International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 5 Issue 12, December - 2021, Pages:82-90

Anti-fuzzy AB-ideals of AB-algebra

Dr. Areej Tawfeeq Hameed[1], Huda Adan Mohammed[2], Dr. Ahmed Hamzah Abed[3]

[1,2,3]Department of Mathematics, Faculty of Education for Girl, University of kufa, Najaf, Iraq.
E-mail: areej.tawfeeq@uokufa.edu.iq[1], areej238@gmail.com[2], ahmedh.abed@uokufa.edu.iq[3]

Abstract: *We introduce the notion of anti-fuzzy AB-ideals on AB-algebra, several appropriate examples are provided and some properties are investigated. The image and the inverse image of anti-fuzzy AB-ideals on AB-algebra are defined and how the image and the inverse image of anti-fuzzy AB-ideals on AB-algebra become anti-fuzzy AB-ideals are studied. Moreover, the Cartesian product of anti-fuzzy AB-ideals are given.*

Keywords: AB-ideals, anti-fuzzy AB-ideals, image and pre-image of anti-fuzzy AB-ideals.

2010 Mathematics Subject Classification: 06F35, 03G25, 08A72.

1. Introduction

BCK-algebras form an important class of logical algebras introduced by K. Iseki [12] and was extensively investigated by several researchers. The class of all BCK- algebras is quasi variety. J. Meng and Y.B. Jun posed an interesting problem (solved in [15]) whether the class of all BCK-algebras is a variety. In connection with this problem, Komori introduced in [14] a notion of BCC-algebras. W.A. Dudek (cf.[3,12]) redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori and studied ideals and congruences of BCC-algebras. In [17,18]), C. Prabpayak and U. Leerawat introduced a new algebraic structure, which is called KU-algebra. They gave the concept of homomorphisms of KU-algebras and investigated some related properties. L.A. Zadeh [20] introduced the notion of fuzzy subsets. At present this concept has been applied to many mathematical branches, such as group, functional analysis, probability theory, topology, and soon. In 1991 , O.G. Xi [19] applied this concept to BCK-algebras, and he introduced the notion of fuzzy sub-algebras (ideals) of the BCK-algebras with respect to minimum, and since then Jun et al studied fuzzy ideals (cf.[1, 11,12,13]), and moreover several fuzzy structures in BCC-algebras are considered (cf.[19,20]). S. Mostafa and et al (in [15]) introduced the notion of fuzzy KUS-ideals of KUS-algebras and they investigated several basic properties which are related to fuzzy KUS-ideals. they described how to deal with the homomorphism image and inverse image of fuzzy KUS-ideals. And in [16], the anti-fuzzy KUS-ideals of KUS-algebras is introduced. Several theorems are stated and proved. In [3], A.T. Hameed introduced and studied new algebraic structure, called AT-algebra and investigate some of its properties. She introduced the notion of fuzzy AT-ideal of AT-algebra, several theorems, properties are stated and proved. A.T. Hameed and et al, ([4-7]) introduced AB-ideals on AB-algebras and introduced the notions fuzzy AB-subalgebras, fuzzy AB-ideals of AB-algebras and investigated relations among them. Also, ([2,8,9]) introduced the notion of fuzzy translation (normalized, maximal) fuzzy extensions and fuzzy magnified of fuzzy AB-subalgebra and fuzzy AB-ideal on AB-algebras and investigate some of their properties. In this paper, we introduce the notion of anti-fuzzy AB-ideals of AB-algebras and then we study the homomorphism image and pre-image of anti-fuzzy AB-ideals. We also prove that the Cartesian product of anti-fuzzy AB-ideals are anti-fuzzy AB-ideals.

2. Preliminaries

We review some definitions and properties that will be useful in our results.

Definition 2.1([5,6]) Let X be a set with a binary operation $*$ and a constant 0. Then $(X;*,0)$ is called **an AB-algebra** if the following axioms satisfied: for all $,y,z \in X$,

(i) $((x*y)*(z*y))*(x*z) = 0,$

(ii) $0*x = 0,$

(iii) $x*0 = x,$

Example 2.2([5,6]) Let X = {0, 1, 2, 3, 4} in which $(*)$ is defined by the following table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

Then $(X;*,0)$ is an AB-algebra.

Remark 2.3([5,6]) Define a binary relation ≤ on AB-algebra $(X;*,0)$ by letting $x \le y$ if and only if $x*y = 0$.

Proposition 2.4([5,6]) In any AB-algebra $(X;*,0)$, the following properties hold: for all $x,y,z \in X$,

(1) $(x * y) * x = 0.$
(2) $(x * y) * z = (x * z) * y.$
(3) $(x * (x * y)) * y = 0.$

Proposition 2.5([5,6]) Let $(X; *, 0)$ be an AB-algebra. X is satisfies for all $x, y, z \in X$,

(1) $x \leq y$ *implies* $x * z \leq y * z$.
(2) $x \leq y$ *implies* $z * y \leq z * x$.

Definition 2.6([5,6]). Let $(X; *, 0)$ be an AB-algebra and let S be a nonempty subset of X. S is called an **AB-subalgebra of** X if $x * y \in S$ whenever $x \in S$ *and* $y \in S$.

Definition 2.7([5,6]). A nonempty subset I of an AB-algebra $(X; *, 0)$ is called **an AB-ideal of *X*** if it satisfies the following conditions: for any $x, y, z \in X$,
(I_1) $0 \in I$,
(I_2) $(x * y) * z \in I$ *and* $y \in I$ *imply* $x * z \in I$.

Proposition 2.8 ([5,6]). Every AB-ideal of AB-algebra is an AB-subalgebra.

Proposition 2.9 ([5,6]). Let $\{I_i \mid i \in \Lambda\}$ be a family of AB-ideals of AB-algebra $(X; *, 0)$. The intersection of any set of AB-ideals of X is also an AB-ideal.

Definition 2.10 ([18,19]). Let $(X ; *, 0)$ and $(Y; *`, 0`)$ be nonempty sets. The mapping $f: (X; *, 0) \rightarrow (Y; *`, 0`)$ is called **a homomorphism** if it satisfies:
$f(x * y) = f(x) *` f(y)$, for all $x, y \in X$. The set $\{x \in X \mid f(x) = 0'\}$ is called **the kernel of** f denoted by ker f.

