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Graceful Coloring of Wheel Graph Family

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***Abstract**: All graph in this paper are connected and simple graph. A proper vertex coloring $f: V(G) \rightarrow \{1, 2, \dots, k\}$, where $k \geq 2$ which induces a proper edge coloring $f: E(G) \rightarrow \{1, 2, \dots, k-1\}$ define by $f'(uv) = |f(u) - f(v)|$ is called a graceful k -coloring. If f is a graceful k-coloring for some $k \in N$, then a vertex coloring f of graph G is a graceful coloring. Graceful chromatic number of G, denoted by $\chi_g(G)$, is the minimum k for which G has a graceful k-coloring. In this paper, we determine the exact value of the graceful chromatic number of some wheel graph family, namely gear graph, helm graph, closed helm graph, flower graph, and web graph.*

Keywords: graceful coloring; graceful chromatic number; wheel graph family

1. INTRODUCTION

Let G be a simple and connected graph with vertex set $V(G)$ and edge set $E(G)$. The detailed information and notation about the graph can be seen in [3,5] There are many types of graph coloring, namely vertex coloring, edge coloring, and maps coloring in Wilson [8]. Vertex coloring of graph G is giving color to each vertex in a graph G such that the colors of any two adjacent vertices are distinct. Minimum k for which G have a k-coloring is called the chromatic number, denoted by $\chi(G)$. Edge coloring of graph G is giving color to each edge in a graph G such that the color of any two adjacent edges are distinct. One of the coloring problems is graceful coloring introduced by Chartrand and defined in [9]. Graceful coloring also studied in [1,2,4,6,7]. For illustration graceful coloring is provided in Figure 1.

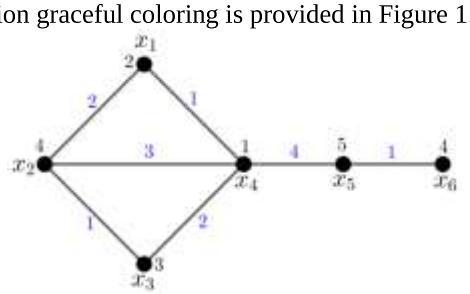

Fig. 1. *Graceful chromatic number,* $\chi_g(G) = 5$

English et.al in [4] discusses the graceful coloring of trees. Mincu et.al in [7] present some graphs with graceful chromatic number smaller than the number of vertices, equal to this order, and greater than the number of vertices. Furthermore, Alfarisi et.al in [1] found graceful chromatic number of unicyclic graphs namely (n, m)-tadpole graph, n-pan graph, and sun graph.

Definition 1.1 [9] A graceful k-coloring of nonempty graph G is a proper vertex coloring $c: V(G) \rightarrow \{1, 2, \dots, k\}$, where $k \geq 2$, that induces a proper edge coloring $: E(G) \rightarrow \{1, 2, \dots, k-1\}$ define by $c'(uv) = |c(u) - c(v)|$. Graceful coloring is a vertex coloring c of a graph G if c is a graceful k-coloring for some $k \in N$.

Definition 1.2 [9] The graceful chromatic number of graph G, denoted by $\chi_g(G)$ is the minimum k for which G has a graceful k-coloring.

Lemma 1.1 [9] If H_n is a sub graph of graph G_n, then $\chi_g(H_n) \leq \chi_g(G_n)$.

Theorem 1.1 [9] If W_n is a wheel graph with $n > 5$, then $\chi_g(W_n) = n + 1$.

In this paper, we study graceful coloring and determined graceful chromatic number of some wheel graph family, namely gear graph, helm graph, closed helm graph, flower graph, and web graph.

2. RESULT

Theorem 2.1 Graceful chromatic number of gear graph G_n is

$$\chi_g(G_n) = \begin{cases} 6, & for\ n = 3\ and\ 4 \\ n+1, & for\ n \geq 5 \end{cases}$$

Proof. The vertex set $V(G_n) = \{x\} \cup \{x_i;\ 1 \leq i \leq 2n\}$ and edge set $E(G_n) = \{xx_i; 1 \leq i \leq 2n, i\ odd\} \cup \{x_{1,i}x_{1,i+1}; 1 \leq i \leq 2n-1\} \cup \{x_1x_{2n}\}$.

Case 1: for $n = 3\ and\ 4$

We prove that $\chi_g(G_n) \geq 6$. Assume that $\chi_g(G_n) < 6$, suppose that $\chi_g(G_n) = 5$. Gear graph G_n contain wheel graph W_n. Suppose that x is a central vertex and x_i is a vertex cycle W_n in gear graph. Let x colored by 1, x_2 colored by 4, and x_3 colored by 3.

When $c(x_3) = 3$, then:

$c'(xx_3) = |c(x) - c(x_3)| = |1 - 3| = 2$

$c'(x_2x_3) = |c(x_2) - c(x_3)| = |4 - 3| = 1$

Next, we determine $c(x_4)$. Because $c(x_3) = 3$ and $x_3x_4 \in E(G_n)$, then $c(x_4) \neq 3$. If $c(x_4) = 1$, then $c'(x_3x_4) = |c(x_3) - c(x_4)| = |3 - 1| = 2$. Because x_3x_4 adjacent with xx_3, and $x_3x_4 = xx_3 = 2$, then contradiction with the definition of graceful coloring, where the color of any two adjacent edges are distinct. Similar with $c(x_4) = 1$, for $c(x_4) = 2, c(x_4) = 4, c(x_4) = 5$ also contradiction with the definition of graceful coloring, where the color of any two adjacent edges are distinct. Hence, $\chi_g(G_n) \geq 6$.

Furthermore, we prove that $\chi_g(G_n) \leq 6$. Define a proper vertex coloring $f: V(G_n) \rightarrow \{1, 2,\ 3 \ldots, 6\}$ as follows:

For $n = 3$

$$f(v) = \begin{cases} 1, & for\ v\ \in \{x\} \\ 2, & for\ v\ \in \{x_1\} \\ 3, & for\ v\ \in \{x_3\} \\ 4, & for\ v\ \in \{x_2, x_5\} \\ 5, & for\ v\ \in \{x_6\} \\ 6, & for\ v\ \in \{x_4\} \end{cases}$$

For $n = 4$

$$f(v) = \begin{cases} 1, & for\ v\ \in \{x\} \\ 2, & for\ v\ \in \{x_1\} \\ 3, & for\ v\ \in \{x_3, x_6\} \\ 4, & for\ v\ \in \{x_2, x_5\} \\ 5, & for\ v\ \in \{x_7\} \\ 6, & for\ v\ \in \{x_4, x_8\} \end{cases}$$

It can see that f also induces edge coloring of $G_n, f: E(G_n) \rightarrow \{1, 2, 3, 4\}$ as follows:
For $n = 3$

$$f(e) = \begin{cases} 1, & for\ e\ \in \{xx_1, x_2x_3, x_5x_6\} \\ 2, & for\ e\ \in \{xx_3, x_1x_2, x_4x_5\} \\ 3, & for\ e\ \in \{xx_5, x_3x_4, x_1x_6\} \end{cases}$$

For $n = 4$

$$f(e) = \begin{cases} 1, & for\ e\ \in \{xx_1, x_2x_3, x_5x_6, x_7x_8\} \\ 2, & for\ e\ \in \{xx_3, x_1x_2, x_4x_5, x_6x_7\} \\ 3, & for\ e\ \in \{xx_5, x_3x_4\} \\ 4, & for\ e\ \in \{xx_7, x_1x_8\} \end{cases}$$

There is 6-graceful coloring of G_3 and G_4. Therefore, it obtained that $\chi_g(G_n) \leq 6$. Hence $\chi_g(G_n) = 6$ for $n = 3,4$.

Case 2: for $n \geq 5$
We prove that $\chi_g(G_n) \geq n + 1$. Wheel graph W_n is a subgraph of gear graph G_n such that based on Lemma 1.1 and Theorem 1.1 that $\chi_g(G_n) \geq \chi_g(W_n) = n + 1$. Thus, we have $\chi_g(G_n) \geq n + 1$. Furthermore, we prove that $\chi_g(G_n) \leq n + 1$. Define a proper vertex coloring $f: V(G_n) \rightarrow \{1, 2,\ 3 \ldots, n + 1\}$ as follows:

$$f(v) = \begin{cases} 1, & for\ v\ \in \{x\} \\ 2, & for\ v\ \in \{x_1\} \\ \frac{i}{2}, & for\ v\ \in \{x_i, 2 < i \leq 2n, i\ even\} \\ \frac{i+3}{2}, & for\ v\ \in \{x_i, 1 < i \leq 2n, i\ odd\} \\ n + 1, & for\ v\ \in \{x_2\} \end{cases}$$

It can see that f also induces edge coloring of $G_n, f: E(G_n) \rightarrow \{1, 2, 3, \ldots, n\}$ as follows:

$$f(e) = \begin{cases} 1, & for\ e\ \in \{x_ix_{i+1}, 1 < i < 2n, i\ odd\} \\ 2, & for\ e\ \in \{x_ix_{i+1}, 1 < i < 2n, i\ even\} \\ \frac{i+1}{2}, & for\ e\ \in \{xx_i, 1 \leq i \leq 2n, i\ odd\} \\ n - 2, & for\ e\ \in \{x_2x_3, x_1x_{2n}\} \\ n - 2, & for\ e\ \in \{x_1x_2\} \end{cases}$$

There is $(n + 1)$-coloring of G_n. Therefore, it obtained that $\chi_g(G_n) \leq n + 1$. Hence $\chi_g(G_n) = n + 1$ for $n \geq 5$.

