we have

$$(\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}})(x_1 * z_1, x_2 * z_2) = min\{(\mu_\alpha^{\mathrm{T}}(x_1 * z_1), \delta_\alpha^{\mathrm{T}}(x_2 * z_2)\}$$
$$\geq min\{min\{\mu_\alpha^{\mathrm{T}}((x_1 * y_1) * z_1), \mu_\alpha^{\mathrm{T}}(y_1)\}, min\{\delta_\alpha^{\mathrm{T}}((x_2 * y_2) * z_2), \delta_\alpha^{\mathrm{T}}(y_2)\}$$
$$= min\{min\{\mu_\alpha^{\mathrm{T}}((x_1 * y_1) * z_1), \delta_\alpha^{\mathrm{T}}((x_2 * y_2) * z_2)\}, min\{\mu_\alpha^{\mathrm{T}}(y_1), \delta_\alpha^{\mathrm{T}}(y_2)\}$$

$= min\{(\mu_\alpha^T \times \delta_\alpha^T)(((x_1 * y_1) * z_1), ((x_2 * y_2) * z_2)), (\mu_\alpha^T \times \delta_\alpha^T)(y_1, y_2)\}$ ------ (A)

If we take $x_1=y_1=z_1=0$ in (A), then

$\delta(x_2 * z_2) + \alpha = \delta_\alpha^T(x_2 * z_2) = (\mu_\alpha^T \times \delta_\alpha^T)(0, x_2 * z_2)$

$\geq min\{(\mu_\alpha^T \times \delta_\alpha^T)(0, ((x_2 * y_2) * z_2)), (\mu_\alpha^T \times \delta_\alpha^T)(0, y_2)\}$

$= \min[min\{(\mu_\alpha^T(0), \delta_\alpha^T((x_2 * y_2) * z_2)\}, min\{\mu_\alpha^T(0), \delta_\alpha^T(y_2)\}]$

$= min\{\delta_\alpha^T((x_2 * y_2) * z_2), \delta_\alpha^T(y_2)\} = min\{\delta((x_2 * y_2) * z_2) + \alpha, \delta(y_2) + \alpha\}$

$= min\{\delta((x_2 * y_2) * z_2), \delta(y_2)\} + \alpha$ this prove that δ is a fuzzy q-ideal of X.

Next, we will prove that μ is a fuzzy q-ideal of X.

Let $\mu_\alpha^T(0) \geq \mu_\alpha^T(x) \Longrightarrow \mu(0) \geq \mu(x)$ **,** since by Theorem (5.4(ii)) either $\delta_\alpha^T(0) \geq \mu_\alpha^T(x)$ or $\delta_\alpha^T(0) \geq \delta_\alpha^T(x)$ **.**

If $\delta_\alpha^T(0) \geq \mu_\alpha^T(x)$for any x∈X. Hence

$(\mu_\alpha^T \times \delta_\alpha^T)(x, 0) = \min\{\mu_\alpha^T(x), \delta_\alpha^T(0)\} = \mu_\alpha^T(x)$, taking $\mu_\alpha^T(x_1)$

$= (\mu_\alpha^T \times \delta_\alpha^T)(x_1, 0)$

If we take $x_2=y_2=z_2=0$ in (A), then

$\mu(x_1 * z_1) + \alpha = \mu_\alpha^T(x_1 * z_1) = (\mu_\alpha^T \times \delta_\alpha^T)(x_1 * z_1, 0)$

$\geq min\{(\mu_\alpha^T \times \delta_\alpha^T)(((x_1 * y_1) * z_1), 0), (\mu_\alpha^T \times \delta_\alpha^T)(y_1, 0)\}$

$= min[min\{(\mu_\alpha^T((x_1 * y_1) * z_1), \delta_\alpha^T(0)\}, min\{\mu_\alpha^T(y_1), \delta_\alpha^T(0)\}]$

$= \min\{\mu_\alpha^T((x_1 * y_1) * z_1), \delta_\alpha^T(y_1)\} = \min\{\mu((x_1 * y_1) * z_1) + \alpha, \delta(y_1) + \alpha\}$

$= min\{\mu((x_1 * y_1) * z_1), \delta(y_1)\} + \alpha$ which proves that μ is a fuzzy q-ideal of X.

Hence either μ or δ is a fuzzy q-ideal of X. □

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