Theorem 2.11 ([5,6]). Let $f: (X; *, 0) \rightarrow (Y; *`, 0`)$ be a homomorphism of an AB-algebra X into an AB-algebra Y, then :
A. $f(0) = 0'$.
B. f is injective if and only if ker $f = \{0\}$.
C. $x \leq y$ implies $f(x) \leq f(y)$.

Theorem 2.12 ([5,6]). Let $f: (X; *, 0) \rightarrow (Y; *`, 0`)$ be a homomorphism of an AB-algebra X into an AB-algebra Y, then:
(F_1) If S is an AB-subalgebra of X, then f (S) is an AB-subalgebra of Y .
(F_2) If I is an AB-ideal of X, then f (I) is an AB-ideal of Y , where f is onto.
(F_3) If H is an AB-subalgebra of Y, then f^{-1} (H) is an AB-subalgebra of .
(F_4) If J is an AB-ideal of Y, then f^{-1} (J) is an AB-ideal of X .
(F_5) ker f is an AB-ideal of X.
(F_6) Im(f) is an AB-subalgebra of Y.

Definition 2.13([21]). Let $(X; *, 0)$ be a nonempty set, a fuzzy subset μ of X is a function $\mu: X \rightarrow [0,1]$.

Definition 2.14 ([21]). Let X be a nonempty set and μ be a fuzzy subset of $(X; *, 0)$, for $t \in [0,1]$, the set $L(\mu, t) = \mu_t = \{x \in X \mid \mu(x) \geq t\}$ is called **a level subset of μ** .

Definition 2.15([7]). Let $(X; *, 0)$ be an AB-algebra, a fuzzy subset μ of X is called **a fuzzy AB-subalgebra of *X*** if for all $x, y \in X$, $\mu(x * y) \geq min\{\mu(x), \mu(y)\}$.

Definition 2.16([7]). Let $(X; *, 0)$ be an AB-algebra, a fuzzy subset μ of X is called **a fuzzy AB-ideal of *X*** if it satisfies the following conditions, for all $x, y, z \in X$,
(FAB_1) $\mu(0) \geq \mu(x)$,
(FAB_2) $\mu(x * z) \geq min\{\mu((x * y) * z), \mu(y)\}$.

Proposition 2.17([7]).
1- The intersection of any set of fuzzy AB-ideals of AB-algebra is also fuzzy AB-ideal.
2- The union of any set of fuzzy AB-ideals of AB-algebra is also fuzzy AB-ideal where is chain.

Proposition 2.18([7]).
1- Let μ be a fuzzy subset of AB-algebra $(X; *, 0)$. If μ is a fuzzy AB-subalgebra of X if and only if for every $t \in [0,1]$, μ_t is an AB-subalgebra of X .
2- Let μ be a fuzzy AB-ideal of AB-algebra $(X; *, 0)$, μ is a fuzzy AB-ideal of X if and only if for every $t \in [0,1]$, μ_t is an AB-ideal of X .

Proposition 2.19([7]). Every fuzzy AB-ideal of AB-algebra is a fuzzy AB-subalgebra.

Lemma 2.20([7]). Let μ be a fuzzy AB-ideal of AB-algebra X and if $\leq y$, then $\mu(x) \geq \mu(y)$, for all $x, y \in X$.

Definition 2.21 ([19]). Let $f: (X; *, 0) \rightarrow (Y; *`, 0`)$ be a mapping nonempty sets X and Y respectively. If μ is a fuzzy subset of X, then the fuzzy subset β of Y defined by: $f(\mu)(y) = \begin{cases} sup\{\mu(x): x \in f^{-1}(y)\} & if\ f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ 0 & otherwise \end{cases}$
is said to be **the image of μ under f.**

Similarly if β is a fuzzy subset of , then the fuzzy subset $\mu = (\beta \circ f)$ of X (i.e the fuzzy subset defined by $\mu(x) = \beta(f(x))$, for all $x \in X$) is called **the pre-image of β under f** .

Definition 2.22 ([19]). A fuzzy subset μ of a set X has sup property if for any subset T of X , there exist $t_0 \in T$ such that $\mu(t_0) = sup\{\mu(t) | t \in T\}$.

Proposition 2.23 ([7]). Let $f: (X; *, 0) \rightarrow (Y; *`, 0`)$ be a homomorphism between AB-algebras X and Y respectively.

1- For every fuzzy AB-subalgebra β of Y, $f^{-1}(\beta)$ is a fuzzy AB-subalgebra of X.
2- For every fuzzy AB-subalgebra μ of X , $f(\mu)$ is a fuzzy AB-subalgebra of Y.
3- For every fuzzy AB-ideal β of Y, $f^{-1}(\beta)$ is a fuzzy AB-ideal of X.
4- For every fuzzy AB-ideal μ of X with sup property, $f(\mu)$ is a fuzzy AB-ideal of Y, where f is onto.

3. Fuzzy AB-subalgebras and Homomorphism of AB-algebras

In this section, we will introduce a new notion called an anti-fuzzy AB-subalgebra

Definition 3.1. Let $(X; *, 0)$ be an AB-algebra, a fuzzy subset μ of X is called **an anti-fuzzy AB-subalgebra of X** if for all $x, y \in X$,
$\mu(x * y) \leq max\{\mu(x), \mu(y)\}$.

Example 3.2. Let X = {0, 1, 2, 3} be a set with the following table:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	3	0

Then $(X; *, 0)$ is an AB-algebra. It is easy to show that I ={0, 1}is an AB-subalgebra of X .

Define a fuzzy subset $\mu: X \rightarrow [0, 1]$ by $\mu(x) = \begin{cases} 0.3 & if x \in I \\ 0.9 & if x \notin I \end{cases}$

Then μ is an anti-fuzzy AB-subalgebra of X.

Routine calculation gives that μ is an anti-fuzzy AB-subalgebra of AB-algebras X.

Theorem 3.3. Let μ be an anti-fuzzy subset of an AB-algebra $(X; *, 0)$. μ is an anti-fuzzy AB-subalgebra of X if and only if it satisfies: for all t∈[0, 1], $U(\mu, t) \neq \emptyset$ implies $U(\mu, t)$ is an AB-subalgebra of X.

Proof: Assume that μ is an anti-fuzzy AB-subalgebra of X, let t∈ [0,1] be such that $U(\mu, t) \neq \emptyset$, and let $x, y \in X$ be such that x, y $\in U(\mu, t)$, then $\mu(x) \leq t$ and $\mu(y) \leq t$, so $\mu(x * y) \leq max\{\mu(x), \mu(y)\} \leq t$, so that $(x * y) \in U(\mu, t)$. Hence $U(\mu, t)$ is an AB-subalgebra of X.

