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Fuzzy Filter Of KK–Algebra

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Abstract*— In the paper, we introduce a new concepts fuzzy filter in KK-algebra, implicative fuzzy filter and positive implicative fuzzy filter of a KK-algebra. Also, we stated and prove some theorems and proposition which determine the relationship between these notions.*

Keywords— KK-algebra, fuzzy filter, implicative fuzzy filter and positive implicative fuzzy filter, the cartesian product of fuzzy filter in KK-algebra.

1. Introduction

In [4], Mang studied filter theory in BE-algebra and gave some characterizations of BE-algebra. In [1], saeid and el ct , studied some types of filters in BE-algebra . In [5], C. prabpyak and U. Leerawat, Introduced a new algebraic structure which is called KU-algebras. They gave the concept of homomorphisms of KU-algebras and investigated some related properties. In [8], L.A. Zadeh, Introduced a concept of fuzzy subset and various operations on it. In [7], A. Rosenfeld, Introduced of the concept of fuzzy subgroups . In [9], S. Asawasamrit and A. Sudprasert, Introduced a new algebraic structure called a KK-algebra and described the relationships between ideals and congruences in this algebra. In [10], S. Asawasmrit, studied isomorphism of a KK-algebra. In [2], A.T. Hameed, H.A. Falh and A.H. Abad, Introduced the notion of fuzzy ideal in KK-agebra and discussed some of its properties. In [3], Introduced the concept of filter and studied a new types of filters. This paper, We introduced the notion of the fuzzy filter in KK-algebra and gave some examples and proofs of it. we also studied new types of fuzzy filter and studied the relationship between them.

2.Preliminaries

Now, we give some definitions and preliminary results need in later section.

Definition 2.1. [2,3] .

Let $(X;*,0)$ be a set with a binary operation $(*)$ and a constant (0). Then X is called a **KK-algebra** if it satisfies the following: for all $x,y,z \in X$,
(1) $(x*y)*((y*z)*(x*z))=0$,
(2) $0*x=x$,
(3) $x*y=0$ and $y*x=0$ if and only if $x=y$.
For brevity we also call X a **KK-algebra,** we can define a binary relation $(\leq)$ by putting $x \leq y$ if and only if $y*x=0$.

Theorem 2.2. [2] .

Let $(X;*,0)$ be a KK-algebra if and only if it satisfies the following condition: for all $x,y,z \in X$,
(1) $(x*y)*\big((y*z)*(x*z)\big)=0$,
(2) $x*\big((x*y)*y\big)=0$,
(3) $x*x=0$,
(4) $x*y=0$ and $y*x=0$ if and only if $x=y$.

Definition 2.3 .[3].

If there is a special element e of a KK-algebra X satisfying $x \leq e$, for all $x \in X$, then e is called **a unit of *X*,** we denoted $e*x$ by x^* for every $x \in X$.

Definition 2.4.[3].

If there is an element e of a KK-algebra X satisfying $x \leq e$ for all x in X, the element e is called unite of X. a KK-algebra with unite is called to be **bounded.**

Proposition 2.5.[3].

In a bounded KK-algebra $(X;*,0)$, we have for all $x,y \in X$,
(1) e^*=0 and 0^*=e ,
(2) $x^* * y = y * x$,
(3) $e^* * x = 0$.

Definition 2.6.[2].

Let $(X;*,0)$ be a KK- algebra and let A be a nonempty subset of X. A is called a **KK- subalgebra of X** if $x*y \in A$ whenever $x \in A$ and $y \in A$.

Definition 2.7.[2]. Let $(X;*,0)$ and $(Y;*,0)$ be nonempty sets. The mapping $f:(X;*,0) \to (Y;*',0')$ is called **a homomorphism** if it satisfies $f(x*y) = f(x)*' f(y)$, for all $x, y \in X$.

Proposition 2.8.[2].

Let $f:(X;*,0) \to (Y;*`,0`)$ be a homomorphism of a KK-algebra into a KK-algebra , then :

(F_1) If A is a KK-subalgebra of X, then f (A) is a KK-subalgebra of Y.

(F_2) If I is filter of X, then f (I) is filter in Y.

(F_3) If B is a KK-subalgebra of Y, then $f^{-1}(B)$ is a KK-subalgebra of X .

(F_4) If J is filter in Y, then $f^{-1}(J)$ is filter in X .

Definition 2.9.[3].