Theorem 2. 2. Graceful chromatic number of helm graph H_n is

$$\chi_g(H_n) = \begin{cases} 8, & for\ n = 4 \\ 8, & for\ n = 5\ and\ 6 \\ n + 1, & for\ n \geq 7 \end{cases}$$

Proof. The vertex set $V(H_n) = \{x\} \cup \{x_{j,i}; 1 \leq j \leq 2; 1 \leq i \leq n\}$ and edge set $E(H_n) = \{xx_{1,i}; 1 \leq i \leq n\} \cup \{x_{1,i}x_{2,i}; 1 \leq i \leq n\} \cup \{x_{1,i}x_{1,i+1}; 1 \leq i \leq n - 1\} \cup \{x_{1,1}x_{1,n}\}$.

Case 1: for $n = 4$

We prove that $\chi_g(H_4) \geq 8$. Assume that $\chi_g(H_4) < 8$, suppose that $\chi_g(H_4) = 7$. Helm graph H_4 contain wheel graph W_4. We define vertex coloring of W_4 is $2, 5, 4, 6$ and the central vertex colored by 1. Suppose that x is a central vertex and $x_{1,i}$ is a vertex cycle W_n in helm graph.

When $c(x_{1,3}) = 4$, then:

$$c'(xx_{1,3}) = |c(x) - c(x_{1,3})| = |1 - 4| = 3$$

$$c'(x_{1,2}x_{1,3}) = |c(x_{1,2}) - c(x_{1,3})| = |5 - 4| = 1$$

$$c'(x_{1,3}x_{1,4}) = |c(x_{1,3}) - c(x_{1,4})| = |4 - 6| = 2$$

Next, we determine $c(x_{2,3})$. Because $c(x_{1,3}) = 4$ and $x_{1,3}x_{2,3} \in E(H_4)$, then $c(x_{2,3}) \neq 4$. If $c(x_{2,3}) = 1$, then $c'(x_{1,3}x_{2,3}) = |c(x_{1,3}) - c(x_{2,3})| = |4 - 1| = 3$. Because $x_{1,3}x_{2,3}$ adjacent with $xx_{1,3}$, and $x_{1,3}x_{2,3} = xx_{1,3} = 3$, then contradiction with the definition of graceful coloring, where the color of any two adjacent edges are distinct. Similar with $c(x_{2,3}) = 1$, for $c(x_{2,3}) = 2, c(x_{2,3}) = 3, c(x_{2,3}) = 5$, $c(x_{2,3}) = 6$, $c(x_{2,3}) = 7$ also contradiction with the definition of graceful coloring, where the color of any two adjacent edges are distinct. Hence, $\chi_g(H_4) \geq 8$.

Furthermore, we prove that $\chi_g(H_4) \leq 8$. Define a proper vertex coloring $f: V(H_4) \rightarrow \{1, 2,\ 3 \ldots, 8\}$ as follows:

$$f(v) = \begin{cases} 1, & for\ v\ \in \{x\} \\ 2, & for\ v\ \in \{x_{1,1}\} \\ 3, & for\ v\ \in \{x_{2,4}\} \\ 4, & for\ v\ \in \{x_{1,3}, x_{2,1}\} \\ 5, & for\ v\ \in \{x_{1,2}\} \end{cases}$$

$$f(v)=\begin{cases}6, & for\ v\ \in\{x_{1,4}\}\\ 7, & for\ v\ \in\{x_{2,2}\}\\ 8, & for\ v\ \in\{x_{2,3}\}\end{cases}$$

It can see that f also induces edge coloring of $H_4, f: E(H_4) \rightarrow \{1,2,3,4,5\}$ as follows:

$$f(e)=\begin{cases}1, & for\ e\ \in\{xx_{1,1}, x_{1,2}x_{1,3}\}\\ 2, & for\ e\ \in\{x_{1,3}x_{1,4}, x_{1,1}x_{2,1}, x_{1,2}x_{2,2}\}\\ 3, & for\ e\ \in\{xx_{1,3}, x_{1,1}x_{1,2}, x_{1,4}x_{2,4}\}\\ 4, & for\ e\ \in\{xx_{1,2}, x_{1,1}x_{1,4}, x_{1,3}x_{2,3}\}\\ 5, & for\ e\ \in\{xx_{1,4}\}\end{cases}$$

There is 8-graceful coloring of H_4. Therefore, it obtained that $\chi_g(H_4) \leq 8$. Hence $\chi_g(H_4) = 8$.

Case 2: for $n = 5\ and\ 6$

We prove that $\chi_g(H_n) \geq 8$. Assume that $\chi_g(H_n) < 8$, suppose that $\chi_g(H_n) = 7$. Helm graph H_n contain wheel graph W_n. We define vertex coloring of W_5 is $2,4,3,6,5$ and the central vertex colored by 1. And vertex coloring of W_6 is $2,4,3,6,5,7$ and the central vertex colored by 1 Suppose that x is a central vertex and $x_{1,i}$ is a vertex cycle W_n in helm graph.

When $c(x_{1,2}) = 4$, then:

$$c'(xx_{1,2}) = |c(x) - c(x_{1,2})| = |1-4| = 3$$

$$c'(x_{1,1}x_{1,2}) = |c(x_{1,1}) - c(x_{1,2})| = |2-4| = 2$$

$$c'(x_{1,2}x_{1,3}) = |c(x_{1,2}) - c(x_{1,3})| = |4-3| = 1$$

Next, we determine $c(x_{2,2})$. Because $c(x_{1,2}) = 4$ and $x_{1,2}x_{2,2} \in E(H_n)$, then $c(x_{2,2}) \neq 4$. If $c(x_{2,2}) = 1$, then $c'(x_{1,2}x_{2,2}) = |c(x_{1,2}) - c(x_{2,2})| = |4-1| = 3$. Because $x_{1,2}x_{2,2}$ adjacent with $xx_{1,2}$, and $x_{1,2}x_{2,2} = xx_{1,2} = 3$, then contradiction with the definition of graceful coloring, where the color of any two adjacent edges are distinct. Similar with $c(x_{2,2}) = 1$, for $c(x_{2,2}) = 2, c(x_{2,2}) = 3, c(x_{2,2}) = 5$, $c(x_{2,2}) = 6$, $c(x_{2,2}) = 7$ also contradiction with the definition of graceful coloring, where the color of any two adjacent edges are distinct. Hence, $\chi_g(H_n) \geq 8$.

Furthermore, we prove that $\chi_g(H_n) \leq 8$. Define a proper vertex coloring $f: V(H_n) \rightarrow \{1,2,\ 3\ldots,8\}$ as follows:

For $n = 5$

$$f(v)=\begin{cases}1, & for\ v\ \in\{x\}\\ 2, & for\ v\ \in\{x_{1,1}\}\\ 3, & for\ v\ \in\{x_{1,3}, x_{2,5}\}\\ 4, & for\ v\ \in\{x_{1,2}, x_{2,4}\}\\ 5, & for\ v\ \in\{x_{1,5}\}\end{cases}$$

$$f(v)=\begin{cases}6, & for\ v\ \in\{x_{1,4}, x_{2,1}\}\\ 7, & for\ v\ \in\{x_{2,3}\}\\ 8, & for\ v\ \in\{x_{2,2}\}\end{cases}$$

For $n = 6$

$$f(v)=\begin{cases}1, & for\ v\ \in\{x\}\\ 2, & for\ v\ \in\{x_{1,1}, x_{2,5}\}\\ 3, & for\ v\ \in\{x_{1,3}\}\\ 4, & for\ v\ \in\{x_{1,2}, x_{2,4}\}\\ 5, & for\ v\ \in\{x_{1,5}\}\\ 6, & for\ v\ \in\{x_{1,4}, x_{2,1}, x_{2,6}\}\\ 7, & for\ v\ \in\{x_{1,6}, x_{2,3}\}\\ 8, & for\ v\ \in\{x_{2,2}\}\end{cases}$$

It can see that f also induces edge coloring of $H_n, f: E(H_n) \rightarrow \{1,2,3,\ldots,6\}$ as follows:

For $n = 5$

$$f(e)=\begin{cases}1, & for\ e\ \in\{xx_{1,1}, x_{1,2}x_{1,3}, x_{1,4}x_{1,5}\}\\ 2, & for\ e\ \in\{xx_{1,3}, x_{1,1}x_{1,2}, x_{1,4}x_{2,4}, x_{1,5}x_{2,5}\}\\ 3, & for\ e\ \in\{xx_{1,2}, x_{1,3}x_{1,4}, x_{1,1}x_{1,5}\}\\ 4, & for\ e\ \in\{xx_{1,5}\}\ and\ e \in \{x_{1,i}x_{2,i}, 1 \leq i \leq 3\}\\ 5, & for\ e\ \in\{xx_{1,4}\}\end{cases}$$

For $n = 6$

$$f(e)=\begin{cases}1, & for\ e\ \in\{xx_{1,1}, x_{1,6}x_{2,6}, x_{1,2}x_{1,3}, x_{1,4}x_{1,5}\}\\ 2, & for\ e\ \in\{xx_{1,3}, x_{1,1}x_{1,2}, x_{1,5}x_{1,6}, x_{1,4}x_{2,4}\}\\ 3, & for\ e\ \in\{xx_{1,2}, x_{1,3}x_{1,4}, x_{1,5}x_{2,5}\}\\ 4, & for\ e\ \in\{xx_{1,5}\}\ and\ e \in \{x_{1,i}x_{2,i}, 1 \leq i \leq 3\}\\ 5, & for\ e\ \in\{xx_{1,4}, x_{1,1}x_{1,6}\}\\ 6, & for\ e\ \in\{xx_{1,6}\}\end{cases}$$

There is 8-graceful coloring of H_5 and H_6. Therefore, it obtained that $\chi_g(H_n) \leq 8$. Hence $\chi_g(H_n) = 8$ for $n = 5, 6$.