Conversely, suppose that μ satisfies $U(\mu, t)$ is an AB-subalgebra of X,
Now, assume $\mu(x * y) > max\{\mu(x), \mu(y)\}$,
taking $\beta_0 = \frac{1}{2}\{\mu(x * y) + max\{\mu(x), \mu(y)\}\}$, we have $\beta_0 \in$ [0, 1] and
$max\{\mu(x), \mu(y)\} < \beta_0 < \mu(x * y)$, it follows that
$max\{\mu(x), \mu(y)\}, \in U(\mu, \beta_0)$ and $x * y \notin U(\mu, \beta_0)$, this is a contradiction and therefore μ is an anti-fuzzy AB-subalgebra of X. ⌂

Corollary 3.4. If a fuzzy subset μ of AB-algebra $(X; *, 0)$ is an anti-fuzzy AB-subalgebra, then for every $t \in Im(\mu)$, $U(\mu, t)$is an AB-subalgebra of X.

Corollary 3.5. Let I be an AB-subalgebra of an AB-algebra $(X; *, 0)$, then for any fixed number t in an open interval (0,1), there exist an anti-fuzzy AB-subalgebra μ of X such that $U(\mu, t)$=I.

Proof: Define $\mu: X \rightarrow [0: 1]$ by $\mu(x) = \begin{cases} 0 & if x \in I \\ t & if x \notin I \end{cases}$.

Where t is a fixed number in (0,1).

Clearly, $\mu(0) \leq \mu(x)$ and we have one two level sets $U(\mu, 0) = I, U(\mu, t) = X$, which are AB-subalgebras of X, then from Theorem (3.3), μ is an anti-fuzzy AB-subalgebra of X. ⌂

Proposition 3.6.

The intersection of any set of anti-fuzzy AB-subalgebras of AB-algebra $(X; *, 0)$ is also anti-fuzzy AB-subalgebra.

Proof:

Let $\{\mu_i \mid i \in \Lambda\}$ be a family of anti-fuzzy AB-subalgebras of AB-algebra X, then for any $x, y \in X$, $i \in \Lambda$,
$(\bigcap_{i \in \Lambda} \mu_i)(x * y) = inf(\mu_i(x * y)) \leq \inf\{max\{\mu_i(x), \mu_i(y)\}\}$

$\leq max\{inf(\mu_i(x)), inf(\mu_i(y)\}$
$= max\{(\bigcap_{i\in\Lambda}\mu_i)(x*y)), (\bigcap_{i\in\Lambda}\mu_i)(x)\}$. ⌂

Proposition 3.7.
The union of any set of anti-fuzzy AB-subalgebras of AB-algebra $(X;*,0)$ is also anti-fuzzy AB-subalgebra of X where is chain.
Proof: Since $\{\mu_i | i\in\Lambda\}$ be a family of anti-fuzzy AB-subalgebras of X. For any $, y \in X$, suppose $x \in\cup_{i\in\Lambda}\mu_i$ and $y \in\cup_{i\in\Lambda}\mu_i$, for all i∈Λ. It follows that $x \in \mu_i$,
$y \in \mu_i$, for some i∈Λ. By assumption $\mu_i \subseteq \mu_k$. Hence $\in \mu_k$, $y \in \mu_k$, but μ_k is a anti-fuzzy AB-subalgebra of X, it follows that $x*y \in \mu_k$, therefore, $x*y \in\cup_{i\in\Lambda}\mu_i$. Hence $\cup_{i\in\Lambda}\mu_i$ is anti-fuzzy AB-subalgebra of X. □

Theorem 3.8. A homomorphic pre-image of anti-fuzzy AB-subalgebra is also an anti-fuzzy AB-subalgebra.
Proof: Let $f:(X;*,0) \to (Y;*`,0`)$ be a homomorphism of AB-algebras, β is an anti-fuzzy AB-subalgebra of Y and μ the pre-image of β under f, let $x, y \in X$, then

$\mu(x*y) = \beta(f(x*y)) = \beta(f(x)*' f(y)) \leq max\{\beta(f(x)), \beta(f(y))\}$
$= max\{\mu(x), \mu(y)\}$, and the proof is completed. ⌂

Definition 3.9. An anti-fuzzy subset μ of AB-algebra $(X;*,0)$ has inf property if for any subset T of X, there exist $t_0 \in$ T such that $\mu(t_0) = inf_{t\in T}\mu(t)$.

Theorem 3.10. Let $f:(X;*,0) \to (Y;*`,0`)$ be an epimorphism between AB-algebras X and Y respectively and f has inf property. For every anti-fuzzy AB-subalgebra of X, $f(\mu)$ is an anti-fuzzy AB-subalgebra of Y.

Proof: By definition $\beta(y') = f(\mu)(y') = inf_{x\in f^{-1}(y')}\mu(x)$, for all $y' \in Y$.
We have to prove that $\beta(x'*y') \leq max\{\beta(x'), \beta(y')\}$, for all $x', y' \in Y$.
Let $f:(X;*,0) \to (Y;*`,0`)$ be an epimorphism of AB-algebras, μ is an anti-fuzzy AB-subalgebra of X with inf property and β the image of μ under f, since μ is anti-fuzzy AB-subalgebra of X, for any x', y'∈ Y, let $x_0 \in f^{-1}(x')$, $y_0 \in f^{-1}(y')$ be such that
$\mu(x_0) = inf_{t\in f^{-1}(x')}\mu(t)$, $\mu(y_0) = inf_{t\in f^{-1}(y')}\mu(t)$ and $\mu(x_0*y_0) = inf_{t\in f^{-1}(x'*y')}\mu(t)$. Then
$\beta(x'*y') = inf_{t\in f^{-1}(x'*y')}\mu(t) = \mu(x_0*y_0)$
$\leq max\{\mu(x_0), \mu(y_0)\}$
$= max\{inf_{t\in f^{-1}(x')}\mu(t), inf_{t\in f^{-1}(y')}\mu(t)\}$
$= max\{\beta(x'), \beta(y')\}$.
Hence β is an anti-fuzzy AB-subalgebra of Y. ⌂

4. Fuzzy AB-ideals and Homomorphism of AB-algebras

In this section, we will introduce a new notion called an anti-fuzzy AB-ideal of AB-algebra and study several basic properties of it.