A nonempty subset F of a KK-algebra $(X;*,0)$ is called **a filter of X** if it satisfies the following condition: for any $x, y \in X$,

(F_1) $e \in F$,

(F_2) $x*y \in F$ and $x \in F$ imply $y \in F$.

Definition 2.10.[3].

A subset F of a KK-algebra $(X;*,0)$ is said to be **implicative filter of X (I-filter)**, if for any $x, y, z \in X$,

(1) $e \in F$,

(2) $x*(y*z) \in F$ and $x*y \in F$ imply $x*z \in F$.

Definition 2.11.[3].

A subset F of a KK-algebra $(X;*,0)$ is said to be **positive implicative filter of X (P-I-filter),** if for any $x, y, z \in X$,

(1) $e \in F$,

(2) $z*((x*y)*x) \in F$ and $z \in F$ imply $x \in F$.

Definition 2.12. [3].

Let (X; ∗ 0) X be nonempty set , a fuzzy subset μ in X is a function μ:X→[0,1].

Definition 2.13.[3].

Let (X; ∗ 0) be a KK-algebra , a fuzzy subset μ in X is called **a fuzzy subalgebra of** X if $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in X$.

Definition 2.14.[3].

Let f: $(X;*,0) \to (Y;*',0')$ be a mapping nonempty sets X and Y respectively. If μ is a fuzzy subset of X, then the fuzzy subset β of Y defined by: $f(\mu)(y) = \begin{cases} \sup\{\mu(x): x \in f^{-1}(y)\} & if\ f^{-1}(y) = \{x \in X, f(x) = y\} \neq \emptyset \\ 0 \quad otherwise \end{cases}$

is said to be **the image of μ under f** .

Similarly, if β is a fuzzy subset of Y , then the fuzzy subset $\mu = (\beta \; o \; f)$ in X (i.e the fuzzy subset defined by $\mu(x) = \beta(f(x))$, for all $x \in X$ is called **the pre-image of β under f** .

3. Fuzzy filter of KK-algebra

In this section, we introduce definition of fuzzy filter, implicative fuzzy filter and positive implicative fuzzy filter on KK-algebra, and we study its relationship with them on KK-algebra.

Definition 3.1

Let (X; ∗ 0) be a KK-algebra , a fuzzy subset μ of X is called a **fuzzy filter of X** if it satisfies the following condition : for all $x, y \in X$,

(F_1) μ (e) ≥ μ (x),

(F_2) μ (y) ≥ $min\{\mu(x*y), \mu(x)\}$.

Example 3.2.

Let $X = \{0, a, b, c\}$ and let * be binary operation defined by the table :

*	0	a	b	c
0	0	a	b	c
a	0	0	b	c
b	0	0	0	c
c	0	0	0	0

Then $(X;*,0)$ is a KK –algebra and F=$\{a,c\}$ is filter of X. Let μ be the fuzzy subset defined as the following

$$\mu(x)=\begin{cases}0.8 & if\ x=a,c\\ 0.5 & if\ x=0,b\end{cases}$$

Then μ is fuzzy filter of X.

Example 3.3.

Let $X=\{0,a,b,c\}$ and binary operation $*$ is defined by the table :

*	0	a	b	c
0	0	a	b	c
a	0	0	a	b
b	0	0	0	a
c	0	0	0	0

Then $(X;*,0)$ is a KK-algebra with unite c and F $=\{a,c\}$ is filter of X. Let μ be the fuzzy subset defined as the following

$$\mu(x)=\begin{cases}0.7 & if\ x=a,c\\ 0.3 & if\ x=0,b\end{cases}$$

Then μ is not fuzzy filter of X a since

$\mu(b)=0.3\geq min\{\mu(a*b),\mu(a)\}=0.7.$

Example 3.4 .

Let $X=\{0,a,b,c\}$ and binary operation $*$ is defined by the table :

0	0	1	2
0	0	1	2
1	0	0	2
2	0	0	0

Then $(X;*,0)$ is a KK- algebra with unite c and F= $\{0,2\}$ is filter of X. Let μ be the fuzzy subset defined as the following $\mu(x)=\begin{cases}0.9 & if\ x=0,2\\ 0.6 & if\ x=1\end{cases}$. Then μ is fuzzy filter of X .

Lemma 3.5.

Let μ be a fuzzy filter of KK-algebra $(X;*,0)$ and if $x\leq y$, then $\mu(x)\geq\mu(y)$, for all $x,y\in X$.