Case 3: for $n \geq 7$

Wheel graph W_n is a subgraph of helm graph H_n such that based on Lemma 1.1 and Theorem 1.1 that $\chi_g(H_n) \geq \chi_g(W_n) = n+1$. Thus, we have $\chi_g(H_n) \geq n+1$. Furthermore, we prove that $\chi_g(H_n) \leq n+1$. Define a proper vertex coloring $f: V(H_n) \rightarrow \{1,2,\ 3\ldots,n+1\}$ as follows:

For n even

$$f(v)=\begin{cases}1, & for\ v\ \in\{x\}\\ 2, & for\ v\ \in\{x_{1,1}\}\\ 6, & for\ v\ \in\{x_{2,1}\}\\ 7, & for\ v\ \in\{x_{2,3}\}\\ 8, & for\ v\ \in\{x_{2,2}\}\\ i-2, & for\ v\ \in\{x_{2,i}, 5 \leq i < n-1, i\ odd\}\\ i, & for\ v\ \in\{x_{1,i}, 1 < i < n, i\ odd\}\\ & and\ v\ \in\{x_{2,i}, 4 \leq i < n, i\ even\}\end{cases}$$

$$f(v)=\begin{cases} i+2, & for\ v\ \in \{x_{1,i}, 1<i<n, i\ even\} \\ n-4, & for\ v\ \in \{x_{2,n-1}\} \\ n, & for\ v\ \in \{x_{2,n}\} \\ n+1, & for\ v\ \in \{x_{1,n}\} \end{cases}$$

For n odd

$$f(v)=\begin{cases} 1, & for\ v\ \in \{x\} \\ 2, & for\ v\ \in \{x_{1,1}\} \\ 6, & for\ v\ \in \{x_{2,1}\} \\ 7, & for\ v\ \in \{x_{2,3}\} \\ 8, & for\ v\ \in \{x_{2,2}\} \\ i-2, & for\ v\ \in \{x_{2,i}, 5\le i\le n, i\ odd\} \\ i, & for\ v\ \in \{x_{1,i}, 1<i\le n, i\ odd\} \\ & and\ v\ \in \{x_{2,i}, 4\le i<n, i\ even\} \\ i+2, & for\ v\ \in \{x_{1,i}, 1<i<n, i\ even\} \end{cases}$$

It can see that f also induces edge coloring of $H_n, f: E(H_n) \to \{1, 2, 3, \dots, n\}$ as follows:
For n even

$$f(e)=\begin{cases} 1, & for\ e\ \in \{xx_{1,1}, x_{1,n}x_{2,n}\} \\ & and\ e \in \{x_{1,i}x_{1,i+1}, 1<i<n, i\ even\} \\ 2, & for\ e\ \in \{x_{1,1}x_{1,2}, x_{1,n-1}x_{1,n}\} \\ & and\ e \in \{x_{1,i}x_{2,i}, 4\le i<n-1\} \\ 3, & for\ e\ \in \{x_{1,n-1}x_{2,n-1}\} \\ & and\ e \in \{x_{1,i}x_{1,i+1}, 1<i<n-1, i\ odd\} \\ 4, & for\ e\ \in \{x_{1,i}x_{2,i}, 1\le i\le 3\} \\ i-1, & for\ e\ \in \{xx_{1,i}, 1<i<n, i\ odd\} \\ i+1, & for\ e\ \in \{xx_{1,i}, 1<i<n, i\ even\} \\ n-1, & for\ e\ \in \{x_{1,1}x_{1,n}\} \\ n, & for\ e\ \in \{xx_{1,n}\} \end{cases}$$

For n odd

$$f(e)=\begin{cases} 1, & for\ e\ \in \{xx_{1,1}\} \\ & and\ e\ \in \{x_{1,i}x_{1,i+1}, 1<i<n, i\ even\} \\ 2, & for\ e\ \in \{x_{1,1}x_{1,2}\} \\ & and\ \in \{x_{1,i}x_{2,i}, 4\le i\le n\} \\ 3, & for\ e\ \in \{x_{1,i}x_{1,i+1}, 1<i<n, i\ odd\} \\ 4, & for\ e \in \{x_{1,i}x_{2,i}, 1\le i\le 3\} \\ i-1, & for\ e\ \in \{xx_{1,i}, 1<i\le n, i\ odd\} \\ i+1, & for\ e\ \in \{xx_{1,i}, 1<i<n, i\ even\} \\ n-2, & for\ e\ \in \{x_{1,1}x_{1,n}\} \end{cases}$$

There is $(n+1)$-coloring of H_n. Therefore, it obtained that $\chi_g(H_n) \le n+1$. Hence $\chi_g(H_n) = n+1$ for $n \ge 7$.

Theorem 2. 3. Graceful chromatic number of closed helm graph CH_n is

$$\chi_g(CH_n)=\begin{cases} 8, & for\ n=4 \\ 8, & for\ n=5\ and\ 6 \\ n+1, & for\ n\ge 7 \end{cases}$$

Proof. The vertex set $V(CH_n) = \{x\} \cup \{x_{j,i}; 1\le j\le 2; 1\le i\le n\}$ and edge set $E(H_n) = \{xx_{1,i}; 1\le i\le n\} \cup \{x_{1,i}x_{2,i}; 1\le i\le n\} \cup \{x_{1,i}x_{1,i+1}; 1\le i\le n-1\} \cup \{x_{2,i}x_{2,i+1}; 1\le i\le n-1\} \cup \{x_{1,1}x_{1,n}, x_{2,1}x_{2,n}\}$.

Case 1: for $n=4$

We prove that $\chi_g(CH_4) \ge 8$. Assume that $\chi_g(CH_4) < 8$, suppose that $\chi_g(CH_4) = 7$. Closed helm graph CH_4 contain wheel graph W_4. We define vertex coloring of W_4 is $2, 5, 4, 6$ and the central vertex colored by 1. Suppose that x is a central vertex and $x_{1,i}$ is a vertex cycle W_n in closed helm graph.

When $c(x_{1,3}) = 4$, then:

$c'(xx_{1,3}) = |c(x) - c(x_{1,3})| = |1-4| = 3$

$c'(x_{1,2}x_{1,3}) = |c(x_{1,2}) - c(x_{1,3})| = |5-4| = 1$

$c'(x_{1,3}x_{1,4}) = |c(x_{1,3}) - c(x_{1,4})| = |4-6| = 2$

Next, we determine $c(x_{2,3})$. Because $c(x_{1,3}) = 4$ and $x_{1,3}x_{2,3} \in E(CH_4)$, then $c(x_{2,3}) \ne 4$. If $c(x_{2,3}) = 1$, then $c'(x_{1,3}x_{2,3}) = |c(x_{1,3}) - c(x_{2,3})| = |4-1| = 3$. Because $x_{1,3}x_{2,3}$ adjacent with $xx_{1,3}$, and $x_{1,3}x_{2,3} = xx_{1,3} = 3$, then contradiction with the definition of graceful coloring, where the color of any two adjacent edges are distinct. Similar with $c(x_{2,3}) = 1$, for $c(x_{2,3}) = 2, c(x_{2,3}) = 3, c(x_{2,3}) = 5$, $c(x_{2,3}) = 6$, $c(x_{2,3}) = 7$ also contradiction with the definition of graceful coloring, where the color of any two adjacent edges are distinct. Hence, $\chi_g(CH_4) \ge 8$.

Furthermore, we prove that $\chi_g(CH_4) \le 8$. Define a proper vertex coloring $f: V(CH_4) \to \{1, 2, 3 \dots, 8\}$ as follows:

$$f(v)=\begin{cases} 1, & for\ v\ \in \{x\} \\ 2, & for\ v\ \in \{x_{1,1}\} \\ 3, & for\ v\ \in \{x_{2,4}\} \\ 4, & for\ v\ \in \{x_{1,3}, x_{2,1}\} \\ 5, & for\ v\ \in \{x_{1,2}\} \\ 6, & for\ v\ \in \{x_{1,4}\} \\ 7, & for\ v\ \in \{x_{2,2}\} \\ 8, & for\ v\ \in \{x_{2,3}\} \end{cases}$$

It can see that f also induces edge coloring of $CH_4, f: E(CH_4) \to \{1, 2, 3, 4, 5\}$ as follows:

$$f(e)=\begin{cases}1, & for\ e\ \in \{xx_{1,1}, x_{1,2}x_{1,3}, x_{2,2}x_{2,3}, x_{2,1}x_{2,4}\}\\ 2, & for\ e\ \in \{x_{1,3}x_{1,4}, x_{1,1}x_{2,1}, x_{1,2}x_{2,2}\}\\ 3, & for\ e\ \in \{xx_{1,3}, x_{1,1}x_{1,2}, x_{1,4}x_{2,4}, x_{2,1}x_{2,2}\}\\ 4, & for\ e\ \in \{xx_{1,2}, x_{1,1}x_{1,4}, x_{1,3}x_{2,3}\}\\ 5, & for\ e\ \in \{xx_{1,4}, x_{2,3}x_{2,4}\}\end{cases}$$

There is 8-graceful coloring of CH_4. Therefore, it obtained that $\chi_g(CH_4) \leq 8$. Hence $\chi_g(CH_4) = 8$.