Definition 4.1. Let $(X;*,0)$ be an AB-algebra, a fuzzy subset μ of X is called **a anti-fuzzy AB-ideal of X** if it satisfies the following conditions, for all $x, y, z \in X$,
($AFAB_1$) $\mu(0) \leq \mu(x)$,
($AFAB_2$) $\mu(x*z) \leq max\{\mu((x*y)*z), \mu(y)\}$.

Example 4.2. Let X = {0, 1, 2, 3} be a set with the following table:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	2	1	0

Then $(X;*,0)$ is an AB-algebra. It is easy to show that I_1 ={0, 1}and I_2={0, 1, 2, 3} are AB-ideals of X.
Define a fuzzy subset $\mu: X \to [0,1]$ by $\mu(0) = t_1$, $\mu(1) = \mu(2) = \mu(3) = t_2$, where $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$.
Routine calculation gives that μ is an anti-fuzzy AB-ideal of AT-algebras X.

Lemma 4.3. Let μ be an anti-fuzzy AB-ideal of AB-algebra$(X;*,0)$ and if $\leq y$, then $\mu(x) \leq \mu(y)$, for all $x, y \in X$.
Proof: Assume that $\leq y$, then $x*y = 0$, and

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 5 Issue 12, December - 2021, Pages:82-90

$\mu(x*0) = \mu(x) \le max\{\mu((x*y)*0), \mu(y)\} = max\{\mu(0), \mu(y)\} = \mu(y)$. Hence $\mu(x) \le \mu(y)$. ⌂

Proposition 4.4. Let μ be an anti-fuzzy AB-ideal of AB-algebra $(X;*,0)$. If the inequality $y*x \le z$ hold in X, then $\mu(y) \le max\{\mu(x), \mu(z)\}$.

Proof: Assume that the inequality $y*x \le z$ hold in X, by Lemma (4.3),
$\mu(y*x) \le \mu(z)$ --- (1).
By ($AFAB_2$), $\mu(y*z) \le max\{\mu((y*x)*z), \mu(x)\}$. Put $z = 0$, then
$\mu(y*0) = \mu(y) \le max\{\mu((y*x)*0), \mu(x)\} = max\{\mu(y*x), \mu(x)\}$ --- (2).
From (1) and (2), we get $\mu(y) \le max\{\mu(x), \mu(z)\}$, for all $x, y, z \in X$. ⌂

Theorem 4.5. A fuzzy subset μ of an AB-algebra $(X;*,0)$ is an anti-fuzzy AB-ideal of X if and only if for every t∈[0,1], $U(\mu,t)$ is an AB-ideal of X, where $U(\mu,t) \neq \emptyset$.

Proof: Assume that μ is an anti-fuzzy AB-ideal of X, by ($AFAB_1$), we have
$\mu(0) \le \mu(x)$ for all $x \in X$, therefore $\mu(0) \le \mu(x) \le t$, for $x \in U(\mu,t)$ and so $0 \in U(\mu,t)$.
Let $((x*y)*z) \in U(\mu,t)$ and $y \in U(\mu,t)$, then $\mu((x*y)*z) \le t$ and
$\mu(y) \le t$, since μ is an anti-fuzzy AB-ideal it follows that
$\mu(x*z) \le max\{\mu((x*y)*z), \mu(y)\} \le t$ and that $x*z \in U(\mu,t)$. Hence $U(\mu,t)$is an AB-ideal of X.

Conversely, we only need to show that ($AFAB_1$) and ($AFAB_2$) are true. If ($AFAB_1$) is false, then there exist $x \in X$such that $\mu(0) > \mu(x)$.
If we take $t = \frac{1}{2}(\mu(x) + \mu(0))$, then $\mu(0) > t$ and $0 \le \mu(x) < t \le 1$ thus $x \in U(\mu,t)$ and $U(\mu,\alpha) \neq \emptyset$. As $U(\mu,t)$is an AB-ideal of X, we have $0 \in U(\mu,t)$ and so $\mu(0) \le t$. This is a contradiction.
Now, assume ($AFAB_2$) is not true, then there exist $x, y, z \in X$such that,
$\mu(x*z) > max\{\mu((x*y)*z), \mu(y)\}$.
Putting $t = \frac{1}{2}\{\mu(x*z) + max\{\mu((x*y)*z), \mu(y)\}\}$,then $\mu(x*z) > t$ and
$0 \le max\{\mu((x*y)*z), \mu(y)\}, < t \le 1$, hence $\mu((x*y)*z) < t$ and $\mu(y) < t$, imply that $x*z \in U(\mu,t)$, since $U(\mu,t)$is an anti-fuzzy AB-ideal, it follows that
$x*z \in U(\mu,t)$ and that $\mu(x*z) \le t$, this is also a contradiction. Hence μ is an anti-fuzzy AB-ideal of X. ⌂

Proposition 4.6.
The intersection of any set of anti-fuzzy AB-ideals of AB-algebra $(X;*,0)$ is also anti-fuzzy AB-ideal.
Proof:
Let $\{\mu_i \mid i \in \Lambda\}$ be a family of anti-fuzzy AB-ideals of AB-algebra X, by ($AFAB_1$), we have
$\mu(0) \le \mu(x)$ for all $x \in X$, therefore $\bigcap_{i\in\Lambda} \mu_i(0) \le \bigcap_{i\in\Lambda} \mu_i(x)$.
By ($AFAB_2$), for any $x, y, z \in X$, i ∈Λ,
$(\bigcap_{i\in\Lambda} \mu_i)(x*z) = inf(\mu_i(x*z)) \le \inf\{max\{\mu_i((x*y)*z), \mu_i(y)\}\}$
$\le max\{inf(\mu_i((x*y)*z)), inf(\mu_i(y))\}$
$= max\{(\bigcap_{i\in\Lambda} \mu_i)((x*y)*z)), (\bigcap_{i\in\Lambda} \mu_i)(x)\}$. ⌂
Hence $\bigcap_{i\in\Lambda} \mu_i$is an anti-fuzzy AB-ideal of X.