Proof :

Assume that $x\leq y$, then $y*x=0$,and $\mu(0*x)=\mu(x)\geq\min\{\mu(0*y),\mu(y*x)\}=min\{\mu(y),\mu(0)\}=\mu(y)$.
Hence $\mu(x)\geq\mu(y)$.

Proposition 3.6.

The intersection of any set of fuzzy filters of a KK-algebra X is also fuzzy filter of X.

Proof :

Let $\{\mu_I\ |i\in\Lambda\}$ be a family of fuzzy filters of KK-algebra X, then for any $x,y\in X$, i $\in\Lambda$,
$(\bigcap_{i\in\Lambda}\mu_i)(e)=\inf(\mu_i(e))\geq\inf(\mu_i(x))=(\bigcap_{i\in\Lambda}\mu_i)(x)$ and
$(\bigcap_{i\in\Lambda}\mu_i)(y)=\inf(\mu_i(y))\geq\inf(\min\{\mu_i(x*y)),\mu_i(x)\})$
$=\min\{\inf(\mu_i(x*y)),\inf(\mu_i(x)\}.$
$=\{(\bigcap_{i\in\Lambda}\mu_i)(x*y),(\bigcap_{i\in\Lambda}\mu_i)(x)\}$.Then completes the proof.

Remark 3.7.

Not that the union of two fuzzy filter of KK-algebra it is not necessarily fuzzy filter of KK-algebra, as it is shown in the following example.

Example 3.8.

Let X= {0, a, b, c} and let $*$ be a binary operation defined by the table:

*	0	a	b	c
0	0	a	b	c
a	0	0	b	c

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b	0	0	0	c
c	0	a	b	0

It is clear that $(X;*,0)$ is a KK- algebra. Let $\mu(x)$ and $\eta(x)$be two fuzzy set defined as the following

$$\mu(x)=\begin{cases}0.9 & if\ x=0,b\\ 0.7 & if\ x=a,c\end{cases}\qquad \eta(x)=\begin{cases}0.9 & if\ x=0,a\\ 0.3 & if\ x=b,c\end{cases}$$

Then $\mu(x)$, $\eta(x)$ are fuzzy filter of X. But $(\mu\cup\eta)(x)=\begin{cases}0.9 & if\ x=0,a,b\\ 0.7 & if\ x=c\end{cases}$. Then $\mu\cup\eta$ are not fuzzy filters of X.

Proposition 3.9.
The union of any set of fuzzy filters of KK-algebra X is also fuzzy filter of X, if μ_{i} is chain. Let $\{\mu_{\mathrm{i}}\ |\mathrm{i}\ \in\Lambda\}$ be a family of fuzzy filters of KK-algebra X, $\mu_1\subseteq\mu_2\subseteq\mu_3\ldots$, for all $\mathrm{i}\in\Lambda$. Then $\bigcup_{i\in\Lambda}\mu_i=\mu_k$, for any $k\in Z$, is fuzzy filter of X.

Definition 3.10.
Let $(X;*,0)$ be a KK –algebra , a fuzzy subset μ of X is called **implicative fuzzy filter of X** if it satisfies the following conditions: for all $x,y,z\in X$,
(F_1) $\mu(e)\geq\mu(x)$,
(F_2) $\mu(x*z)\geq\min\{\mu(x*(y*z)),\mu(x*y)\}$.

Proposition 3.11.
Any implicative fuzzy filter of a KK-algebra $(X;*,0)$ is fuzzy filter of X.
Proof: Assume that μ implicative fuzzy filter of X and put $x=0$ in (F_2), we get
$\mu(z)\geq\min\{\mu(0*(y*z),\mu(0*y)\}\geq\min\{\mu(y*z),\mu(y)\}$ This means that μ is fuzzy filter of X.

Remark 3.12.
the converse of proposition (**3.11**) need not be true in general as in the example.