Case 2: for $n = 5\ and\ 6$

We prove that $\chi_g(CH_n) \geq 8$. Assume that $\chi_g(CH_n) < 8$, suppose that $\chi_g(CH_n) = 7$. Closed helm graph CH_n contain wheel graph W_n. We define vertex coloring of W_5 is $2, 4, 3, 6, 5$ and the central vertex colored by 1. And vertex coloring of W_6 is $2, 4, 3, 6, 5, 7$ and the central vertex colored by 1Suppose that x is a central vertex and $x_{1,i}$ is a vertex cycle W_n in helm graph.

When $c(x_{1,2}) = 4$, then:

$c'(xx_{1,2}) = |c(x) - c(x_{1,2})| = |1 - 4| = 3$

$c'(x_{1,1}x_{1,2}) = |c(x_{1,1}) - c(x_{1,2})| = |2 - 4| = 2$

$c'(x_{1,2}x_{1,3}) = |c(x_{1,2}) - c(x_{1,3})| = |4 - 3| = 1$

Next, we determine $c(x_{2,2})$. Because $c(x_{1,2}) = 4$ and $x_{1,2}x_{2,2} \in E(CH_n)$, then $c(x_{2,2}) \neq 4$. If $c(x_{2,2}) = 1$, then $c'(x_{1,2}x_{2,2}) = |c(x_{1,2}) - c(x_{2,2})| = |4 - 1| = 3$. Because $x_{1,2}x_{2,2}$ adjacent with $xx_{1,2}$, and $x_{1,2}x_{2,2} = xx_{1,2} = 3$, then contradiction with the definition of graceful coloring, where the color of any two adjacent edges are distinct. Similar with $c(x_{2,2}) = 1$, for $c(x_{2,2}) = 2, c(x_{2,2}) = 3, c(x_{2,2}) = 5$, $c(x_{2,2}) = 6$, $c(x_{2,2}) = 7$ also contradiction with the definition of graceful coloring, where the color of any two adjacent edges are distinct. Hence, $\chi_g(CH_n) \geq 8$.

Furthermore, we prove that $\chi_g(CH_n) \leq 8$. Define a proper vertex coloring $f: V(CH_n) \to \{1, 2,\ 3 \ldots, 8\}$ as follows:

For $n = 5$

$$f(v)=\begin{cases}1, & for\ v\ \in \{x\}\\ 2, & for\ v\ \in \{x_{1,1}\}\\ 3, & for\ v\ \in \{x_{1,3}, x_{2,5}\}\\ 4, & for\ v\ \in \{x_{1,2}, x_{2,4}\}\\ 5, & for\ v\ \in \{x_{1,5}\}\\ 6, & for\ v\ \in \{x_{1,4}, x_{2,1}\}\\ 7, & for\ v\ \in \{x_{2,3}\}\\ 8, & for\ v\ \in \{x_{2,2}\}\end{cases}$$

For $n = 6$

$$f(v)=\begin{cases}1, & for\ v\ \in \{x\}\\ 2, & for\ v\ \in \{x_{1,1}\}\\ 3, & for\ v\ \in \{x_{1,3}, x_{2,6}\}\\ 4, & for\ v\ \in \{x_{1,2}, x_{2,4}\}\\ 5, & for\ v\ \in \{x_{1,5}\}\\ 6, & for\ v\ \in \{x_{1,4}, x_{2,1}\}\\ 7, & for\ v\ \in \{x_{1,6}, x_{2,3}\}\\ 8, & for\ v\ \in \{x_{2,2}, x_{2,5}\}\end{cases}$$

It can see that f also induces edge coloring of $CH_n, f: E(CH_n) \to \{1, 2, 3, \ldots, 6\}$ as follows:
For $n = 5$

$$f(e)=\begin{cases}1, & for\ e\ \in \{xx_{1,1}, x_{1,2}x_{1,3}, x_{1,4}x_{1,5}, x_{2,2}x_{2,3}\}\\ & and\ e \in \{x_{2,4}x_{2,5}\}\\ 2, & for\ e\ \in \{xx_{1,3}, x_{1,1}x_{1,2}, x_{2,1}x_{2,2}, x_{1,4}x_{2,4}\}\\ & and\ e\ \in \{x_{1,5}x_{2,5}\}\\ 3, & for\ e\ \in \{xx_{1,2}, x_{1,3}x_{1,4}, x_{1,1}x_{1,5}, x_{2,3}x_{2,4}\}\\ & and\ e\ \in \{x_{2,1}x_{2,5}\}\\ 4, & for\ e\ \in \{xx_{1,5}\}\ and\ e \in \{x_{1,i}x_{2,i}, 1 \leq i \leq 3\}\\ 5, & for\ e\ \in \{xx_{1,4}\}\end{cases}$$

For $n = 6$

$$f(e)=\begin{cases}1, & for\ e\ \in \{xx_{1,1}, x_{2,2}x_{2,3}, x_{1,2}x_{1,3}, x_{1,4}x_{1,5}\}\\ 2, & for\ e\ \in \{xx_{1,3}, x_{1,1}x_{1,2}, x_{2,1}x_{2,2}, x_{1,4}x_{2,4}\}\\ & and\ e\ \in \{x_{1,5}x_{1,6}\}\\ 3, & for\ e\ \in \{xx_{1,2}, x_{1,3}x_{1,4}, x_{1,5}x_{2,5}, x_{2,3}x_{2,4}\}\\ & and\ e\ \in \{x_{2,1}x_{2,6}\}\\ 4, & for\ e\ \in \{xx_{1,5}, x_{1,6}x_{2,6}, x_{2,4}x_{2,5}\}\\ & and\ e \in \{x_{1,i}x_{2,i}, 1 \leq i \leq 3\}\\ 5, & for\ e\ \in \{xx_{1,4}, x_{1,1}x_{1,6}, x_{2,5}x_{2,6}\}\\ 6, & for\ e\ \in \{xx_{1,6}\}\end{cases}$$

There is 8-graceful coloring of CH_5 and CH_6. Therefore, it obtained that $\chi_g(CH_n) \leq 8$. Hence $\chi_g(CH_n) = 8$ for $n = 5, 6$.

Case 3: for $n \geq 7$
We prove that $\chi_g(CH_n) \geq n + 1$. Wheel graph W_n is a subgraph of closed helm graph CH_n such that based on Lemma 1.1 and Theorem 1.1 that $\chi_g(CH_n) \geq \chi_g(W_n) = n + 1$. Thus, we have $\chi_g(CH_n) \geq n + 1$. Furthermore, we prove that $\chi_g(CH_n) \leq n + 1$. Define a proper vertex coloring $f: V(CH_n) \to \{1, 2,\ 3 \ldots, n + 1\}$ as follows:
For $n = 7$

$$f(v)=\begin{cases}1, & for\ v\ \in \{x\}\\ 2, & for\ v\ \in \{x_{1,1}, x_{2,4}\}\\ 3, & for\ v\ \in \{x_{1,3}, x_{2,7}\}\\ 4, & for\ v\ \in \{x_{1,2}, x_{2,6}\}\end{cases}$$

$$f(v)=\begin{cases}5, & for\ v\ \in\{x_{1,5}\}\\ 6, & for\ v\ \in\{x_{1,4},x_{2,1}\}\\ 7, & for\ v\ \in\{x_{1,6},x_{2,3}\}\\ 8, & for\ v\ \in\{x_{1,7},x_{2,2},x_{2,5}\}\end{cases}$$

For n even

$$f(v)=\begin{cases}1, & for\ v\ \in\{x\}\\ 2, & for\ v\ \in\{x_{1,1},x_{2,n-1}\}\\ 3, & for\ v\ \in\{x_{2,n}\}\\ 6, & for\ v\ \in\{x_{2,1}\}\\ 7, & for\ v\ \in\{x_{2,3}\}\\ 8, & for\ v\ \in\{x_{2,2}\}\\ i-2, & for\ v\ \in\{x_{2,i},5\le i<n-1,i\ odd\}\\ i, & for\ v\ \in\{x_{1,i},1<i<n,i\ odd\}\\ & and\ v\ \in\{x_{2,i},4\le i<n-1,i\ even\}\\ i+2, & for\ v\ \in\{x_{1,i},1<i<n,i\ even\}\\ n+1, & for\ v\ \in\{x_{1,n}\}\end{cases}$$

For n odd, $n\ge 9$

$$f(v)=\begin{cases}1, & for\ v\ \in\{x\}\\ 2, & for\ v\ \in\{x_{1,1}\}\\ 7, & for\ v\ \in\{x_{2,3}\}\\ 8, & for\ v\ \in\{x_{2,2}\}\\ i-2, & for\ v\ \in\{x_{2,i},5\le i\le n,i\ odd\}\\ i, & for\ v\ \in\{x_{1,i},1<i\le n,i\ odd\}\\ & and\ v\ \in\{x_{2,i},4\le i<n,i\ even\}\\ i+2, & for\ v\ \in\{x_{1,i},1<i<n,i\ even\}\\ n+1, & for\ v\ \in\{x_{2,1}\}\end{cases}$$