Proposition 4.7.
The union of any set of anti-fuzzy AB-ideals of AB-algebra $(X;*,0)$ is also anti-fuzzy AB-ideal of X where is chain.
Proof: Since $\{\mu_i \mid i\in\Lambda\}$ be a family of anti-fuzzy AB-ideals of X. For any $, y \in X$, suppose $((x*y)*z) \in \cup_{i\in\Lambda} \mu_i$ and $y \in \cup_{i\in\Lambda} \mu_i$, for all i∈Λ. It follows that $((x*y)*z) \in \mu_i$, $y \in \mu_i$, for some i∈Λ. By assumption $\mu_i \subseteq \mu_k$. Hence $((x*y)*z) \in \mu_k$, $y \in \mu_k$, but μ_k is a anti-fuzzy AB-ideal of X , it follows that $x*z \in \mu_k$, therefore, $x*z \in \cup_{i\in\Lambda} \mu_i$. Hence $\cup_{i\in\Lambda} \mu_i$ is anti-fuzzy AB-ideal of X. □

Proposition 4.8. Every anti-fuzzy AB-ideal of AB-algebra $(X;*,0)$ is a anti-fuzzy AB-subalgebra of .
Proof: Since μ is an anti-fuzzy AB-ideal of AB-algebra X, then by Theorem (4.5) , for every t ∈ [0,1] , $U(\mu,t)$ is AB-ideal of X . By Proposition (2.9), for every t ∈ [0,1], $U(\mu,t)$ is AB-subgalgebra of X . Hence μ is an anti-fuzzy AB-subalgebra of X by Theorem (3.3). □

Remark 4.9. The converse of proposition (4.8) is not true as the following example:
Example 4.10. Let X = {0, 1, 2, 3, 4} in which $(*)$ is defined by the following table:

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 5 Issue 12, December - 2021, Pages:82-90

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	0	0	0	0
3	3	2	1	0	0
4	4	3	4	3	0

Then $(X;*,0)$ is an AB-algebra. It is easy to show that I ={0, 1, 2} is AB-subalgebra of .

Define a fuzzy subset $\mu: X \to [0,1]$ by μ (x) =$\begin{cases} 0.2 & if x \in I \\ 0.7 & if x \notin I \end{cases}$

Then μ is an anti-fuzzy AB-subalgebra of X, but μ is an anti-fuzzy AB-ideal of X since

$\mu\,(4*2) = \mu(4) = 0.7 > \max\{\mu((4*1)*2), \mu(1)\} = \max\{\mu(3*2), \mu(1)\} = \max\{\mu(1), \mu(1)\} = \mu(1) = 0.2$.

Corollary 4.11. If a fuzzy subset μ of AB-algebra $(X;*,0)$ is an anti-fuzzy AB-ideal, then for every $t \in Im(\mu)$, $U(\mu,t)$is an AB-ideal of X.

Corollary 4.12. Let I be an AB-ideal of an AB-algebra $(X,*,0)$, then for any fixed number t in an open interval (0,1), there exist an anti-fuzzy AB-ideal μ of X such that $U(\mu,t)$=I.

Proof: Define $\mu: X \to [0:1]$ by μ (x) =$\begin{cases} 0 & if x \in I \\ t & if x \notin I \end{cases}$. Where t is a fixed number in (0,1).

Clearly, μ (0) ≤ μ (x) and we have one two level sets $U(\mu,0) = I, U(\mu,t) = X$, which are AB-ideals of X, then from Theorem (4.10), μ is an anti-fuzzy AB-ideal of X. ⌂

Theorem 4.13. A homomorphic pre-image of anti-fuzzy AB-ideal is also an anti-fuzzy AB-ideal.

Proof: Let $f: (X; *, 0) \to (Y; *`, 0`)$ be a homomorphism of AB-algebras, β is an anti-fuzzy AB-ideal of Y and μ the pre-image of β under f , then

$\beta(f(x)) = \mu(x)$, for all $x \in X$.

Let $x \in X$, then $\mu(0) = \beta(f(0)) \leq \beta(f(x)) = \mu(x)$.

Now, let $x, y, z \in X$, then

$\mu\,(x*z) = \beta(f\,(x*z)) = \beta(f\,(x)*'f\,(z))$

$\leq max\{\beta((f\,(x)*'f\,(y))*'f\,(z)), \beta(f\,(y))\}$

$= max\{\beta(f\,((x*y)*z)), \beta(f\,(y))\}$

$= max\{\mu\,((x*y)*z), \mu\,(y)\}$, and the proof is completed. ⌂

Theorem 4.14. Let $f: (X; *, 0) \to (Y; *`, 0`)$ be an epimorphism between AB-algebras X and Y respectively and f has inf property. For every anti-fuzzy AB-ideal of X, $f(\mu)$ is an anti-fuzzy AB-ideal of Y.

Proof: By definition β(y') = $f\,(\mu)$(y')= $inf_{x \in f^{-1}(y')}\,\mu\,(x)$, for all y'∈ Y.

We have to prove that $\beta\,(x'*z') \leq max\{\beta((x'*y')*z'), \beta(y')\}$, for all $x', y', z' \in Y$.

Let $f: (X; *, 0) \to (Y; *`, 0`)$ be an epimorphism of AB-algebras , μ is an anti-fuzzy AB-ideal of X with inf property and β the image of μ under f , since μ is anti-fuzzy AB-ideal of X, we have $\mu\,(0) \leq \mu\,(x)$ for all $x \in X$. Note that $0 \in f^{-1}$ (0'), where 0, 0'are the zero of X and Y, respectively.

Thus β (0') = $inf_{t \in f^{-1}(x')}\,\mu\,(t)$ = β (x'), for all x ∈ X, which implies that

β (0') ≤$inf_{t \in f^{-1}(x')}\,\mu\,(t)$ = β (x'), for any x'∈ Y.

For any x', y', z'∈ Y, let $x_0 \in f^{-1}$ (x'), $y_0 \in f^{-1}$ (y'), $z_0 \in f^{-1}$ (z') be such that

$\mu\,((x_0*y_0)*z_0) = inf_{t \in f^{-1}((x'*y')*z')}\,\mu\,(t)$, $\mu\,(y_0) = inf_{t \in f^{-1}(y')}\,\mu\,(t)$ and

$\mu\,(x_0*z_0)$=$inf_{t \in f^{-1}(x'*z')}\,\mu\,(t)$. Then

$\beta\,(x'*z') = inf_{t \in f^{-1}(x'*z')}\,\mu\,(t) = \mu\,(x_0*z_0)$

$\leq max\{\mu\,((x_0*y_0)*z_0)\,, \mu\,(y_0)\}$

$= \max\{inf_{t \in f^{-1}((x'*y')*z')}\,\mu\,(t), inf_{t \in f^{-1}(y')}\,\mu\,(t)\}$

$= max\{\beta((x'*y')*z')), \beta(y')\}$.