Example 3.13.
Let $X=\{0,a,b,c\}$ and let $*$ be a binary operation defined by the table:

*	0	a	b	c
0	0	a	b	c
a	0	0	b	c
b	0	0	0	c
c	0	0	0	0

Then it can be easily that verified that $(X;*,0)$ is a KK-algebra and $F=\{0,a,c\}$ is a implicative filter of X. Let μ be the fuzzy subset of X defined as the following

$$\mu(\mathrm{x})=\begin{cases}0.9 & if x=0,a,c\\ 0.4 & if\ x=b\end{cases}$$

for all $x\in X$, then clearly μ is fuzzy filter of X, but μ is not a implicative fuzzy filter of X since
$\mu(0*b)=0.4\geq\min\{\mu(0*(c*b),\mu(0*c)\}$ =0.9

Proposition 3.14. The intersection of any set of implicative fuzzy filters of KK-algebra X is also implicative fuzzy filters of X.
Proof :
Let $\{\mu_{\mathrm{I}}\ |\mathrm{i}\ \in\Lambda\}$ be a family of implicative fuzzy filters of a KK-algebra X, then for any $x,y\in X$, $\mathrm{i}\in\Lambda$,
$(\bigcap_{i\in\Lambda}\mu_i)(\mathrm{e})=\inf(\mu_i(e))\geq\inf(\mu_i(x))=(\bigcap_{i\in\Lambda}\mu_i)(x)$ and
$(\bigcap_{i\in\Lambda}\mu_i)(x*\mathrm{z})=\inf(\mu_i(x*z))\geq\inf(\min\{\mu_i(x*y),\mu_i(x*(y*z))\})$
$=\min\{\inf(\mu_i(x*y)),\inf(\mu_i(x*(y*z)))\}$
$=\{(\bigcap_{i\in\Lambda}\mu_i)(x*y),(\bigcap_{i\in\Lambda}\mu_i)(x*(y*z))\}$.
This completes the proof .

Remark 3.15.
Not that the union of two implicative fuzzy filter of KK-algebra it is not necessarily implicative fuzzy filter of KK-algebra, as it is shown in the following example.

Example 3.16.
Let X= {0, a, b, c} and let $*$ be a binary operation defined by the table:

*	0	a	b	c
0	0	a	b	c

a	0	0	b	c
b	0	0	0	c
c	0	a	b	0

It is clear that $(X; *, 0)$ is a KK- algebra. Let $\mu(x)$ and $\eta(x)$be two fuzzy set defined as the following

$\mu(x) = \begin{cases} 0.8 & if\ x = 0, a \\ 0.6 & if\ x = b, c \end{cases}$ $\eta(x) = \begin{cases} 0.8 & if\ x = 0, b \\ 0.5 & if\ x = a, c \end{cases}$. Then $\mu(x), \eta(x)$ are implicative fuzzy filter of X.

But $(\mu \cup \eta)(x) = \begin{cases} 0.8 & if\ x = 0, a, b \\ 0.6 & if\ x = c \end{cases}$. Then $\mu \cup \eta$ are not implicative fuzzy filters of X.

Proposition 3.17.

The union of any set of implicative fuzzy filter of KK-algebra X is also implicative fuzzy filter, if μ_i is chain **.**

Proof:

Let $\{\mu_i \mid i \in \Lambda\}$be family of implicative fuzzy filter of KK-algebra X, $\mu_1 \subseteq \mu_2 \subseteq \mu_3 \ldots$ then for any $k \in z$, for all $i \in \Lambda$.

Then$\bigcup_{i \in \Lambda} \mu_i = \mu_k$, for any $k \in Z$, is implicative fuzzy filter of X.

Definition 3.18 .

Let $(X; * ,0)$ be a KK- algebra, a fuzzy subset μ of X is called **a positive implicate fuzzy filter of X** if it satisfies the following condition: for all $x, y, z \in X$,

(F_1) $\mu(e) \geq \mu(x)$,

(F_2) $\mu(x) \geq \min\{(\mu(z * ((x * y) * x)\}, \mu(z)\}$.

Proposition 3.19 .

Any positive implicative fuzzy filter of a KK-algebra X is a fuzzy filter of X.

Proof:

Assume that μ positive implicative fuzzy filter of X and put z=0 in (F_2) ,we get $\mu(y) \geq \min\{\mu(x), \mu(x * ((y * 0) * y)\}$

$\geq min\{\mu(x), \mu(x * (0 * y)\}$

$\geq \min\{\mu(x), \mu(x * y)\}$. Hence μ is fuzzy filter of X.

Remark 3.20.

The converse of proposition (3.19) need not be true in general as in the example.

Example 3.21.