It can see that f also induces edge coloring of $CH_n, f:E(CH_n)\to\{1,2,3,\dots,n\}$ as follows:
For $n=7$

$$f(e)=\begin{cases}1, & for\ e\ \in\{xx_{1,1},x_{1,2}x_{1,3},x_{1,4}x_{1,5},x_{1,6}x_{1,7}\}\\ & and\ e\in\{x_{2,2}x_{2,3},x_{2,6}x_{2,7}\}\\ 2, & for\ e\ \in\{xx_{1,3},x_{1,1}x_{1,2},x_{1,5}x_{1,6},x_{2,1}x_{2,2}\}\\ 3, & for\ e\ \in\{xx_{1,2},x_{1,3}x_{1,4},x_{1,5}x_{2,5},x_{2,1}x_{2,7}\}\\ & and\ e\ \in\{x_{1,6}x_{2,6}\}\\ 4, & for\ e\ \in\{xx_{1,5},x_{2,5}x_{2,6}\}\\ & and\ e\in\{x_{1,i}x_{2,i},1\le i\le 4\}\\ 5, & for\ e\ \in\{xx_{1,4},x_{1,7}x_{2,7},x_{2,3}x_{2,4}\}\\ 6, & for\ e\ \in\{xx_{1,6},x_{1,1}x_{1,6},x_{2,4}x_{2,5}\}\\ 7, & for\ e\ \in\{xx_{1,7}\}\end{cases}$$

For n even

$$f(e)=\begin{cases}1, & for\ e\ \in\{xx_{1,1},x_{2,n}x_{2,n-1}\},\\ & e\in\{x_{1,i}x_{1,i+1},1<i<n,i\ even\}\\ & and\ e\ \in\{x_{2,i}x_{2,i+1},1<i<n-2,i\ odd\}\\ 2, & for\ e\ \in\{x_{1,1}x_{1,2},x_{1,n-1}x_{1,n},x_{2,1}x_{2,2}\}\\ & and\ e\in\{x_{1,i}x_{2,i},4\le i<n-1\}\\ 3, & for\ e\ \in\{x_{2,1}x_{2,n}\},\\ & e\in\{x_{1,i}x_{1,i+1},1<i<n-1,i\ odd\}\\ & and\ e\ \in\{x_{2,i}x_{2,i+1},1<i<n-1,i\ odd\}\\ 4, & for\ e\ \in\{x_{1,i}x_{2,i},1\le i\le 3\}\\ i-1, & for\ e\ \in\{xx_{1,i},1<i<n,i\ odd\}\\ i+1, & for\ e\ \in\{xx_{1,i},1<i<n,i\ even\}\\ n-4, & for\ e\ \in\{x_{2,n-2}x_{2,n-1}\}\\ n-3, & for\ e\ \in\{x_{1,n-1}x_{2,n-1}\}\\ n-2, & for\ e\ \in\{x_{1,n}x_{2,n}\}\\ n-1, & for\ e\ \in\{x_{1,1}x_{1,n}\}\\ n, & for\ e\ \in\{xx_{1,n}\}\end{cases}$$

For n odd, $n\ge 9$

$$f(e)=\begin{cases}1, & for\ e\ \in\{xx_{1,1}\},\\ & e\ \in\{x_{1,i}x_{1,i+1},1<i<n,i\ even\}\\ & and\ e\ \in\{x_{2,i}x_{2,i+1},1<i<n,i\ even\}\\ 2, & for\ e\ \in\{x_{1,1}x_{1,2}\}\\ & and\ \in\{x_{1,i}x_{2,i},4\le i\le n\}\\ 3, & for\ e\ \in\{x_{1,i}x_{1,i+1},1<i<n,i\ odd\}\\ & and\ e\ \in\{x_{2,i}x_{2,i+1},1<i<n,i\ odd\}\\ 4, & for\ e\in\{x_{1,i}x_{2,i},1<i\le 3\}\\ i-1, & for\ e\ \in\{xx_{1,i},1<i\le n,i\ odd\}\\ i+1, & for\ e\ \in\{xx_{1,i},1<i<n,i\ even\}\\ n-7, & for\ e\ \in\{x_{2,1}x_{2,2}\}\\ n-2, & for\ e\ \in\{x_{1,1}x_{1,n}\}\\ n-1, & for\ e\ \in\{x_{1,1}x_{2,1}\}\end{cases}$$

There is $(n+1)$-coloring of CH_n. Therefore, it obtained that $\chi_g(CH_n)\le n+1$. Hence $\chi_g(CH_n)=n+1$ for $n\ge 7$.

Theorem 2.4 Graceful chromatic number of flower graph Fl_n is $n+1$ for $n\ge 4$

Proof. The vertex set $V(Fl_n)=\{x\}\cup\{x_{j,i}; 1\le j\le 2; 1\le i\le n\}$ and edge set $E(Fl_n)=\{xx_{1,i}; 1\le i\le n\}\cup\{x_{1,i}x_{2,i}; 1\le i\le n\}\cup\{x_{1,i}x_{1,i+1}; 1\le i\le n-1\}\cup\{xx_{2,i}; 1\le i\le n-1\}\cup\{x_{1,1}x_{1,n}\}$.

We prove that $\chi_g(Fl_n)\ge 2n+1$. Suppose that x is a central vertex of $Fl_n, x_{1,i}$ is a vertex cycle of Fl_n, and $x_{2,i}$ is a pendant that connected in a central vertex. Assume that $\chi_g(Fl_n)<2n+1$. Take $\chi_g(Fl_n)=2n, Fl_n$ colored with $2n$ colors. Fl_n have $2n+1$ vertices, so that, if Fl_n colored with

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$2n$ colors, then there are two vertices that have the same color. Suppose that the same color is $2n$. Then, vertex x colored by 1. Based on the assumption below, there are 3 possibilities, namely

a. Color $2n$ lies on not adjacent vertices $x_{1,i}$. Take $x_{1,1}$ and $x_{1,3}$.
$c(x) = 1$ and $c(x_{1,1}) = 2n$, then $c'(xx_{1,1}) = |c(x) - c(x_{1,1})| = |1 - 2n|$.
$c(x) = 1$ and $c(x_{1,3}) = 2n$, then $c'(xx_{1,3}) = |c(x) - c(x_{1,3})| = |1 - 2n|$.
$xx_{1,1}$ adjacent with $xx_{1,3}$, and $c'(xx_{1,1}) = c'(xx_{1,3}) = |1 - 2n|$, then contradiction with the definition of graceful coloring. This also applies to all $x_{1,i}$, if color $2n$ lies on not adjacent vertices $x_{1,i}$, then there must be two adjacent edges $xx_{1,i}$ with the same color.

b. Color $2n$ lies on not adjacent vertices $x_{2,i}$. Take $x_{2,1}$ and $x_{2,3}$.
$c(x) = 1$ and $c(x_{2,1}) = 2n$, then $c'(xx_{2,1}) = |c(x) - c(x_{2,1})| = |1 - 2n|$.
$c(x) = 1$ and $c(x_{2,3}) = 2n$, then $c'(xx_{2,3}) = |c(x) - c(x_{2,3})| = |1 - 2n|$.
$xx_{2,1}$ adjacent with $xx_{2,3}$, and $c'(xx_{2,1}) = c'(xx_{2,3}) = |1 - 2n|$, then contradiction with the definition of graceful coloring. This also applies to all $x_{2,i}$, if color $2n$ lies on not adjacent vertices $x_{2,i}$, then there must be two adjacent edges $xx_{2,i}$ with the same color.

c. Color $2n$ lies on not adjacent vertices $x_{1,i}$ and $x_{2,i}$. Take $x_{1,1}$ and $x_{2,3}$.
$c(x) = 1$ and $c(x_{1,1}) = 2n$, then $c'(xx_{1,1}) = |c(x) - c(x_{1,1})| = |1 - 2n|$.
$c(x) = 1$ and $c(x_{2,3}) = 2n$, then $c'(xx_{2,3}) = |c(x) - c(x_{2,3})| = |1 - 2n|$.
$xx_{1,1}$ adjacent with $xx_{2,3}$, and $c'(xx_{1,1}) = c'(xx_{2,3}) = |1 - 2n|$, then contradiction with the definition of graceful coloring. This also applies to all $x_{1,i}$ and $x_{2,i}$, if color $2n$ lies on not adjacent vertices $x_{1,i}$ and $x_{2,i}$, then there must be two adjacent edges $xx_{1,i}$ and $xx_{2,i}$ with the same color.