Hence β is an anti-fuzzy AB-ideal of Y. ⌂

6. Cartesian product of anti-fuzzy AB-ideals

In this section, we discuss the Cartesian product of anti-fuzzy AB-ideals on AB-algebras and establish some of its properties in detail on the basis of anti-fuzzy AB-ideal as [3, 17].

Definition 6.1 ([1],[16]). A fuzzy relation R on any set S is a fuzzy subset

R: $S \times S \rightarrow [0,1]$.

Definition 6.2 ([1]). If R is a fuzzy relation on sets S and β is a fuzzy subset of , then R is a fuzzy relation on β if $R(x,y) \geq max\{\beta(x), \beta(y)\}$,, for all $x, y \in S$.

Definition 6.3([1]). Let μ and β be fuzzy subsets of a set . The Cartesian product of μ and β is defined by $(\mu \times \beta)(x,y) = max\{\mu(x), \beta(y)\}$, , for all $, y \in S$.

Lemma 6.4([1]). Let S be a set and μ and β be fuzzy subsets of S . Then

(1) $\mu \times \beta$is a fuzzy relation on S,

(2) $(\mu \times \beta)_t = \mu_t \times \beta_t$, for all $t \in [0,1]$.

Definition 6.5([1]). Let S be a set and β be fuzzy subset of S. The strongest fuzzy relation on S, that is, a fuzzy relation on β is R_β given by

$R_\beta(x,y) = max\{\beta(x), \beta(y)\}$, for all $x, y \in S$.

Lemma 6.6([1]). For a given fuzzy subset β of a set S, let $R_\beta$be the strongest fuzzy relation on S. Then for t $\in$[0,1], we have $(R_\beta)_t = \beta_t \times \beta_t$.

Proposition 6.7. For a given fuzzy subset β of an AB-algebra $(X; *, 0)$, let R_β be the strongest fuzzy relation on X . If β is an anti-fuzzy AB-ideal of $\times X$, then

R_β (x,x) $\geq R_\beta$ (0,0) , for all $x \in X$.

Proof: Since R_β is a strongest fuzzy relation of $X \times X$, it follows from that,

R_β (x, x) = max{ β (x), β (x)}$\geq$ max{ β (0), β (0)}= R_β (0,0),which implies that

R_β (x,x) $\geq R_\beta$ (0,0).⌂

Proposition 6.8. For a given fuzzy subset β of an AB-algebra $(X; *, 0)$, let R_β be the strongest fuzzy relation on X . If $R_\beta$is an anti-fuzzy AB-ideal of $X \times X$, then

$\beta(0) \leq \beta(x)$, for all $\in X$.

Proof: Since R_β is an anti-fuzzy AB-ideal of $X \times X$, it follows from ($AFAB_1$),

R_β (x,x) $\geq R_\beta$ (0,0), where (0,0) is the zero element of $X \times X$. But this means that, $max\{\beta(x), \beta(x)\} \geq max\{\beta(0), \beta(0)\}$which implies that $\beta(0) \leq \beta(x)$. ⌂

Remark 6.9([9]). Let $(X; *, 0)$ and $(Y; *', 0')$ be AB-algebras, we define * on $X \times Y$ by: for all $(x,y), (u,v) \in X \times Y$, $(x,y) * (u,v) = (x * u, y * v)$. Then clearly $(X \times Y; *, (0,0))$ is an AB-algebra.

Theorem 6.10. Let μ and β be an anti-fuzzy AB-ideals of AB-algebra $(X; *, 0)$. Then $\mu \times \beta$ is an anti-fuzzy AB-ideal of $X \times X$.

Proof: Note first that for every $(x,y) \in X \times X$,

$(\mu \times \beta)(0,0) = max\{\mu(0), \beta(0)\} \leq max\{\mu(x), \beta(y)\} = (\mu \times \beta)(x,y)$.

Now, let (x_1,x_2) , (y_1,y_2) , $(z_1,z_2) \in X \times X$. Then

$(\mu \times \beta)(x_1 * z_1, x_2 * z_2)$ = max { μ ($x_1 * z_1$) , β ($x_2 * z_2$)}

$\leq$ max { max{ μ (($x_1 * y_1$) $* z_1$) , μ (y_1)}, max{ β (($x_2 * y_2$) $* z_2$), β (y_2)}}

= max { max{ μ (($x_1 * y_1$) $* z_1$) , β (($x_2 * y_2$) $* z_2$)}, max{ μ (y_1), β (y_2)}}

= max { $(\mu \times \beta)$(($x_1 * y_1$) $* z_1$) ,(($x_2 * y_2$) $* z_2$) , $(\mu \times \beta)$ (y_1,y_2)}

Hence $(\mu \times \beta)$ is an anti-fuzzy AB-ideal of $X \times X$.⌂

Theorem 6.11. Let μ and β be anti-fuzzy subsets of AB-algebra $(X; *, 0)$ such that $\mu \times \beta$ is an anti-fuzzy AB-ideal of $X \times X$. Then for all $x \in X$,

(i) either $\mu(0) \leq \mu(x)$ or $\beta(0) \leq \beta(x)$.

(ii) $\mu(0) \leq \mu(x)$ for all x$\in$X ,then either $\beta(0) \leq \beta(x)$ or $\beta(0) \leq \mu(x)$.

(iii) If $\beta(0) \leq \beta(x)$for all x$\in$X , then either $\mu(0) \leq \mu(x)$ or $\mu(0) \leq \beta(x)$.

(iv) Either μ or β is an anti-fuzzy AB-ideal of .

Proof.