Let $X = \{0, a, b, c\}$ and let* be a binary operation by the table:

*	0	a	b	c
0	0	a	b	c
a	0	0	b	c
b	0	0	0	c
c	0	0	0	0

Then $(X; * ,0)$ is a KK-algebra and $F = \{0, a, c\}$ is a positive implicative filter of X . Let μ be the fuzzy subset of X defined as the following $\mu(x) = \begin{cases} 0.7 & if\ x = 0,\ a, c \\ 0.3 & if\ x = b \end{cases}$

for all $x \in X$ .Then clearly μ is a fuzzy filter of X, but μ is not a positive implicative fuzzy filter of X since $\mu(b) = 0.3 \geq \min\{\mu(c * ((b * 0) * b), \mu(c)\} = 0.7$.

Proposition 3.22.

The intersection of any set of positive implicative fuzzy filters of a KK-algebra X is also positive implicative fuzzy filter of X **.**

Proof:

Let $\{\mu_I \mid i \in \Lambda\}$ be a family of fuzzy filters of KK-algebra X, then for any $x, y \in X$, $i \in \Lambda$,

$(\bigcap_{i \in \Lambda} \mu_i)(e) = \inf(\mu_i(e)) \geq \inf(\mu_i(x)) = (\bigcap_{i \in \Lambda} \mu_i)(x)$ and

$(\bigcap_{i \in \Lambda} \mu_i)(y) = \inf(\mu_i(y)) \geq \inf(\min\{\mu_i(x * (y * z) * y)), \mu_i(x)\})$

$\geq \min\{\inf(\mu_i(x * (y * z) * y)), \inf(\mu_i(x)\}$

$= \min\{(\bigcap_{i \in \Lambda} \mu_i)(x * (y * z) * y)), (\bigcap_{i \in \Lambda} \mu_i)(x)\}$.

This completes the proof.

Remark 3.23.

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In general, the union of two fuzzy filter of KK-algebra it is not necessarily positive implicative fuzzy filter of KK-algebra, as it is shown in the following example.

Example 3.24.

Let X= {0, a, b, c} and let $*$ be a binary operation defined by the table:

*	0	a	b	c
0	0	a	b	c
a	0	0	b	c
b	0	0	0	c
c	0	0	b	0

It is clear that $(X; *, 0)$ is a KK- algebra. Let $\mu(x)$ and $\eta(x)$be two fuzzy set defined as the following $\mu(x) = \begin{cases} 0.7 & if\ x = 0, a \\ 0.5 & if\ x = c, b \end{cases}$ $\eta(x) = \begin{cases} 0.7 & if\ x = 0, b \\ 0.3 & if\ x = a, c \end{cases}$. Then $\mu(x), \eta(x)$ are positive implicative fuzzy filters of X.

But $(\mu \cup \eta)(x) = \begin{cases} 0.7 & if\ x = 0, a, b \\ 0.5 & if\ x = c \end{cases}$. Then $\mu \cup \eta$ are not positive implicative fuzzy filters of X.

Proposition 2.25.

The union of any set of positive implicative fuzzy filter of a KK-algebra X is also positive implicative fuzzy filter, if μ_i is chain .

Proof:

Let $\{\mu_i \mid i \in \Lambda\}$be family of positive implicative fuzzy filter of KK-algebra X, $\mu_1 \subseteq \mu_2 \subseteq \mu_3 \ldots$, for all i $\in \Lambda$. Then $\bigcup_{i \in \Lambda} \mu_i = \mu_k$, for any $k \in Z$ is positive implicative fuzzy filter of X .

4. Homomorphism of fuzzy filter of KK-algebras

In this section, we describe how to deal with the homomorphism of image and inverse of fuzzy filter, implicative fuzzy filter and positive implicative fuzzy filter.

Theorem 4.1

An into homomorphic pre-image of fuzzy filter of a KK-algebra X is also a fuzzy filter of X.

Proof:

$\beta(e_y) = \beta(f(e_x)) = \mu(e_x)$ and so $\mu(e_x) \geq \mu(x)$ for all $x \in X$. Let $x, y \in Y$ such that $f(a) = x$, $f(b) = y$, for all $a, b \in X$.

$\mu(y) = \beta(f(b)) \geq \min\{\beta(f(a * b)), \beta(f(a))\} = \min\{\beta(f(a)) * f(b)), \beta(f(a))\} = \min\{\mu(x * y), \mu(x)\}$.

Therefore μ is fuzzy filter of X.

Definition 4.2.

Let $f: (X; *, 0) \to (Y; *', 0')$ be an epimorphism of KK-algebra and μ is fuzzy subset of X , $f(\mu)$ is fuzzy subset of X such that $f(\mu)(x) = \mu(f(a))$, for all $a \in X, x \in Y$.