It's possible that (a), (b), and (c) contradict with the definition of graceful coloring. Hence, $\chi_g(Fl_n) \geq 2n$. Furthermore, we prove that $\chi_g(Fl) \leq 2n + 1$. Define a proper vertex coloring $f: V(Fl_n) \to \{1, 2, 3 \dots, 2n + 1\}$ as follows

For $n = 4$

$$f(v) = \begin{cases} 1, & for\ v \in \{x\} \\ 2, & for\ v \in \{x_{2,4}\} \\ 3, & for\ v \in \{x_{2,3}\} \\ 4, & for\ v \in \{x_{2,2}\} \\ 5, & for\ v \in \{x_{2,1}\} \\ 6, & for\ v \in \{x_{1,1}\} \\ 7, & for\ v \in \{x_{1,3}\} \\ 8, & for\ v \in \{x_{1,2}\} \\ 9, & for\ v \in \{x_{1,4}\} \end{cases}$$

For n even, $n \geq 6$

$$f(v) = \begin{cases} 1, & for\ v \in \{x\} \\ 2, & for\ v \in \{x_{1,1}\} \\ i, & for\ v \in \{x_{1,i}, 1 < i < n, i\ odd\} \\ i+2, & for\ v \in \{x_{1,i}, 1 < i < n, i\ even\} \\ n+1, & for\ v \in \{x_{1,n}\} \\ n+i, & for\ v \in \{x_{2,i}, 1 < i < n-1\} \\ 2n-1, & for\ v \in \{x_{2,n}\} \\ 2n, & for\ v \in \{x_{2,n-1}\} \\ 2n+1, & for\ v \in \{x_{2,1}\} \end{cases}$$

For n odd

$$f(v) = \begin{cases} 1, & for\ v \in \{x\} \\ 2, & for\ v \in \{x_{1,1}\} \\ i, & for\ v \in \{x_{1,i}, 1 < i \leq n, i\ odd\} \\ i+2, & for\ v \in \{x_{1,i}, 1 < i < n, i\ even\} \\ n+i+1, & for\ v \in \{x_{2,i}, 1 \leq i \leq n\} \end{cases}$$

It can see that f also induces edge coloring of $Fl_n, f: E(Fl_n) \to \{1, 2, 3, \dots, 2n\}$ as follows:

$$f(e) = \begin{cases} 1, & for\ e \in \{xx_{2,4}, x_{1,2}x_{1,3}, x_{1,1}x_{2,1}\} \\ 2, & for\ e \in \{xx_{2,3}, x_{1,3}x_{1,4}, x_{1,1}x_{1,2}\} \\ 3, & for\ e \in \{xx_{1,2}, x_{1,1}x_{1,4}\} \\ 4, & for\ e \in \{xx_{2,1}, x_{1,2}x_{2,2}, x_{1,3}x_{2,3}\} \\ 5, & for\ e \in \{xx_{1,1}\} \\ 6, & for\ e \in \{xx_{1,3}\} \\ 7, & for\ e \in \{xx_{1,1}, x_{1,4}x_{2,4}\} \\ 8, & for\ e \in \{xx_{1,4}\} \end{cases}$$

For n even, $n \geq 6$

$$f(e) = \begin{cases} 1, & for\ e \in \{xx_{1,1}\} \\ i-1, & for\ e \in \{xx_{1,i}, 1 < i < n, i\ odd\} \\ i+1, & for\ e \in \{xx_{1,i}, 1 < i < n, i\ even\} \\ n-2, & for\ e \in \{x_{1,i}x_{2,i}, 1 < i \leq n, i\ even\} \\ n, & for\ e \in \{xx_{1,n}\}, \\ & e \in \{x_{1,i}x_{2,i}, 1 < i < n-1, i\ odd\} \end{cases}$$

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$$f(e)=\begin{cases} n+1, & for\ e\ \in\{x_{1,n-1}x_{2,n-1}\} \\ n+1-i, & for\ e\ \in\{xx_{2,i}, 1<i<n-1\} \\ 2n-2, & for\ e\ \in\{xx_{2,n}\} \\ 2n-1, & for\ e\ \in\{x_{1,1}x_{2,1}, xx_{2,n-1}\} \\ 2n, & for\ e\ \in\{xx_{2,1}\} \end{cases}$$

For n odd

$$f(e)=\begin{cases} 1, & for\ e\ \in\{xx_{1,1}\} \\ i-1, & for\ e\ \in\{xx_{1,i}, 1<i\le n, i\ odd\} \\ i+1, & for\ e\ \in\{xx_{1,i}, 1<i<n, i\ even\} \\ n-1, & for\ e\ \in\{x_{1,i}x_{2,i}, 1<i<n, i\ even\} \\ n, & for\ e\ \in\{x_{1,1}x_{2,1}\} \\ n+1, & for\ e\ \in\{x_{1,i}x_{2,i}, 1<i\le n, i\ odd\} \\ n+i, & for\ e\ \in\{xx_{2,i}, 1\le i\le n\} \end{cases}$$

There is $(2n+1)$-coloring of Fl_n. Therefore, it obtained that $\chi_g(Fl_n)\le 2n+1$. Hence $\chi_g(Fl_n)=2n+1$ for $n\ge 4$. For illustration graceful coloring of flower graph is provided in Figure 2.

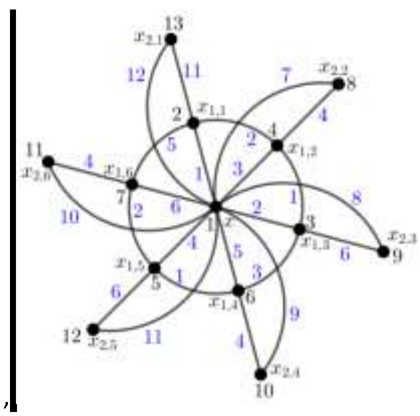

Fig. 2. *Flower graph,* $\chi_g(Fl_6)=7$

Theorem 2. 5. Graceful chromatic number of web graph Wb_n is

$$\chi_g(Wb_n)=\begin{cases} 8, & for\ n=4 \\ 8, & for\ n=5\ and\ 6 \\ n+1, & for\ n\ge 7 \end{cases}$$

Proof. The vertex set $V(Wb_n)=\{x\}\cup\{x_{j,i}; 1\le j\le 3; 1\le i\le n\}$ and edge set $E(Wb_n)=\{xx_{1,i}; 1\le i\le n\}\cup\{x_{1,i}x_{2,i}; 1\le i\le n\}\cup\{x_{2,i}x_{3,i}; 1\le i\le n\}\cup\{x_{1,i}x_{1,i+1}; 1\le i\le n-1\}\cup\{x_{2,i}x_{2,i+1}; 1\le i\le n-1\}\cup\{x_{1,1}x_{1,n},\ x_{2,1}x_{2,n}\}$.

Case 1: for $n=4$

We prove that $\chi_g(Wb_4)\ge 8$. Assume that $\chi_g(Wb_4)<8$, suppose that $\chi_g(Wb_4)=7$. Web graph Wb_4 contain wheel graph W_4. We define vertex coloring of W_4 is $2, 5, 4, 6$ and the central vertex colored by 1. Suppose that x is a central vertex and $x_{1,i}$ is a vertex cycle W_n in web graph.

When $c(x_{1,3})=4$, then:

$c'(xx_{1,3})=|c(x)-c(x_{1,3})|=|1-4|=3$

$c'(x_{1,2}x_{1,3})=|c(x_{1,2})-c(x_{1,3})|=|5-4|=1$

$c'(x_{1,3}x_{1,4})=|c(x_{1,3})-c(x_{1,4})|=|4-6|=2$

Next, we determine $c(x_{2,3})$. Because $c(x_{1,3})=4$ and $x_{1,3}x_{2,3}\in E(Wb_4)$, then $c(x_{2,3})\ne 4$. If $c(x_{2,3})=1$, then $c'(x_{1,3}x_{2,3})=|c(x_{1,3})-c(x_{2,3})|=|4-1|=3$. Because $x_{1,3}x_{2,3}$ adjacent with $xx_{1,3}$, and $x_{1,3}x_{2,3}=xx_{1,3}=3$, then contradiction with the definition of graceful coloring, where the color of any two adjacent edges are distinct. Similar with $c(x_{2,3})=1$, for $c(x_{2,3})=2, c(x_{2,3})=3, c(x_{2,3})=5$, $c(x_{2,3})=6$, $c(x_{2,3})=7$ also contradiction with the definition of graceful coloring, where the color of any two adjacent edges are distinct. Hence, $\chi_g(Wb_4)\ge 8$.

Furthermore, we prove that $\chi_g(Wb_4)\le 8$. Define a proper vertex coloring $f: V(Wb_4)\to\{1, 2,\ 3\dots, 8\}$ as follows:

$$f(v)=\begin{cases} 1, & for\ v\ \in\{x\} \\ 2, & for\ v\ \in\{x_{1,1}\} \\ 3, & for\ v\ \in\{x_{2,4}, x_{3,2}\} \\ 4, & for\ v\ \in\{x_{1,3}, x_{2,1}\} \\ 5, & for\ v\ \in\{x_{1,2}, x_{3,4}\} \\ 6, & for\ v\ \in\{x_{1,4}, x_{3,3}\} \\ 7, & for\ v\ \in\{x_{2,2}\} \\ 8, & for\ v\ \in\{x_{2,3}, x_{3,1}\} \end{cases}$$

It can see that f also induces edge coloring of $Wb_4, f: E(Wb_4)\to\{1, 2, 3, 4, 5\}$ as follows:

$$f(e)=\begin{cases} 1, & for\ e\ \in\{xx_{1,1}, x_{1,2}x_{1,3}, x_{2,2}x_{2,3}, x_{2,1}x_{2,4}\} \\ 2, & for\ e\ \in\{x_{1,3}x_{1,4}, x_{1,1}x_{2,1}, x_{1,2}x_{2,2}, x_{2,3}x_{3,4}\} \\ & and\ e\ \in\{x_{2,4}x_{3,4}\} \\ 3, & for\ e\ \in\{xx_{1,3}, x_{1,1}x_{1,2}, x_{1,4}x_{2,4}, x_{2,1}x_{2,2}\} \\ 4, & for\ e\ \in\{xx_{1,2}, x_{1,1}x_{1,4}, x_{1,3}x_{2,3}, x_{2,2}x_{3,2}\} \\ 5, & for\ e\ \in\{xx_{1,4}, x_{2,3}x_{2,4}\} \end{cases}$$

There is 8-graceful coloring of Wb_4. Therefore, it obtained that $\chi_g(Wb_4)\le 8$. Hence $\chi_g(Wb_4)=8$.