International Journal of Engineering and Information Systems (IJEAIS)
ISSN: 2643-640X
Vol. 5 Issue 12, December - 2021, Pages:82-90

(i) Suppose that $\mu(0) > \mu(x)$ and $\beta(0) > \beta(y)$, for some $, y \in X$. Then
$(\mu \times \beta)(x, y) = max\{\mu(x), \beta(y)\} < max\{\mu(0), \beta(0)\} = (\mu \times \beta)(0,0)$. This is a contradiction and we obtain (i).
(ii) Assume that there exist $x, y \in X$ such that $\beta(0) > \mu(x)$ and $(0) > \beta(y)$. Then $(\mu \times \beta)(0,0) = max\{\mu(0), \beta(0)\} = \beta(0)$ it follows that
$(\mu \times \beta)(x, y) = max\{\mu(x), \beta(y)\} < \beta(0) = (\mu \times \beta)(0,0)$ which is a contradiction. Hence (ii) holds.
(iii) Is by similar method to part (ii).
(iv) Suppose $\beta(0) \leq \beta(x)$ by (i), then form (iii) either $\mu(0) \leq \mu(x)$ *or*
$\mu(0) \leq \beta(x)$, for all $\in X$.
If $\mu(0) \leq \beta(x)$, for any $x \in X$, then $(\mu \times \beta)(0, x) = max\{\mu(0), \beta(x)\} = \beta(x)$. Let (x_1,x_2), (y_1,y_2), $(z_1,z_2) \in X \times X$, since$(\mu \times \beta)$ is an anti-fuzzy AB-ideal of $X \times X$, we have
$(\mu \times \beta)(x_1 * z_1, x_2 * z_2) \leq \max\{(\mu \times \beta)((x_1 * y_1) * z_1),((x_2 * y_2) * z_2), (\mu \times \beta)(y_1,y_2)\}$---- (A)
If we take $x_1 = y_1 = z_1 = 0$, then
$\beta(x_2 * z_2) = (\mu \times \beta)(0, x_2 * z_2)$
$\leq \max\{(\mu \times \beta)(0,((x_2 * y_2) * z_2), (\mu \times \beta)(0,y_2)\}$
$= \max\{\max\{\mu(0), \beta((x_2 * y_2) * z_2)\}, \max\{\mu(0), \beta(y_2)\}\}$
$= \max\{\beta((x_2 * y_2) * z_2), \beta(y_2)\}$
This prove that β is an anti-fuzzy AB-ideal of .
Now we consider the case $(0) \leq \mu(x)$, for all $x \in X$. Suppose that $\mu(0) > \mu(y)$ for some $\in X$. then $\beta(0) \leq \beta(y) < \mu(0)$.
Since $\mu(0) \leq \mu(x)$, for all $x \in X$, it follows that $\beta(0) < \mu(x)$ for any $x \in X$.
Hence $(\mu \times \beta)(x, 0) = max\{\mu(x), \beta(0)\} = \mu(x)$ taking $x_2 = y_2 = z_2 = 0$ in (A), then
$\mu(x_1 * z_1) = (\mu \times \beta)(x_1 * z_1, 0)$
$\leq \max\{(\mu \times \beta)((x_1 * y_1) * z_1), 0), (\mu \times \beta)(y_1, 0)\}$
$= \max\{\max\{\mu(x_1 * (y_1 * z_1)), \beta(0)\}, \max\{\mu(y_1), \beta(0)\}\}$
$= \max\{\mu(x_1 * (y_1 * z_1)), \mu(y_1)\}$
Which proves that μ is an anti-fuzzy AB-ideal of X. Hence either μ or β is an anti-fuzzy AB-ideal of X. ⌂

Theorem 6.12. Let β be a fuzzy subset of an AB-algebra $(X; *, 0)$ and let R_β be the strongest fuzzy relation on , then β is an anti-fuzzy AB-ideal of X if and only if R_β is an anti-fuzzy AB-ideal of $X \times X$.

Proof: Assume that β is an anti-fuzzy AB-ideal of . By Proposition (6.7), we get, $R_\beta(0,0) \leq R_\beta(x,y)$, for any $(x,y) \in X \times X$.
Let (x_1,x_2), (y_1,y_2), $(z_1,z_2) \in X \times X$, we have from ($AFAB_2$) :
$R_\beta(x_1 * z_1, x_2 * z_2) = \max\{\beta(x_1 * z_1), \beta(x_2 * z_2)\}$
$\leq \max\{\max\{\beta((x_1 * y_1)*z_1), \beta(y_1)\}, \max\{\beta((x_2 * y_2)*z_2), \beta(y_2)\}\}$
$= \max\{\max\{\beta((x_1 * y_1)*z_1), \beta((x_2 * y_2)*z_2))\}, \max\{\beta(y_1), \beta(y_2)\}\}$
$= \max\{R_\beta(((x_2 * y_2)*z_2), ((x_2 * y_2)*z_2))), R_\beta(y_1, y_2)\}$
Hence R_β is an anti-fuzzy AB-ideal of $X \times X$.
Conversely, suppose that R_β is an anti-fuzzy AB-ideal of $X \times X$, by Proposition (6.8) $\beta(0) \leq \beta(x)$, for all $x \in X$, which prove ($AFAB_1$).
Now, let (x_1,x_2), (y_1,y_2), $(z_1,z_2) \in X \times X$. Then,
$\max\{\beta(x_1 * z_1), \beta(x_2 * z_2)\} = R_\beta(x_1 * z_1, x_2 * z_2)$
$\leq \max\{R_\beta(((x_1,x_2) * (y_1, y_2)) * (z_1, z_2)), R_\beta(y_1,y_2)\}$
$= \max\{R_\beta(((x_1 * y_1)*z_1),((x_2 * y_2)*z_2)), R_\beta(y_1,y_2)\}$
$= \max\{\max\{\beta((x_1 * y_1)*z_1), \beta((x_2 * y_2)*z_2)\}, \max\{\beta(y_1), \beta(y_2)\}\}$
In particular if we take $x_2 = y_2 = z_2 = 0$, then
$\beta(z_1 * x_1) \leq \max\{\beta(z_1 * (y_1 * x_1)), \beta(y_1)\}$.
This proves ($AFAB_2$) and β is an anti-fuzzy AB-ideal of . ⌂

References

[1] P. Bhattacharye and N.P. Mukheriee, **Fuzzy Relations and Fuzzy Group**, Inform Sci., vol. 36 (1985), pp:267-282.
[2] A.T. Hameed and B.H. Hadi, **Anti-Fuzzy AT-Ideals on AT-algebras,** Journal Al-Qadisyah for Computer Science and Mathematics , vol.10, no.3(2018), 63-74.
[3] A.T. Hameed and B.H. Hadi, **Cubic Fuzzy AT-subalgebras and Fuzzy AT-Ideals on AT-algebra,** World Wide Journal of Multidisciplinary Research and Development, vol.4, no.4(2018), 34-44.
[4] A.T. Hameed and B.H. Hadi, **Intuitionistic Fuzzy AT-Ideals on AT-algebras,** Journal of Adv Research in Dynamical & Control Systems, vol.10, 10-Special Issue, 2018
[5] A.T. Hameed and B.N. Abbas**, AB-ideals of AB-algebras,** Applied Mathematical Sciences, vol.11, no.35 (2017), pp:1715-1723.