Theorem 4.3.

Let $f: (X; *, 0) \to (Y; *', 0')$ be an epimorphism of a KK-algebra and μ is fuzzy subset of X , If μ is a fuzzy filter of X. Then $f(\mu)$ is a fuzzy filter of Y.

Proof :

Assume that μ is a fuzzy filter of X. For any $x \in X$, we have

$f(\mu)(e_x) = \mu(f(e_x)) = \mu(e_y) \geq \mu(f(x)) = f(\mu)(x)$.

Let $x, y \in Y$ such that $f(a) = x$, $f(b) = y$ for all $a, b \in X$.

$$\begin{aligned} f(\mu)(y) &= \mu(f(b)) \\ &\geq \min\{\mu(f(a * b)), \mu(f(a))\} \\ &= \min\{\mu(f(a) *' f(b)), \mu(f(a))\} \\ &= \min\{f(\mu)(x *' y), f(\mu)(x)\} \end{aligned}$$

Therefore $f(\mu)$ is fuzzy filter of Y.

Theorem 4.4

An into homomorphic pre-image of a implicative fuzzy filter of a KK-algebra is also a implicative fuzzy filter of X.

Proof:

$\beta(e_y) = \beta(f(e_x)) = \mu(e_x)$ and so $\mu(e_x) \geq \mu(x)$ for all $x \in X$.

Let $x, y, z \in Y$ such that $f(a) = x$, $f(b) = y$ and $f(c) = z$ for all $a, b, c \in X$.

$\mu(x * z) = \beta(f(a * c))$

$\geq \min\{ \beta(f(a*(b*c)), \beta(f(a*b))\}$
$= \min\{ \beta(f(a) *' (f(b) *' f(c))), \beta(f(a) *' f(b))\}$
$= \min\{ \mu(x*(y*z), \mu(x*y)\}$

Therefore μ is a implicative fuzzy filter of X.

Theorem 4.5

Let $f:(X;*,0) \to (Y;*',0')$ be an epimorphism of a KK-algebra and μ is fuzzy subset of X , If μ is a implicative fuzzy filter of X. Then $f(\mu)$ is a implicative fuzzy filter of Y.

Proof :

Assume that μ is aimplicative fuzzy filter of X. For any $x \in X$, we have
$f(\mu)(e_x) = \mu(f(e_x)) = \mu(e_y) \geq \mu(f(x)) = f(\mu)(x)$.
Let $x, y \in Y$ such that $f(a) = x$, $f(b) = y$, for all $a, b \in X$.

$f(\mu)(y) = \mu(f(a*c))$
$= \min\{\mu(f(a*(b*c)), \mu(f(a*b))\}$
$= \min\{ \mu(f(a) *' (f(b) *' f(c))), \mu(f(a))\}$
$= \min\{ f(\mu)(x*y), f(\mu)(x)\}$

Therefore $f(\mu)$ is a implicative fuzzy filter of Y.

Theorem 4.6

An into homomorphic pre-image of a positive implicative fuzzy filter of KK-algebra is also a positive implicative fuzzy filter of X.

Proof:

$\beta(e_y) = \beta(f(e_x)) = \mu(e_x)$ and so $\mu(e_x) \geq \mu(x)$, for all $x \in X$. Let $x, y, z \in Y$ such that $f(a) = x$, $f(b) = y$ and $f(c) = z$, for all $a, b, c \in X$.

$\mu(y) = \beta(f(b))$
$\geq \min\{\beta(f(a*(b*c)*b)), \beta(f(a))\}$
$= \min\{ \beta(f(a) *' (f(b) *' f(c)) *' f(b)), \beta(f(a))\}$
$= \min\{\mu(x*(y*z)*y)), \mu(x)\}$

Therefore μ is a positive implicative fuzzy filter of X.

Theorem 4.7.

Let $f:(X;*,0) \to (Y;*',0')$ be an epimorphism of KK-algebra and μ is fuzzy subset of X , If μ is a positive implicative fuzzy filter of X. Then $f(\mu)$ is a positive implicative fuzzy filter of Y.

Proof :

Assume that μ is a positive implicative fuzzy filter of X. For any $x \in X$, we have
$f(\mu)(e_x) = \mu(f(e_x)) = \mu(e_y) \geq \mu(f(x)) = f(\mu)(x)$.
Let $x, y, z \in Y$ such that $f(a) = x$, $f(b) = y$ and $f(c) = z$ for all $a, b, c \in X$.