Case 2: for $n=5\ and\ 6$

Assume that $\chi_g(Wb_n)<8$, suppose that $\chi_g(Wb_n)=7$. Web graph Wb_n contain wheel graph W_n. We define vertex coloring of W_5 is $2, 4, 3, 6, 5$ and the central vertex colored by 1. And vertex coloring of W_6 is $2, 4, 3, 6, 5, 7$ and the central vertex colored by 1. Suppose that x is a central vertex and $x_{1,i}$ is a vertex cycle W_n in web graph.

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When $c(x_{1,2}) = 4$, then:

$$c'(xx_{1,2}) = |c(x) - c(x_{1,2})| = |1-4| = 3$$

$$c'(x_{1,1}x_{1,2}) = |c(x_{1,1}) - c(x_{1,2})| = |2-4| = 2$$

$$c'(x_{1,2}x_{1,3}) = |c(x_{1,2}) - c(x_{1,3})| = |4-3| = 1$$

Next, we determine $c(x_{2,2})$. Because $c(x_{1,2}) = 4$ and $x_{1,2}x_{2,2} \in E(CH_n)$, then $c(x_{2,2}) \neq 4$. If $c(x_{2,2}) = 1$, then $c'(x_{1,2}x_{2,2}) = |c(x_{1,2}) - c(x_{2,2})| = |4-1| = 3$. Because $x_{1,2}x_{2,2}$ adjacent with $xx_{1,2}$, and $x_{1,2}x_{2,2} = xx_{1,2} = 3$, then contradiction with the definition of graceful coloring, where the color of any two adjacent edges are distinct. Similar with $c(x_{2,2}) = 1$, for $c(x_{2,2}) = 2, c(x_{2,2}) = 3, c(x_{2,2}) = 5$, $c(x_{2,2}) = 6$, $c(x_{2,2}) = 7$ also contradiction with the definition of graceful coloring, where the color of any two adjacent edges are distinct. Hence, $\chi_g(Wb_n) \geq 8$.

Furthermore, we prove that $\chi_g(Wb_n) \leq 8$. Define a proper vertex coloring $f: V(Wb_n) \to \{1, 2, 3 \dots, 8\}$ as follows:

For $n = 5$

$$f(v) = \begin{cases} 1, \ for\ v \in \{x\} \\ 2, \ for\ v \in \{x_{1,1}\} \\ 3, \ for\ v \in \{x_{1,3}, x_{2,5}\} \\ 4, \ for\ v \in \{x_{1,2}, x_{2,4}\} \\ 5, \ for\ v \in \{x_{1,5}\}\ and\ v \in \{x_{3,i}, 1 \leq i \leq 3\} \\ 6, \ for\ v \in \{x_{1,4}, x_{2,1}\} \\ 7, \ for\ v \in \{x_{2,3}, x_{3,5}\} \\ 8, \ for\ v \in \{x_{2,2}, x_{3,4}\} \end{cases}$$

For $n = 6$

$$f(v) = \begin{cases} 1, \ for\ v \in \{x\} \\ 2, \ for\ v \in \{x_{1,1}\} \\ 3, \ for\ v \in \{x_{1,3}, x_{2,6}, x_{3,4}\} \\ 4, \ for\ v \in \{x_{1,2}, x_{2,4}\} \\ 5, \ for\ v \in \{x_{1,5}\}\ and\ v \in \{x_{3,i}, 1 \leq i \leq 3\} \\ 6, \ for\ v \in \{x_{1,4}, x_{2,1}, x_{3,5}\} \\ 7, \ for\ v \in \{x_{1,6}, x_{2,3}, x_{3,6}\} \\ 8, \ for\ v \in \{x_{2,2}, x_{2,5}\} \end{cases}$$

It can see that f also induces edge coloring of $Wb_n, f: E(Wb_n) \to \{1, 2, 3, \dots, 6\}$ as follows:

For $n = 5$

$$f(e) = \begin{cases} 1, \ for\ e \in \{xx_{1,1}, x_{1,2}x_{1,3}, x_{1,4}x_{1,5}, x_{2,2}x_{2,3}\} \\ \quad and\ e \in \{x_{2,4}x_{2,5}, x_{2,1}x_{3,1}\} \\ 2, \ for\ e \in \{xx_{1,3}, x_{1,1}x_{1,2}, x_{2,1}x_{2,2}, x_{2,3}x_{3,3}\} \\ \quad and\ e \in \{x_{1,4}x_{2,4}, x_{1,5}x_{2,5}\} \\ 3, \ for\ e \in \{xx_{1,2}, x_{1,3}x_{1,4}, x_{1,1}x_{1,5}, x_{2,3}x_{2,4}\} \\ \quad and\ e \in \{x_{2,1}x_{2,5}, x_{2,2}x_{3,2}\} \\ 4, \ for\ e \in \{xx_{1,5}, x_{2,4}x_{3,4}, x_{2,5}x_{3,5}\} \\ \quad and\ e \in \{x_{1,i}x_{2,i}, 1 \leq i \leq 3\} \\ 5, \ for\ e \in \{xx_{1,4}\} \end{cases}$$

For $n = 6$

$$f(e) = \begin{cases} 1, \ for\ e \in \{xx_{1,1}, x_{1,2}x_{1,3}, x_{1,4}x_{1,5}, x_{2,2}x_{2,3}\} \\ \quad and\ e \in \{x_{2,1}x_{3,1}, x_{2,4}x_{3,4}\} \\ 2, \ for\ e \in \{xx_{1,3}, x_{1,1}x_{1,2}, x_{2,1}x_{2,2}, x_{1,4}x_{2,4}\} \\ \quad and\ e \in \{x_{1,5}x_{1,6}, x_{2,3}x_{3,3}, x_{2,5}x_{3,5}\} \\ 3, \ for\ e \in \{xx_{1,2}, x_{1,3}x_{1,4}, x_{1,5}x_{2,5}, x_{2,3}x_{2,4}\} \\ \quad and\ e \in \{x_{2,1}x_{2,6}, x_{2,2}x_{3,2}\} \\ 4, \ for\ e \in \{xx_{1,5}, x_{1,6}x_{2,6}, x_{2,4}x_{2,5}, x_{2,6}x_{3,6}\} \\ \quad and\ e \in \{x_{1,i}x_{2,i}, 1 \leq i \leq 3\} \\ 5, \ for\ e \in \{xx_{1,4}, x_{1,1}x_{1,6}, x_{2,5}x_{2,6}\} \\ 6, \ for\ e \in \{xx_{1,6}\} \end{cases}$$

There is 8-graceful coloring of Wb_5 and Wb_6. Therefore, it obtained that $\chi_g(Wb_n) \leq 8$. Hence $\chi_g(Wb_n) = 8$ for $n = 5, 6$.

Case 3: for $n \geq 7$

We prove that $\chi_g(Wb_n) \geq n+1$. Wheel graph W_n is a subgraph of web graph Wb_n such that based on Lemma 1.1 and Theorem 1.1 that $\chi_g(Wb_n) \geq \chi_g(W_n) = n+1$. Thus, we have $\chi_g(Wb_n) \geq n+1$. Furthermore, we prove that $\chi_g(Wb_n) \leq n+1$. Define a proper vertex coloring $f: V(Wb_n) \to \{1, 2, 3 \dots, n+1\}$ as follows:

For $n = 7$

$$f(v) = \begin{cases} 1, \ for\ v \in \{x\} \\ 2, \ for\ v \in \{x_{1,1}, x_{2,4}, x_{3,6}\} \\ 3, \ for\ v \in \{x_{1,3}, x_{2,7}\} \\ 4, \ for\ v \in \{x_{1,2}, x_{2,6}\} \\ 5, \ for\ v \in \{x_{1,5}, x_{3,7}\}\ and\ v \in \{x_{3,i}, 1 \leq i \leq 4\} \\ 6, \ for\ v \in \{x_{1,4}, x_{2,1}\} \\ 7, \ for\ v \in \{x_{1,6}, x_{2,3}, x_{3,5}\} \\ 8, \ for\ v \in \{x_{1,7}, x_{2,2}, x_{2,5}\} \end{cases}$$

For n even

$$f(v) = \begin{cases} 1, \ for\ v \in \{x\} \\ 2, \ for\ v \in \{x_{1,1}, x_{2,n-1}\} \\ 3, \ for\ v \in \{x_{2,n}\} \\ 4, \ for\ v \in \{x_{3,n-1}\} \\ 5, \ for\ v \in \{x_{3,i}, 1 \leq i \leq 3\} \\ 6, \ for\ v \in \{x_{2,1}\} \\ 7, \ for\ v \in \{x_{2,3}, x_{3,n}\} \\ 8, \ for\ v \in \{x_{2,2}, x_{3,4}\} \\ i-4, \ for\ v \in \{x_{3,i}, 6 \leq i < n-2, i\ even\} \end{cases}$$

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$$f(v)=\begin{cases} i-2, & for\ v\ \in\{x_{2,i}, 5\le i<n-1, i\ odd\} \\ i, & for\ v\ \in\{x_{1,i}, 1<i<n, i\ odd\} \\ & and\ v\ \in\{x_{2,i}, 4\le i<n, i\ even\} \\ i+2, & for\ v\ \in\{x_{1,i}, 1<i<n, i\ even\} \\ & and\ v\ \in\{x_{3,i}, 5\le i<n-1, i\ odd\} \\ n-3, & for\ v\ \in\{x_{3,n-2}\} \\ n+1, & for\ v\ \in\{x_{1,n}\} \end{cases}$$