[6] A.T. Hameed and B.N. Abbas**, Derivation of AB-ideals and fuzzy AB-ideals of AB-algebra**, LAMBERT Academic Publishing, 2018.
[7] A.T. Hameed and B.N. Abbas**, On Some Properties of AB-algebras,** Algebra Letters, vol.7 (2017), pp:1-12.
[8] A.T. Hameed and B.N. Abbas**, Some properties of fuzzy AB-ideal of AB-algebras,** Journal of AL-Qadisiyah for Computer Science and Mathematics, vol.10, no. 1(2018), pp:1-7.
[9] A.T. Hameed and E.K. Kadhim, **Interval-valued IFAT-ideals of AT-algebra**, Journal of Physics: Conference Series (IOP Publishing), 2020, pp:1-5.
[10] A.T. Hameed and N.H. Malik, (2021), **(β, α)-Fuzzy Magnified Translations of AT-algebra**, Journal of Physics: Conference Series (IOP Publishing), 2021, pp:1-13.
[11] A.T. Hameed and N.J. Raheem, (2020), **Hyper SA-algebra**, International Journal of Engineering and Information Systems (IJEAIS), vol.4, Issue 8, pp.127-136.
[12] A.T. Hameed and N.J. Raheem, (2021), **Interval-valued Fuzzy SA-ideals with Degree (λ,κ) of SA-algebra**, Journal of Physics: Conference Series (IOP Publishing), 2021, pp:1-13.
[13] A.T. Hameed, , R.A. Flayyih and S.H. Ali, (2021), **Bipolar hyper Fuzzy AT-ideals of AT-Algebra**, International Journal of Engineering and Information Systems (IJEAIS), vol.4, Issue 3, pp.145-152.
[14] A.T. Hameed, A.H. Abed and I.H. Ghazi, (2020), **Fuzzy β-magnified AB-ideals of AB-algebras**, International Journal of Engineering and Information Systems (IJEAIS), vol.4, Issue 6, pp.8-13.
[15] A.T. Hameed, A.S. abed and A.H. Abed, (2018), **TL-ideals of BCC-algebras**, Jour of Adv Research in Dynamical & Control Systems, Vol. 10, 11- Special Issue, 2018
[16] A.T. Hameed, **AT-ideals and Fuzzy AT-ideals of AT-algebras**, Journal of Iraqi AL-Khwarizmi Society, vol.1, no.2, (2018).
[17] A.T. Hameed, F. F. Kareem and S.H. Ali, **Hyper Fuzzy AT-ideals of AT-algebra**, Journal of Physics: Conference Series (IOP Publishing), 2021, pp:1-15.
[18] A.T. Hameed, **Fuzzy ideals of some algebras**, PH. Sc. Thesis, Ain Shams University, Faculty of Sciences, Egypt, 2015.
[19] A.T. Hameed, H.A. Faleh and A.H. Abed, (2021), **Fuzzy Ideals of KK-algebra**, Journal of Physics: Conference Series (IOP Publishing), 2021, pp:1-7.
[20] A.T. Hameed, H.A. Mohammed and A.H. Abed, **Anti-fuzzy AB-ideals of AB-algebras,** to appear (2021).
[21] A.T. Hameed, I.H. Ghazi and A.H. Abed, (2019), **Big Generalized fuzzy AB-Ideal of AB-algebras**, Jour of Adv Research in Dynamical & Control Systems, vol. 11, no.11(2019), pp:240-249.
[22] A.T. Hameed, I.H. Ghazi and A.H. Abed, (2020), **Fuzzy α-translation AB-ideal of AB-algebras**, Journal of Physics: Conference Series (IOP Publishing), 2020, pp:1-19.
[23] A.T. Hameed, S. Mohammed and A.H. Abed, (2018), **Intuitionistic Fuzzy KUS-ideals of KUS-Algebras**, Jour of Adv Research in Dynamical & Control Systems, Vol. 10, 11-Special Issue, 2018, pp:154-160.
[24] A.T. Hameed, S.H. Ali and , R.A. Flayyih, **The Bipolar-valued of Fuzzy Ideals on AT-algebra**, Journal of Physics: Conference Series (IOP Publishing), 2021, pp:1-9.
[25] S.M. Mostafa, A.T. Hameed and A.H. Abed, **Fuzzy KUS-ideals of KUS-algebra**, Basra Journal of Science (A), vol.34, no.2 (2016), 73-84.
[26] S.M. Mostafa, M.A. Abdel Naby, F. Abdel-Halim and A.T. Hameed, **On KUS-algebra**, International Journal of Algebra, vol.7, no.3(2013), 131-144.
[27] S.M. Mostafa, M.A. Abdel Naby, F. Abdel-Halim and A.T. Hameed, **Interval-valued fuzzy KUS-ideal** ,IOSR Journal of Mathematics (IOSR-JM), vol.5, Issue 4(2013) , 61-66 .
[28] W.A. Dudek, **The Number of Sub-algebras of Finite BCC-algebras**, Bull. Inst. Math. Academia Sonica 20 (1992), pp:129-136.
[29] K. Is´eki and S. Yanaka, **An Introduction to Theory of BCK-algebras**, Math. Japonica, vol. 23 (1979), pp:1-20.
[30] Y. B. Jun, S. M. Hong and E. H. Roh, **Fuzzy characteristic sub-algebras /ideals of a BCK-algebra**, Pusan Kyongnam Math. J.(presently East Asian Math. J.), vol. 9, no.1 (1993), 127-132.
[31] Y. Komori, **The Class of BCC-algebras isnot A Variety**, Math. Japon., vol. 29 (1984), pp:391-394.
[32] J. Meng and Y. B. Jun, **BCK-algebras**, Kyung Moon Sa Co., Korea, 1994.
[33] C. Prabpayak and U. Leerawat, **On Ideals and Congurences in KU-algebras**, Scientia Magna J., vol. 5, no.1 (2009), pp:54-57.
[34] C. Prabpayak and U. Leerawat, **On Isomorphisms of KU-algebras**, Scientia Magna J., vol. 5, no.3 (2009), pp:25-31.
[35] O.G. Xi, **Fuzzy BCK-algebras**, Math. Japan., vol. 36(1991), pp:935-942.
[36] L. A. Zadeh, **Fuzzy sets**, Inform. Control, vol.8 (1965), pp:338-353.