$f(\mu)(y) = \mu(f(b))$
$\geq \min\{ \mu(f(a*(b*c)*b)), \mu(f(a))\}$
$= \min\{ \mu(f(a) *' (f(b) *' f(c)) *' f(b)), \mu(f(a))\}$
$= \min\{ f(\mu)(x*(y*z)*y)), f(\mu)(x))\}$.

Therefore $f(\mu)$ is a positive implicative fuzzy filter of Y.

5. Cartesian Product of fuzzy filter

In this section, we will discuss, investigate a new notion called Cartesian product of fuzzy filter, implicative fuzzy filer and positive implicative fuzzy filter, and study several basic properties which related to fuzzy filter .

Definition 5.1.[7].

A fuzzy relation R on any set S is a fuzzy subset $R: S \times S \to [0,1]$.

Definition 5.2.[7].

If R is a fuzzy relation on sets S and β is a fuzzy subset of S , then R is a fuzzy relation on β if $R(x, y) \leq min\{\beta(x), \beta(y)\}$, for all $x, y \in S$.

Definition 5.3.[7].

Let μ and β be fuzzy subsets of a set S . The Cartesian product of μ and β is defined by $(\mu \times \beta)(x, y) = \min\{\mu(x), \beta(y)\}$, for all $x, y \in S$.

Remark 5.4.[7].

Let X and Y be KK-algebras, we define $(*)$ on $X \times Y$ by: for all $(x, y), (u, v) \in X \times Y, (x, y) * (u, v) = (x*u, y*v)$. Then clearly $(X \times Y, *, (0,0))$ is a KK-algebra .

Theorem 5.5 .

If μ and β are fuzzy subalgebras of a KK-algebra X, then $\mu \times \beta$ is also a fuzzy subalgebra of $X \times X$.

Proof :

Now let (x_1, x_2) , (y_1, y_2) $\in$ $X \times X$. Then

$(\mu \times \beta)$ $(x_1, x_2) *$ (y_1, y_2) =$(\mu \times \beta)$ $(x_1 *$ x_2 , $y_1 * y_2)$

= min { $\mu(x_1 *$ $x_2)$, $\beta(y_1 * y_2)$}

= min {min {μ (x_1) , $\beta(y_1)$} , min{ $\mu(x_2)$, $\beta(y_2)$}}

= min { $(\mu \times \beta)$ (x_1, y_1) , $(\mu \times \beta)$ (x_2, y_2)}.

This completes the proof .

Theorem 5.6.

Let μ and β be fuzzy filters of KK-algebra X. Then μ $\times \beta$ is a fuzzy filters of X $\times$ X.

Proof:

Note first that for every (x, y) $\in$ $X \times X$,

$(\mu \times \beta)(e_x, e_x)$ =min {$\mu(e_x), \beta(e_x)$} $\geq$ min {$\mu(x), \beta(y)$} = $(\mu \times \beta)$ = (x, y)

Now let (x_1, x_2) , (y_1, y_2) $\in$ $X \times X$. Then

$(\mu \times \beta)($ $y_1, y_2)$ = min { $\mu(y_1)$,$\beta(y_2)$}

$\geq$ min { min{$\mu(x_1 * y_1)$, $\mu(x_1)$}, min{$\beta(x_2 *$ y_2)$\beta(x_2)$}}

= min { min{$\mu(x_1 * y_1)$, $\beta(x_2 *$ $y_2)$ }, min{ $\mu(x_1)$, $\beta(x_2)$}}

= min {$(\mu \times \beta)($ $x_1 * y_1$, $x_2 *$ $y_2)$, $(\mu \times \beta)(x_1, x_2)$}.

Hence $(\mu \times \beta)$ is a fuzzy filter of $X \times X$.

Theorem 5.7.

Let μ and β be fuzzy subset of KK-algebra X such that $\mu \times \beta$ is a fuzzy filter of $X \times X$. Then , for all $x \in X$,

(i) either $\mu(e_x) \geq \mu(x)$ or $\beta(e_x) \geq$ $\beta(x)$.

(ii) $\mu(e_x) \geq \mu(x)$, for all $x \in X$, then either $\beta(e_x) \geq$ $\mu(x)$ or $\beta(e_x) \geq$ $\beta(x)$.