For n odd, $n\ge 9$

$$f(v)=\begin{cases} 1, & for\ v\ \in\{x\} \\ 2, & for\ v\ \in\{x_{1,1}\} \\ 5, & for\ v\ \in\{x_{3,2}, x_{3,3}\} \\ 7, & for\ v\ \in\{x_{2,3}\} \\ 8, & for\ v\ \in\{x_{2,2}, x_{3,4}\} \\ i-4, & for\ v\ \in\{x_{3,i}, 6\le i<n, i\ even\} \\ i-2, & for\ v\ \in\{x_{2,i}, 5\le i\le n, i\ odd\} \\ i, & for\ v\ \in\{x_{1,i}, 1<i\le n, i\ odd\} \\ & and\ v\ \in\{x_{2,i}, 4\le i<n, i\ even\} \\ i+2, & for\ v\ \in\{x_{1,i}, 1<i<n, i\ even\} \\ & and\ v\ \in\{x_{3,i}, 5\le i<n, i\ odd\} \\ n-6, & for\ v\ \in\{x_{3,n}\} \\ n, & for\ v\ \in\{x_{3,1}\} \\ n+1, & for\ v\ \in\{x_{2,1}\} \end{cases}$$

It can see that f also induces edge coloring of $Wb_n, f: E(Wb_n)\to\{1,2,3,\ldots,n\}$ as follows:
For $n=7$

$$f(e)=\begin{cases} 1, & for\ e\ \in\{xx_{1,1}, x_{2,2}x_{2,3}, x_{2,6}x_{2,7}, x_{2,1}x_{3,1}\}, \\ & e\in\{x_{2,5}x_{3,5}\} \\ & and\ e\ \in\{x_{1,i}x_{1,i+1}, 2\le i\le 6, i\ even\} \\ 2, & for\ e\ \in\{xx_{1,3}, x_{1,1}x_{1,2}, x_{1,5}x_{1,6}, x_{2,1}x_{2,2}\} \\ & and\ e\ \in\{x_{2,3}x_{3,3}, x_{2,6}x_{3,6}, x_{2,7}x_{3,7}\} \\ 3, & for\ e\ \in\{xx_{1,2}, x_{1,3}x_{1,4}, x_{1,5}x_{1,6}, x_{2,1}x_{2,7}\} \\ & and\ e\ \in\{x_{1,6}x_{2,6}, x_{2,2}x_{3,2}, x_{2,4}x_{3,4}\} \\ 4, & for\ e\ \in\{xx_{1,5}, x_{2,5}x_{2,6}\} \\ & and\ e\in\{x_{1,i}x_{2,i}, 1\le i\le 4\} \\ 5, & for\ e\ \in\{xx_{1,4}, x_{1,7}x_{2,7}, x_{2,3}x_{2,4}\} \\ 6, & for\ e\ \in\{xx_{1,6}, x_{1,1}x_{1,7}, x_{2,4}x_{2,5}\} \\ 7, & for\ e\ \in\{xx_{1,7}\} \end{cases}$$

For n even

$$f(e)=\begin{cases} 1, & for\ e\ \in\{xx_{1,1}, x_{2,1}x_{3,1}, x_{2,n-1}x_{2,n}\}, \\ & e\in\{x_{2,n-2}x_{3,n-2}\}, \\ & e\in\{x_{1,i}x_{1,i+1}, 1<i<n, i\ even\} \\ & and\ e\ \in\{x_{2,i}x_{2,i+1}, 1<i<n-2, i\ even\} \\ 2, & for\ e\ \in\{x_{1,1}x_{1,2}, x_{1,n-1}x_{1,n}, x_{2,1}x_{2,2}\}, \\ & e\ \in\{x_{2,3}x_{3,3}, x_{2,n-1}x_{3,n-1}\} \\ & and\ e\in\{x_{1,i}x_{2,i}, 4\le i<n-1\} \\ 3, & for\ e\ \in\{x_{2,1}x_{2,n}, x_{2,2}x_{3,2}\}, \\ & e\in\{x_{1,i}x_{1,i+1}, 1<i<n-1, i\ odd\} \\ & and\ e\ \in\{x_{2,i}x_{2,i+1}, 1<i<n-1, i\ odd\} \\ 4, & for\ e\ \in\{x_{2,n}x_{3,n}\} \\ & e\ \in\{x_{1,i}x_{2,i}, 1\le i\le 3\} \\ & and\ e\ \in\{x_{2,i}x_{3,i}, 4\le i<n-2\} \\ i-1, & for\ e\ \in\{xx_{1,i}, 1<i<n, i\ odd\} \\ i+1, & for\ e\ \in\{xx_{1,i}, 1<i<n, i\ even\} \\ n-4, & for\ e\ \in\{x_{2,n-2}x_{2,n-1}\} \\ n-3, & for\ e\ \in\{x_{1,n-1}x_{2,n-1}\} \\ n-2, & for\ e\ \in\{x_{1,n}x_{2,n}\} \\ n-1, & for\ e\ \in\{x_{1,1}x_{1,n}\} \\ n, & for\ e\ \in\{xx_{1,n}\} \end{cases}$$

For n odd, $n\ge 9$

$$f(e)=\begin{cases} 1, & for\ e\ \in\{xx_{1,1}, x_{2,1}x_{3,1}\}, \\ & e\ \in\{x_{1,i}x_{1,i+1}, 1<i<n, i\ even\} \\ & and\ e\ \in\{x_{2,i}x_{2,i+1}, 1<i<n, i\ even\} \\ 2, & for\ e\ \in\{x_{1,1}x_{1,2}, x_{2,3}x_{3,3}\} \\ & and\ \in\{x_{1,i}x_{2,i}, 4\le i\le n\} \\ 3, & for\ e\ \in\{x_{2,1}x_{2,n}, x_{2,2}x_{3,2}\}, \\ & e\ \in\{x_{1,i}x_{1,i+1}, 1<i<n, i\ odd\} \\ & and\ e\ \in\{x_{2,i}x_{2,i+1}, 1<i<n, i\ odd\} \\ 4, & for\ \ e\in\{x_{1,2}x_{2,2}, x_{1,3}x_{2,3}\} \\ & for\ \ e\in\{x_{2,i}x_{3,i}, 4\le i\le n\} \\ i-1, & for\ e\ \in\{xx_{1,i}, 1<i\le n, i\ odd\} \\ i+1, & for\ e\ \in\{xx_{1,i}, 1<i<n, i\ even\} \\ n-7, & for\ e\ \in\{x_{2,1}x_{2,2}\} \\ n-2, & for\ e\ \in\{x_{1,1}x_{1,n}\} \\ n-1, & for\ e\ \in\{x_{1,1}x_{2,1}\} \end{cases}$$

There is $(n+1)$-coloring of Wb_n. Therefore, it obtained that $\chi_g(Wb_n)\le n+1$. Hence $\chi_g(Wb_n)=n+1$ for $n\ge 7$. For illustration graceful coloring of web graph is provided in Figure 3.

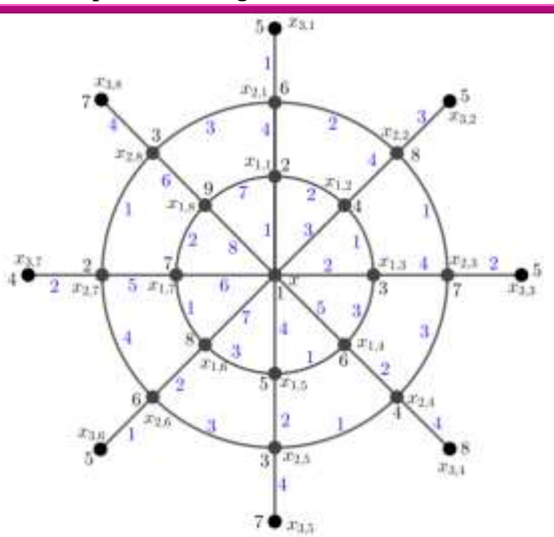

Fig. 3. *Web graph*, $\chi_g(Wb_8) = 9$

3. CONCLUSION

In this paper, we have investigated the graceful coloring of wheel related graphs, namely gear graph G_n, helm graph H_n, closed helm graph CH_n, flower graph Fl_n, and web graph Wb_n. We have the exact value of the graceful chromatic number of some wheel graph family, namely $\chi_g(G_n) = 6$ for $n = 4$ and $\chi_g(G_n) = n + 1$ for $n \geq 5$, $\chi_g(H_n) = 8$ for $n = 4$, $\chi_g(H_n) = 8$ for $n = 5,6$ and $\chi_g(H_n) = n + 1$ for $n \geq 7$, $\chi_g(CH_n) = 8$ for $n = 4$, $\chi_g(CH_n) = 8$ for $n = 5,6$ and $\chi_g(CH_n) = n + 1$ for $n \geq 7$, $\chi_g(Fl_n) = 2n + 1$ for $n \geq 4$, $\chi_g(Wb_n) = 8$ for $n = 4$, $\chi_g(Wb_n) = 8$ for $n = 5,6$ and $\chi_g(Wb_n) = n + 1$ for $n \geq 7$.

4. ACKNOWLEDGMENT

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