(iii) If $\beta(e_x) \geq$ $\beta(x)$, for all $x \in X$, then either $\mu(e_x) \geq \mu(x)$ or $\mu(e_x) \geq \beta(x)$.

Proof :

(i) suppose that $\mu(x) > \mu(e_x)$ and $\beta(y) > \beta(e_x)$, for some $x, y \in$ X.

Then $(\mu \times \beta)$ (x, y) = min { $\mu(x)$, $\beta(y)$} > min{ $\mu(e_x)$, $\beta(e_x)$ } = $(\mu \times \beta)(e_x, e_x)$ .This is a contradiction .Hence $\mu(e_x) \geq$ $\mu(x)$ or $\beta(e_x) \geq$ $\beta(x)$.

(ii) Assume that there exist $x, y \in$ X such that $\mu(x) > \beta(e_x)$ and $\beta(y) > \beta(e_x)$. Then $(\mu \times \beta)(e_x, e_x)$ = min{ $\mu(e_x)$, $\beta(e_x)$} = $\beta(e_x)$. Hence $(\mu \times \beta)$ (x, y) = min { $\mu(x)$, $\beta(y)$} > $\beta(e_x)$ = $(\mu \times \beta)(e_x, e_x)$. Which is a contradiction .Hence (ii) holds .

(iii) is by similar method to part (ii) .

Theorem 5.8.

Let μ and β be implicative fuzzy filters of KK-algebra X.Then μ $\times \beta$ is an implicative fuzzy filters of X $\times$ X.

Proof:

Note first that for every $(x, y) \in$ $X \times X$

$(\mu \times \beta)(e_x, e_x)$ =min {$\mu(e_x), \beta(e_x)$} $\geq$min {$\mu(x), \beta(y)$} = $(\mu \times \beta)$ = (x, y)

Now let (x_1, x_2) , (y_1, y_2), (z_1, z_2) $\in$ $X \times X$. Then

$(\mu \times \beta)($ $x_1, x_2) * (z_1, z_2)$ = min { $\mu(($ $x_1 * z_1)$, β $(x_2 * z_2)$

$\geq$ min { min{$\mu(x_1 * y_1)$, $\mu(x_1 * (y_1 * z_1))$, min{$\beta(x_2$ $*$ $y_2)$, $\beta(x_2 *$ $(y_2 * z_2))$}

= min { min{$\mu(x_1 * y_1)$, $\beta(x_2 *$ $y_2)$ }, min{ $\mu(x_1 * (y_1$ $* z_1))$, $\beta(x_2 *$ $(y_2 * z_2))$}}

= min { $(\mu \times \beta)($ $x_1 * y_1)$, $(x_2 *$ $y_2)$, $(\mu \times \beta)($ $x_1 * (y_1$ $* z_1))$, $(x_2 *$ $(y_2 * z_2))$.

Theorem 5.9.

Let μ and β be a positive implicative fuzzy filters of KK-algebra X. Then μ $\times \beta$ is a positive implicative fuzzy filters of X $\times$ X.

Proof:

Note first that for every $(x, y) \in X \times X$,

$(\mu \times \beta)(e_x, e_x)$ =min {$\mu(e_x), \beta(e_x)$} $\geq$min {$\mu(x), \beta(y)$} = $(\mu \times \beta)$ = (x, y)

Now let (x_1, x_2), (y_1, y_2), (z_1, z_2) $\in$ $X \times X$. Then

$(\mu \times \beta)($ $y_1, y_2)$ = min{ $\mu(y_1)$,$\beta(y_2)$}

$\geq$ min{min{$\mu(x_1 * (y_1 *$ $z_1) *$ $y_1))$, $\mu(x_1)$}, min{$(\beta(x_2 * ($ $y_2 *$ $z_2) *$ $y_2))$ $\beta(x_2)$}}

= min{min{$\mu(x_1 * (y_1 *$ $z_1) *$ $y_1))$, $\beta(x_2 * ($ $y_2 *$ $z_2) *$ $y_2))$ }, min{$\mu(x_1), \beta(x_2)$}

= min { $(\mu$ $\times \beta$ $)($ $x_1 * (y_1 *$ $z_1) *$ $y_1)$, $($ $x_2 * ($ $y_2 *$ $z_2) *$ $y_2)$, $(\mu$ $\times \beta)($ $x_1, x_2)$}. This completes the proof.

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